Antinormal ordering of Susskind-Glogower quantum phase operators

C. Brif and Y. Ben-Aryeh
Department of Physics, Technion–Israel Institute of Technology, Haifa 32000, Israel
(Received 9 February 1994)

A quantum phase description is developed by the use of the antinormal ordering of Susskind-Glogower phase operators. The theory is studied in the infinite-dimensional Hilbert space and it is closely related to the Pegg-Barnett quantum phase description in the finite-dimensional Hilbert space with a proper limiting procedure. Various applications of the antinormal ordering technique are presented.

PACS number(s): 42.50.Lc

I. INTRODUCTION

The problem of defining a Hermitian phase operator is an old one [1–3]. In quantum mechanics any observable should be related to a Hermitian operator. The problem with the use of Susskind and Glogower (SG) cos and sin Hermitian phase operators [1–3] follows from the fact that the energy spectrum is restricted from below. A very important development in this field has been made by Pegg and Barnett (PB) [4–6]. They have defined a Hermitian phase operator in a finite-dimensional (but arbitrarily large) Hilbert space Ψ. In the PB quantum phase formalism the problems about the unitarity of the exponential phase operator are solved by using a cyclic relation which connects the highest number state $|s\rangle$ and the vacuum state $|0\rangle$. Since the natural description of the electromagnetic field means the use of infinite-dimensional Hilbert space, one must take the infinite-dimensional limit at the end of expectation value calculations performed in the Ψ space [4–6]. The properties of such limiting procedures were discussed by Vaccaro and Pegg [7].

Lukš and Peřinová [8,9] have suggested the use of antinormal ordering for SG quasiunitary quantum phase operators. We will show later that use of this antinormal ordering is equivalent to defining the vacuum as a state of a random phase, analogous to the case of the familiar normal ordering of $\hat{a}$ and $\hat{a}^\dagger$ operators that defines the vacuum to be a state of zero energy. A similar idea for explaining the physical nature of the antinormal ordering was discussed by one of the authors [10]. Recently Vaccaro has shown [11] that the algebra of PB phase operators is conserved in the infinite-dimensional limit (the weak limit) when the antinormal ordering is used. The weak limit of a PB phase operator can be represented by a Fourier-like series in the SG quasiunitary operators $\widehat{e^{i\phi}_{\rm SG}}$ and $\widehat{e^{-i\phi}_{\rm SG}}$ [11]. The idea for using such Fourier-like operatoric series in the SG formalism was first introduced by Lukš and Peřinová [8,9].

In the present work we have made a systematic study of various properties of the antinormally ordered SG quantum phase operators and their relation to the PB quantum phase formalism. In Sec. II we discuss the basic theory and explain the physical meaning of the antinormal ordering. In Sec. III we derive additional useful properties of the antinormal ordering. We show that the antinormal ordering effectively replaces the orthogonality of phase states in the SG formalism. We develop a technique that simplifies the calculations with the antinormal ordering. Also, we derive a Parseval-like identity that is connected with the algebraic properties of the SG phase operators under the antinormal ordering. In Sec. IV we discuss some applications of the antinormal ordering. We derive the PB phase operators, which give in the weak limit the antinormally ordered Carruthers and Nieto (CN) phase operators [2]. Also, we discuss the use of a phase distribution function $Q(\theta)$ and its relation to the antinormal ordering. In Sec. V we summarize our conclusions.

II. ANTINORMAL ORDERING AND THE RELATION BETWEEN PB AND SG QUANTUM PHASE FORMALISMS

Pegg and Barnett have developed [4–6] the theory of a Hermitian phase operator by using a finite-dimensional Hilbert space Ψ. Such $(s+1)$-dimensional state space Ψ is spanned by the number states $|n\rangle$ $(n = 0, 1, \ldots, s)$ or, equivalently, by the complete orthonormal set of phase states [4],

$$|\theta_m\rangle = \frac{1}{\sqrt{s+1}} \sum_{n=0}^{s} e^{in\theta_m} |n\rangle \ , \tag{2.1}$$

where

$$\theta_m = \theta_0 + \frac{2\pi m}{s+1}, \quad m = 0, 1, \ldots, s \ . \tag{2.2}$$

By the choice of θ_0 it is possible to build the Hermitian phase operator $\hat{\phi}_\theta$. Pegg and Barnett [4–6] have studied extensively the properties of this operator. We quote here only some important formulas that are relevant to the present research:

$$\hat{\phi}_\theta = \sum_{m=0}^{s} \theta_m |\theta_m\rangle\langle\theta_m| \ , \tag{2.3}$$

$$\hat{\phi}_\theta |\theta_m\rangle = \theta_m |\theta_m\rangle \ . \tag{2.4}$$

By using the Hermitian operator $\hat{\phi}_\theta$ it is possible to construct the unitary phase operator,

$$\exp(i\hat{\phi}_\theta) = |0\rangle\langle 1| + |1\rangle\langle 2| + \cdots + |s-1\rangle\langle s| + \exp[i(s+1)\theta_0]|s\rangle\langle 0| \ . \qquad (2.5)$$

Commuting Hermitian phase operators $\cos(\hat{\phi}_\theta)$ and $\sin(\hat{\phi}_\theta)$ can be formed in the usual way from the unitary phase operator $\exp(i\hat{\phi}_\theta)$ and its Hermitian conjugate $\exp(-i\hat{\phi}_\theta)$.

Since physical reality is related to the infinite-dimensional Hilbert space $\mathcal{H}$, the procedure for using the desirable properties of the Ψ space is to calculate expectation values in the finite-dimensional space and only after that to take the infinite-dimensional limit $s \to \infty$. The exact relations between operators and states in Ψ space and those in conventional $\mathcal{H}$ space have been studied in great detail by Vaccaro and Pegg [7]. They have shown that in the limit $s \to \infty$ the expectation values of Hermitian operators remain real, in agreement with a basic postulate of quantum mechanics. There are two types of limiting procedures in the transition from Ψ space to $\mathcal{H}$ space: strong and weak limits. The strong limit is valid, for example, for any power series in $\hat{a}$ and $\hat{a}^\dagger$ that is bounded. Here an operator $\hat{A}_s$ belonging to the Ψ space and its strong limit $\hat{A}$ on the $\mathcal{H}$ space are called bounded if $\| \hat{A}_s|f\rangle \| < \infty \ \forall s$, and $\|\hat{A}|f\rangle\| < \infty$, $\forall |f\rangle \in \mathcal{H}$. Under the strong limit the algebra of bounded operators is conserved [7],

$$\left(\lim_{s\to\infty_{\rm strong}} \hat{B}_s\right)\left(\lim_{s\to\infty_{\rm strong}} \hat{A}_s\right)|f\rangle = \lim_{s\to\infty_{\rm strong}} (\hat{B}_s\hat{A}_s)|f\rangle \ , \qquad (2.6)$$

where $|f\rangle \in \mathcal{H}$.

A sequence of operators $\hat{A}_s$ converges only weakly to $\hat{A}_w$ on $\mathcal{H}$ if

$$\langle g|\hat{A}_s|f\rangle \to \langle g|\hat{A}_w|f\rangle \quad \text{as } s\to\infty \qquad (2.7)$$

for all $|g\rangle, |f\rangle \in \mathcal{H}$. For example, the well-known SG quantum phase operators $\widehat{e^{i\phi}_{\rm SG}}$ and $\widehat{e^{-i\phi}_{\rm SG}}$ are the weak limits of the corresponding PB operators $\exp(i\hat{\phi}_\theta)$ and $\exp(-i\hat{\phi}_\theta)$. The PB phase operator $\hat{\phi}_\theta$ and its functions converge only weakly to operators on $\mathcal{H}$. Vaccaro and Pegg [7] have shown that $\mathcal{H}$ space and Ψ space formalisms are consistent in the weak limit due to the relation

$$\lim_{s\to\infty_{\rm weak}} \langle \hat{A}_s\rangle_s = \langle \hat{A}_w\rangle \ , \qquad (2.8)$$

where an $\mathcal{H}$-space operator $\hat{A}_w$ is the weak limit of $\hat{A}_s$. So, the two formalisms give the same expectation values. However, the main problem is nonconservation of algebra under the weak limit,

$$\lim_{s\to\infty_{\rm weak}} \langle \hat{A}_s\hat{B}_s\rangle_s \neq \langle \hat{A}_w\hat{B}_w\rangle \ . \qquad (2.9)$$

As a simple example we note that the PB operators $\exp(i\hat{\phi}_\theta)$ and $\exp(-i\hat{\phi}_\theta)$ are unitary, but their weak limits are not:

$$\widehat{e^{-i\phi}_{\rm SG}}\,\widehat{e^{i\phi}_{\rm SG}} = \hat{1} - |0\rangle\langle 0| \ . \qquad (2.10)$$

However, it still holds that

$$\widehat{e^{i\phi}_{\rm SG}}\,\widehat{e^{-i\phi}_{\rm SG}} = \hat{1} \ . \qquad (2.11)$$

One can define [8,9] *antinormal ordering* of SG quantum phase operators as a procedure that places all raising operators $\widehat{e^{-i\phi}_{\rm SG}}$ to the right of all lowering operators $\widehat{e^{i\phi}_{\rm SG}}$. Our notation of the antinormal ordering consists of two $\substack{*\\*}$ on either side of an expression,

$$\substack{*\\*}(\widehat{e^{-i\phi}_{\rm SG}})^{m_1}(\widehat{e^{i\phi}_{\rm SG}})^{n_1}(\widehat{e^{-i\phi}_{\rm SG}})^{m_2}(\widehat{e^{i\phi}_{\rm SG}})^{n_2}\cdots(\widehat{e^{-i\phi}_{\rm SG}})^{m_k}(\widehat{e^{i\phi}_{\rm SG}})^{n_k}\substack{*\\*} = (\widehat{e^{i\phi}_{\rm SG}})^{n_1+n_2+\cdots+n_k}(\widehat{e^{-i\phi}_{\rm SG}})^{m_1+m_2+\cdots+m_k} \ , \qquad (2.12)$$

where m_i, n_i $(i=1,2,\ldots,k)$, and k are arbitrary positive integers. The example given by Eqs. (2.10) and (2.11) indicates that the antinormal ordering realizes in the SG description the advantages of the PB description and we will see it later. For instance, the unitarity is restored for $\widehat{e^{i\phi}_{\rm SG}}$ and $\widehat{e^{-i\phi}_{\rm SG}}$ operators:

$$\widehat{e^{i\phi}_{\rm SG}}\,\widehat{e^{-i\phi}_{\rm SG}} = \substack{*\\*}\widehat{e^{-i\phi}_{\rm SG}}\,\widehat{e^{i\phi}_{\rm SG}}\substack{*\\*} = \hat{1} \ . \qquad (2.13)$$

Analogously, by using the antinormal ordering, $\hat{C}$ and $\hat{S}$ (the SG cos and sin) operators become commuting:

$$\substack{*\\*}[\hat{C},\hat{S}]\substack{*\\*} = 0 \ . \qquad (2.14)$$

The vacuum state may then be described as a state of a random phase, similar to all other numbers states,

$$\langle n|\substack{*\\*}\hat{C}^2\substack{*\\*}|n\rangle = \langle n|\substack{*\\*}\hat{S}^2\substack{*\\*}|n\rangle = \tfrac{1}{2} \quad (\text{including } n=0) \ , \qquad (2.15)$$

and therefore,

$$\substack{*\\*}\hat{C}^2 + \hat{S}^2\substack{*\\*} = \hat{1} \ . \qquad (2.16)$$

The question must be raised: What is the physical nature of the antinormal ordering of the SG quantum phase operators? Our answer is that by antinormal ordering we define the vacuum to be a state of a random phase (in the same way as for all other number states) corresponding to our physical perceptions. In the calculations of the electromagnetic-field energy, we use *normal ordering* for $\hat{a}$ and $\hat{a}^\dagger$ operators to exclude the infinite energy of the vacuum and actually define the vacuum to be the state of zero energy. In a similar way, by using the antinormal ordering of $\widehat{e^{i\phi}_{\rm SG}}$ and $\widehat{e^{-i\phi}_{\rm SG}}$ operators we exclude nonrandom phase properties for the vacuum.

Vaccaro has shown [11] that a function of phase in Ψ space, which is given by

$$G(\hat{\phi}_\theta) = \sum_{m=0}^{s} G(\theta_m)|\theta_m\rangle\langle\theta_m| \ , \tag{2.17}$$

has the weak limit

$$\lim_{s\to\infty_{\rm weak}} G(\hat{\phi}_\theta) = {}^*_* \sum_{k=-\infty}^{\infty} \tilde{G}_k (\widehat{e^{-i\phi}_{\rm SG}})^k {}^*_* \equiv \hat{G}_{\rm SG} \ , \tag{2.18}$$

where

$$ {}^*_*(\widehat{e^{-i\phi}_{\rm SG}})^k{}^*_* = \begin{cases} (\widehat{e^{-i\phi}_{\rm SG}})^k, & k>0 \\ \hat{1}, & k=0 \\ (\widehat{e^{i\phi}_{\rm SG}})^{|k|}, & k<0 \ , \end{cases} \tag{2.19}$$

and Fourier coefficients are given by

$$\tilde{G}_k = \frac{1}{2\pi}\int_{\theta_0}^{\theta_0+2\pi} d\theta \, G(\theta) e^{ik\theta} \ . \tag{2.20}$$

The idea of using Fourier-like series in operators $\widehat{e^{i\phi}_{\rm SG}}$ and $\widehat{e^{-i\phi}_{\rm SG}}$ to represent phase-dependent operators in the SG formalism was introduced by Lukš and Peřinová [8]. Such expansion [Eq. (2.18)] is valid for an operator that corresponds to a 2π–periodic function $G(\theta)$ that has convergent Fourier series. For a function $G(\theta)$ that is not intrinsically 2π periodic (e.g,. θ related to operator $\hat{\phi}_\theta$ and its weak limit $\hat{\phi}_{\rm SG}$) we must use the periodic expansion on the entire real axis, e.g.,

$$\begin{aligned}\theta_{\rm per} = \theta \text{ for } \theta \in [\theta_0, \theta_0+2\pi) \\ +2\pi\text{-periodic expansion on } \mathbb{R} \ .\end{aligned} \tag{2.21}$$

An operator $\hat{G}_{\rm SG}$ defined by (2.18) can also be written in another form by using the function $G(\theta)$ and the states

$$|\theta\rangle\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} e^{in\theta}|n\rangle \ . \tag{2.22}$$

One gets [8,9]

$$\hat{G}_{\rm SG} = \int_{\theta_0}^{\theta_0+2\pi} d\theta \, G(\theta)|\theta\rangle\rangle\langle\langle\theta| \ . \tag{2.23}$$

The form (2.23) for $\hat{G}_{\rm SG}$ and the idea of the antinormal ordering of SG phase operators have been discussed by Lukš and Peřinová [8,9].

The use of the weak limit (2.18) leads to a very important theorem [11] that relates the weak limit and the antinormal ordering:

$$\lim_{s\to\infty_{\rm weak}} G(\hat{\phi}_\theta)F(\hat{\phi}_\theta) = {}^*_*\hat{G}_{\rm SG}\hat{F}_{\rm SG}{}^*_* \ . \tag{2.24}$$

This theorem shows that the algebra is conserved in the weak limit with the antinormal ordering. Using Eq. (2.23), Eq. (2.24) can be written also as

$$\int_{\theta_0}^{\theta_0+2\pi} d\theta \, G(\theta)F(\theta)|\theta\rangle\rangle\langle\langle\theta| = {}^*_*\hat{G}_{\rm SG}\hat{F}_{\rm SG}{}^*_* \ . \tag{2.25}$$

III. ADDITIONAL PROPERTIES OF THE ANTINORMAL ORDERING

The phase states $|\theta\rangle\rangle$ defined in Eq. (2.22) are not orthogonal. The orthogonality problem (as emphasized by Carruthers and Nieto [2]) is related to the restriction of the energy spectrum from below by the vacuum state. If we had continued the spectrum of the number operator $\hat{N}$ until $-\infty$ (like that of angular momentum for a plane rotator) we would obtain orthonormalized (by δ function) phase states and solve the problems related to the SG quantum phase formalism. But such a continuation of the number operator spectrum is not correct physically.

Nevertheless, the antinormal ordering effectively replaces the orthogonality of $|\theta\rangle\rangle$ states. From Eq. (2.25) we see that for any two operators $\hat{G}_{\rm SG}$ and $\hat{F}_{\rm SG}$, which may be written in the form (2.23), the antinormal ordering provides the commutation relation

$${}^*_*[\hat{G}_{\rm SG}, \hat{F}_{\rm SG}]^*_* = 0 \ . \tag{3.1}$$

The important thing is that operators $\hat{G}_{\rm SG}$ and $\hat{F}_{\rm SG}$ would be commutive if the states $|\theta\rangle\rangle$ were orthogonal. Indeed, according to Eq. (2.25), we get

$$\begin{aligned} &{}^*_*\int_{\theta_0}^{\theta_0+2\pi} d\theta \int_{\theta_0}^{\theta_0+2\pi} d\theta' G(\theta)F(\theta')|\theta\rangle\rangle\langle\langle\theta|\theta'\rangle\rangle\langle\langle\theta'|{}^*_* \\ &\qquad = \int_{\theta_0}^{\theta_0+2\pi} d\theta \, G(\theta)F(\theta)|\theta\rangle\rangle\langle\langle\theta| \ . \end{aligned} \tag{3.2}$$

It is easy to see that we would have gotten the right-hand side of this equation if instead of using the antinormal ordering we had assumed the orthogonality of the states $|\theta\rangle\rangle$:

$$\langle\langle\theta|\theta'\rangle\rangle = \delta(\theta-\theta') \ \ (\text{hypothetic}) \ . \tag{3.3}$$

So, we find that the antinormal ordering replaces effectively the orthogonality of the phase states $|\theta\rangle\rangle$, which are essentially not orthogonal.

To explore the phase properties of a state of the electromagnetic field, we need to calculate expectation value of a phase operator,

$$\langle\hat{G}_{\rm SG}\rangle = {\rm Tr}(\hat{\rho}\hat{G}_{\rm SG}) \ , \tag{3.4}$$

where $\hat{\rho}$ is the density operator of the state. It is simple to treat pure states for which

$$\hat{\rho} = |\psi\rangle\langle\psi| \ . \tag{3.5}$$

So, we get

$$\langle\hat{G}_{\rm SG}\rangle = \int_{\theta_0}^{\theta_0+2\pi} d\theta \, G(\theta)|\langle\psi|\theta\rangle\rangle|^2 \ . \tag{3.6}$$

It is convenient to introduce [8,9] [by using an analogy to the well-known $Q(\alpha)$ function of coherent states] phase distribution function

$$Q(\theta) = \mathrm{Tr}(\hat{\rho}|\theta\rangle\rangle\langle\langle\theta|) \overset{\text{pure state}}{=} |\langle\psi|\theta\rangle\rangle|^2 . \tag{3.7}$$

Hence, Eq. (3.6) may be written in the form

$$\langle \hat{G}_{\rm SG}\rangle = \int_{\theta_0}^{\theta_0+2\pi} d\theta\, G(\theta)Q(\theta) . \tag{3.8}$$

Properties of the antinormal ordering cause the phase distribution $Q(\theta)$ to be very useful, in spite of nonorthogonality of $|\theta\rangle\rangle$ states since, according to Eq. (2.25),

$$\langle {}^*_*\hat{G}_{\rm SG}\hat{F}_{\rm SG}{}^*_*\rangle = \int_{\theta_0}^{\theta_0+2\pi} d\theta\, G(\theta)F(\theta)Q(\theta) . \tag{3.9}$$

So, we may investigate the phase properties of different states of light without leaving the infinite-dimensional Hilbert space. The results are the same as those derived in the frames of the PB theory [6] taking the infinite-dimensional limit after expectation values are calculated. Therefore, with the antinormal ordering of SG phase operators, the order in which the infinite-dimensional limit and expectation value calculations occur is unimportant. Hence these two operations become commuting, as required from a standard physical point of view.

Any state in Hilbert space may be represented by a superposition of the number states:

$$|\psi\rangle = \sum_{n=0}^{\infty} C_n|n\rangle . \tag{3.10}$$

Then the $Q(\theta)$ function defined by (3.7) is

$$Q(\theta) = \frac{1}{2\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C_n^* C_m e^{i(n-m)\theta} . \tag{3.11}$$

According to Barnett and Pegg [5,6], a state $|\psi\rangle$ is "physically accessible" if the series

$$\langle\psi|\hat{N}^p|\psi\rangle = \sum_{n=0}^{\infty} n^p |C_n|^2 \tag{3.12}$$

is convergent for any given finite integer p. It is evident that for any physically accessible state $|\psi\rangle$ the $Q(\theta)$ function is well defined and we may find the phase properties of such states by calculating the corresponding expectation values. This is not the case for a state $|\psi\rangle$ that is not physically accessible; for example, the formalism does not work for the phase state $|\theta\rangle\rangle$ itself, since it does not possess a finite norm. But this discrepancy is of no importance in any physical application.

It is interesting to note that there are operators that have phase representation (2.23) and that can create single-mode physical states from the vacuum. For any operator $\hat{D}$ of the form (2.23), we get

$$\begin{aligned}\hat{D}|0\rangle &= \int_{\theta_0}^{\theta_0+2\pi} d\theta\, D(\theta)|\theta\rangle\rangle\langle\langle\theta|0\rangle \\ &= \frac{1}{2\pi}\sum_{n=0}^{\infty}\int_{\theta_0}^{\theta_0+2\pi} d\theta\, D(\theta)e^{in\theta}|n\rangle = \sum_{n=0}^{\infty}\tilde{D}_n|n\rangle .\end{aligned} \tag{3.13}$$

In order for $\hat{D}$ to be a *state-creating operator*, obeying

$$\hat{D}|0\rangle = |\psi\rangle = \sum_{n=0}^{\infty} C_n|n\rangle , \tag{3.14}$$

we must choose a corresponding function $D(\theta)$, such that its Fourier coefficients are

$$\tilde{D}_n = \begin{cases} C_n, & n \geq 0 \\ 0, & n < 0 . \end{cases} \tag{3.15}$$

The Fourier coefficients $\tilde{D}_n$ with negative values of n must vanish in order to prevent an ambiguity in the choice of $\hat{D}$. Also, Eq. (3.15) gives the correct normalization condition for the function $D(\theta)$. By using the Parseval identity [12], we get for the normalized state $|\psi\rangle$,

$$\frac{1}{2\pi}\int_{\theta_0}^{\theta_0+2\pi} d\theta |D(\theta)|^2 = \sum_{n=-\infty}^{\infty} |\tilde{D}_n|^2 = \sum_{n=0}^{\infty} |C_n|^2 = 1 . \tag{3.16}$$

It must be emphasized that state-creating operators, defined in this way, are generally not unitary and are different from the usually used unitary displacement operators. However, when *acting on the vacuum*, our state-creating operators and the corresponding unitary displacement operators give the same results. For example, the coherent state-creating operator

$$\hat{D}_\alpha = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\left(\widehat{e^{-i\phi}_{\rm SG}}\right)^n \tag{3.17}$$

and the Glauber displacement operator

$$\hat{D}(\alpha) = e^{-|\alpha|^2/2}e^{\alpha\hat{a}^\dagger}e^{-\alpha^*\hat{a}} \tag{3.18}$$

both give the coherent state by acting on the vacuum. The state-creating operators, in spite of their nonunitarity, are useful for describing phase properties of light. By expressing the state-creating operator in the form (2.23), we change the basis $|n\rangle$ with coefficients C_n into the basis $|\theta\rangle\rangle$ with the function $D(\theta)$. We can find a relation between the function $D(\theta)$ corresponding to the state-creating operator and the $Q(\theta)$ function that contains information about phase properties of the state $|\psi\rangle$. From Eq. (3.11), we see that

$$\begin{aligned}Q(\theta) &= \frac{1}{2\pi}\sum_{n=0}^{\infty} C_n^* e^{in\theta}\sum_{m=0}^{\infty} C_m e^{-im\theta} \\ &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\tilde{D}_n^* e^{in\theta}\sum_{m=-\infty}^{\infty}\tilde{D}_m e^{-im\theta} \\ &= \frac{1}{2\pi}|D(\theta)|^2 .\end{aligned} \tag{3.19}$$

It is interesting to note that the properties of the weak limit connection between the Ψ space and the $\mathcal{H}$ space can be used as a mathematical tool to derive various results for the SG phase operators on the $\mathcal{H}$ space. Equation (2.25) is one example and we also provide another example of such results. Let $F(x,y)$ be a polynomial in x and y, i.e.,

$$F(x,y) = \sum_{n,m} F_{nm} x^n y^m \ , \tag{3.20}$$

with arbitrary ordering of x and y. Analogously we define $F(\widehat{e^{i\phi}_{\rm SG}}, \widehat{e^{-i\phi}_{\rm SG}})$, $G(\widehat{e^{i\phi}_{\rm SG}}, \widehat{e^{-i\phi}_{\rm SG}})$, and $H(\widehat{e^{i\phi}_{\rm SG}}, \widehat{e^{-i\phi}_{\rm SG}})$ as polynomials in operators $\widehat{e^{i\phi}_{\rm SG}}$ and $\widehat{e^{-i\phi}_{\rm SG}}$ (with arbitrary ordering of these operators) obeying the relation

$$H(\widehat{e^{i\phi}_{\rm SG}}, \widehat{e^{-i\phi}_{\rm SG}}) = F(\widehat{e^{i\phi}_{\rm SG}}, \widehat{e^{-i\phi}_{\rm SG}}) G(\widehat{e^{i\phi}_{\rm SG}}, \widehat{e^{-i\phi}_{\rm SG}}) \ . \tag{3.21}$$

Then we are able to prove (see Appendix A) the following result.

Theorem 1. For the antinormal ordering of a product that contains SG phase operators it is possible to take, in an intermediate stage, the antinormal ordering of each multiplier.

That is,

$$\begin{aligned} {}^*_*H(\widehat{e^{i\phi}_{\rm SG}}, \widehat{e^{-i\phi}_{\rm SG}})^*_* &= {}^*_*F(\widehat{e^{i\phi}_{\rm SG}}, \widehat{e^{-i\phi}_{\rm SG}}) G(\widehat{e^{i\phi}_{\rm SG}}, \widehat{e^{-i\phi}_{\rm SG}})^*_* \\ &= {}^*_*[{}^*_*F(\widehat{e^{i\phi}_{\rm SG}}, \widehat{e^{-i\phi}_{\rm SG}})^*_*] \\ &\quad \times [{}^*_*G(\widehat{e^{i\phi}_{\rm SG}}, \widehat{e^{-i\phi}_{\rm SG}})^*_*]^*_* \ . \end{aligned} \tag{3.22}$$

This theorem shows that it is possible to simplify the calculations with the antinormal ordering of complicated functions of $\widehat{e^{i\phi}_{\rm SG}}$ and $\widehat{e^{-i\phi}_{\rm SG}}$ by performing the antinormal ordering in intermediate steps and, only at the end of calculations, to complete the antinormal ordering procedure. Assuming the special case $G = F^{n-1}$, we get

$${}^*_*F^n{}^*_* = {}^*_*[{}^*_*F^*_*][{}^*_*F^{n-1}{}^*_*]^*_* \ , \tag{3.23}$$

and by a simple recursion relation, we obtain

$${}^*_*[F(\widehat{e^{i\phi}_{\rm SG}}, \widehat{e^{-i\phi}_{\rm SG}})]^n{}^*_* = {}^*_*[{}^*_*F(\widehat{e^{i\phi}_{\rm SG}}, \widehat{e^{-i\phi}_{\rm SG}})^*_*]^n{}^*_* \ . \tag{3.24}$$

In a similar way, we find from Eq. (2.24), by using the above theorem, that

$$\lim_{s\to\infty_{\rm weak}} [G(\hat{\phi}_\theta)]^n \equiv \widehat{G^n}_{\rm SG} = {}^*_*[\hat{G}_{\rm SG}]^n{}^*_* \ , \tag{3.25}$$

or, in another form corresponding to Eq. (2.25),

$$\int_{\theta_0}^{\theta_0+2\pi} d\theta [G(\theta)]^n |\theta\rangle\rangle\langle\langle\theta| = {}^*_*\left[\int_{\theta_0}^{\theta_0+2\pi} d\theta\, G(\theta)|\theta\rangle\rangle\langle\langle\theta|\right]^n {}^*_* \ . \tag{3.26}$$

We have obtained in Eqs. (2.25) and (3.26) very important formulas in the $\mathcal{H}$ space. They are valid without any relation to the existence of the PB description. However, the very effective and convenient way to derive these equations is by using the properties of the weak limit [Eqs. (2.18) and (2.24)], which connects the SG and PB descriptions. In the following section, we show some applications to the above results and to the general theory presented in the preceding section.

In Eqs. (2.18) and (2.23) we have shown that an operator in the SG quantum phase formalism can be written in different forms,

$$\hat{G}_{\rm SG} = \sum_{k=-\infty}^{\infty} \tilde{G}_k (\widehat{e^{-i\phi}_{\rm SG}})^k = \int_{\theta_0}^{\theta_0+2\pi} d\theta\, G(\theta)|\theta\rangle\rangle\langle\langle\theta| \ , \tag{3.27}$$

where $\tilde{G}_k$ are given by Eq. (2.20). Replacing $G(\theta)$ by $G^*(\theta)$ in Eq. (2.20), we get

$$\frac{1}{2\pi}\int_{\theta_0}^{\theta_0+2\pi} d\theta\, G^*(\theta) e^{ik\theta} = \tilde{G}^*_{-k} \ , \tag{3.28}$$

and the Fourier-like series in (3.27) with these coefficients becomes

$$\sum_{k=-\infty}^{\infty} \tilde{G}^*_{-k} (\widehat{e^{-i\phi}_{\rm SG}})^k = \sum_{k=-\infty}^{\infty} \tilde{G}^*_k (\widehat{e^{i\phi}_{\rm SG}})^k = \hat{G}^\dagger_{\rm SG} \ . \tag{3.29}$$

Therefore, we get

$$\hat{G}^\dagger_{\rm SG} = \int_{\theta_0}^{\theta_0+2\pi} d\theta\, G^*(\theta)|\theta\rangle\rangle\langle\langle\theta| \ . \tag{3.30}$$

So, a Hermitian operator $\hat{G}_{\rm SG}$ corresponds to a real function $G(\theta)$. Using Eq. (3.30) we prove in Appendix B the following result.

Theorem 2 (generalized Parseval-like identity). For any two operators $\hat{G}_{\rm SG}$ and $\hat{F}_{\rm SG}$, which satisfy Eq. (3.27), and for any number state $|n\rangle$ ($n = 0,1,2,\ldots$), the following identity is valid:

$$\langle n|{}^*_*\hat{F}^\dagger_{\rm SG}\hat{G}_{\rm SG}{}^*_*|n\rangle = \sum_{k=-\infty}^{\infty} \tilde{F}^*_k \tilde{G}_k \ . \tag{3.31}$$

In Appendix B, we also generalize this result to nondiagonal matrix elements:

$$\langle n|{}^*_*\hat{F}^\dagger_{\rm SG}\hat{G}_{\rm SG}{}^*_*|n'\rangle = \sum_{k=-\infty}^{\infty} \hat{F}^*_k \tilde{G}_{k+n-n'} \ . \tag{3.32}$$

So, we can get the matrix elements in the number-state representation for the antinormal ordering of any SG phase operator or product of such operators by using the corresponding Fourier coefficients. These relations provide us with a mathematical tool to calculate numerous infinite series, but we would get similar results in the frames of the ordinary Fourier expansions [12]. However, as we will soon see, the importance of the Parseval-like identity (3.31) is its connection with the algebraic properties of the SG phase operators. This connection helps in the explanation of the physical nature of the antinormal ordering. In the SG description, we are considering a set of operators

$$\{\mathcal{O}_{\rm SG}\} = \{\ldots, (\widehat{e^{-i\phi}_{\rm SG}})^k, \ldots, \widehat{e^{-i\phi}_{\rm SG}}, \hat{1}, \widehat{e^{i\phi}_{\rm SG}}, \ldots, (\widehat{e^{i\phi}_{\rm SG}})^k, \ldots\} . \tag{3.33}$$

It is interesting to note that this set of operators has all the properties of a group if and only if we use the antinormal ordering. Without the antinormal ordering we find, for example,

$$\widehat{e^{-i\phi}_{\rm SG}}\widehat{e^{i\phi}_{\rm SG}} = \hat{1} - |0\rangle\langle 0| , \tag{3.34}$$

i.e., the multiplication of two operators belonging to the set $\{\mathcal{O}_{\rm SG}\}$ produces the projection operator for the vacuum, which does not belong to this set. Introducing the antinormal ordering, we close the set $\{\mathcal{O}_{\rm SG}\}$ and provide it with all the properties of an Abelian group. Indeed, the identity elements is $\hat{1}$, the inverse of $(\widehat{e^{i\phi}_{\rm SG}})^k$ is $(\widehat{e^{-i\phi}_{\rm SG}})^k$, and commutative and associative properties are fulfilled as is evident from Eq. (2.13).

The use of the antinormal ordering of the SG phase operators is related to the theory of ordinary Fourier series [12]. Consider a Euclidean space $\mathcal{W}$, which consists of continuous in parts and 2π-periodic functions $G(\theta)$. The scalar product in $\mathcal{W}$ is defined to be

$$(G,F) = \frac{1}{2\pi}\int_{\theta_0}^{\theta_0+2\pi} d\theta\, G(\theta)F^*(\theta) . \tag{3.35}$$

Any orthonormal set $\{\varphi_k\}$ in $\mathcal{W}$ is closed if, and only if, the Parseval identity is fulfilled [12]:

$$\|F\|^2 = (F,F) = \sum_k |(F,\varphi_k)|^2 . \tag{3.36}$$

It is known [12] that $\{e^{-ik\theta}\}$ $(-\infty < k < \infty)$ is such an orthonormal and closed set, which may be used to construct the Fourier series

$$G(\theta) = \sum_{k=-\infty}^{\infty} \tilde{G}_k e^{-ik\theta} , \tag{3.37}$$

where

$$\tilde{G}_k = (G(\theta), e^{-ik\theta}) = \frac{1}{2\pi}\int_{\theta_0}^{\theta_0+2\pi} d\theta\, G(\theta)e^{ik\theta} . \tag{3.38}$$

The Parseval identity may be written in the generalized form,

$$(G,F) = \sum_{k=-\infty}^{\infty} \tilde{G}_k\tilde{F}_k^* . \tag{3.39}$$

Corresponding to each function $G(\theta)$ in $\mathcal{W}$ we can define the operator $\hat{G}_{\rm SG}$ acting on the Hilbert space $\mathcal{H}$,

$$\hat{G}_{\rm SG} = \int_{\theta_0}^{\theta_0+2\pi} d\theta\, G(\theta)|\theta\rangle\rangle\langle\langle\theta| , \tag{3.40}$$

where $|\theta\rangle\rangle$ are given by Eq. (2.22). We find that Eqs. (3.37)–(3.39) for the $\mathcal{W}$ space of classical functions $G(\theta)$ are analogous to the properties of the Fourier-like series, which represent operators $\hat{G}_{\rm SG}$ on $\mathcal{H}$ [see Eqs. (2.18)–(2.20) and (3.31)]. For example, the orthornomal set of functions $\{e^{-ik\theta}\}$ is transformed by (3.40) into the set $\{\mathcal{O}_{\rm SG}\}$, defined by Eq. (3.33); and the Parseval identity (3.39) based on the functions $\{e^{-ik\theta}\}$ is analogous to the Parseval-like identity (3.31), based on the operators belonging to $\{\mathcal{O}_{\rm SG}\}$. This Parseval-like identity (3.31) is valid only with the use of the antinormal ordering, and the set $\{\mathcal{O}_{\rm SG}\}$ is closed also due to the use of the antinormal ordering. Such a relation is very similar to the classical connection between the Parseval identity and a closed set of orthonormal functions.

IV. APPLICATIONS OF THE ANTINORMAL ORDERING

Carruthers and Nieto have defined [2] two independent Hermitian phase operators $\hat{\phi}_S$ and $\hat{\phi}_C$,

$$\hat{\phi}_S \equiv \sin^{-1}\hat{S} = \sum_{k=0}^{\infty} a_k \hat{S}^{2k+1} , \tag{4.1}$$

$$\hat{\phi}_C \equiv \cos^{-1}\hat{C} = \frac{\pi}{2} - \sin^{-1}\hat{C} = \frac{\pi}{2} - \sum_{k=0}^{\infty} a_k\hat{C}^{2k+1} , \tag{4.2}$$

where $\hat{S}$ and $\hat{C}$ are the SG sin and cos Hermitian phase operators and a_k is a short notation for the Taylor expansion coefficients:

$$a_k = \frac{(-1)^k}{2k+1}\begin{pmatrix} -\frac{1}{2} \\ k \end{pmatrix} = \frac{(2k+1)!!}{(2k)!!(2k+1)^2} . \tag{4.3}$$

By using $\hat{\phi}_S$ and $\hat{\phi}_C$ it is possible to build unitary phase operators [2]:

$$\hat{U}_C = e^{i\hat{\phi}_C}, \quad \hat{U}_C^\dagger\hat{U}_C = \hat{U}_C\hat{U}_C^\dagger = \hat{1} , \tag{4.4}$$

$$\hat{U}_S = e^{i\hat{\phi}_S}, \quad \hat{U}_S^\dagger\hat{U}_S = \hat{U}_S\hat{U}_S^\dagger = \hat{1} . \tag{4.5}$$

But these definitions do not solve the quantum phase problem since $\hat{\phi}_S$ and $\hat{\phi}_C$ do not commute:

$$[\hat{\phi}_C, \hat{\phi}_S] \neq 0 . \tag{4.6}$$

We can use the results of the previous sections in order to find a relation between the PB phase operators defined in the Ψ space and the antinormally ordered CN operators $\hat{\phi}_S$ and $\hat{\phi}_C$ in the $\mathcal{H}$ space. Applying the antinormal ordering to the CN operators we get, by using Eq. (3.26),

$$\begin{aligned} {}^*_*\hat{\phi}_S{}^*_* &= \sum_{k=0}^{\infty} a_k {}^*_*\hat{S}^{2k+1}{}^*_* \\ &= \int_{\theta_0}^{\theta_0+2\pi} d\theta \sum_{k=0}^{\infty} a_k(\sin\theta)^{2k+1}|\theta\rangle\rangle\langle\langle\theta| \\ &= \int_{\theta_0}^{\theta_0+2\pi} d\theta \sin^{-1}(\sin\theta)|\theta\rangle\rangle\langle\langle\theta| , \end{aligned} \tag{4.7}$$

and in a similar way,

$$\overset{*}{*}\hat{\phi}_C\overset{*}{*} = \int_{\theta_0}^{\theta_0+2\pi} d\theta \cos^{-1}(\cos\theta)|\theta\rangle\rangle\langle\langle\theta| \ . \qquad (4.8)$$

We note that the functions $\sin^{-1}(\sin\theta)$ and $\cos^{-1}(\cos\theta)$ do *not* coincide with the function θ in the whole 2π-wide region $(\theta_0, \theta_0 + 2\pi)$ for any choice of θ_0. We define

$$\theta_C \equiv \cos^{-1}(\cos\theta) = |\theta| \quad \text{for } \theta \in [-\pi, \pi)$$
$$+2\pi\text{-periodic expansion on } \mathbb{R} \ , \qquad (4.9)$$

$$\theta_S \equiv \sin^{-1}(\sin\theta) = \frac{\pi}{2} - \left|\theta - \frac{\pi}{2}\right| \quad \text{for} \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right)$$
$$+2\pi\text{-periodic expansion on } \mathbb{R} \ . \qquad (4.10)$$

Accordingly to Eq. (2.18), we find

$$\overset{*}{*}\hat{\phi}_C\overset{*}{*} = \sum_{k=-\infty}^{\infty} \tilde{\theta}_C(k)(\widehat{e_{\rm SG}^{-i\phi}})^k \ , \qquad (4.11)$$

where

$$\tilde{\theta}_C(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta|\theta|e^{ik\theta} = \begin{cases} -2/(\pi k^2), & k \text{ odd} \\ 0, & k \text{ even } (k \neq 0) \\ \pi/2, & k = 0 \ , \end{cases} \qquad (4.12)$$

and similarly

$$\overset{*}{*}\hat{\phi}_S\overset{*}{*} = \sum_{k=-\infty}^{\infty} \tilde{\theta}_S(k)(\widehat{e_{\rm SG}^{-i\phi}})^k \ , \qquad (4.13)$$

where

$$\begin{aligned} \tilde{\theta}_S(k) &= \frac{1}{2\pi}\int_{-\pi/2}^{3\pi/2} d\theta(\pi/2 - |\theta - \pi/2|)e^{ik\theta} \\ &\overset{k\neq 0}{=} e^{-ik\pi/2}\tilde{\theta}_C(k) \\ &= \begin{cases} i(-1)^{(k-1)/2}2/(\pi k^2), & k \text{ odd} \\ 0, & k \text{ even} \ . \end{cases} \end{aligned} \qquad (4.14)$$

These results for $\overset{*}{*}\hat{\phi}_S\overset{*}{*}$ and $\overset{*}{*}\hat{\phi}_C\overset{*}{*}$ can be checked by a direct calculation without using Eq. (3.26). By the binomial expansion in Eqs. (4.1) and (4.2) we get with the antinormal ordering,

$$\overset{*}{*}\hat{\phi}_S\overset{*}{*} = \sum_{k=0}^{\infty} \frac{(-1)^k a_k}{i2^{2k+1}} \sum_{p=0}^{2k+1}(-1)^{p+1}\frac{(2k+1)!}{p!(2k+1-p)!}(\widehat{e_{\rm SG}^{-i\phi}})^{2k+1-2p} \ , \qquad (4.15)$$

$$\overset{*}{*}\hat{\phi}_C\overset{*}{*} = \frac{\pi}{2} - \sum_{k=0}^{\infty}\frac{a_k}{2^{2k+1}}\sum_{p=0}^{2k+1}\frac{(2k+1)!}{p!(2k+1-p)!}(\widehat{e_{\rm SG}^{-i\phi}})^{2k+1-2p} \ . \qquad (4.16)$$

We substitute

$$q = 2k + 1 - 2p \ , \qquad (4.17)$$

and then find

$$\overset{*}{*}\hat{\phi}_S\overset{*}{*} = \sum_{\text{odd } q=-\infty}^{\infty} i(-1)^{(q-1)/2}\mathcal{A}(q)(\widehat{e_{\rm SG}^{-i\phi}})^q \ , \qquad (4.18)$$

$$\overset{*}{*}\hat{\phi}_C\overset{*}{*} = \frac{\pi}{2} - \sum_{\text{odd } q=-\infty}^{\infty} \mathcal{A}(q)(\widehat{e_{\rm SG}^{-i\phi}})^q \ , \qquad (4.19)$$

where

$$\mathcal{A}(q) = \sum_{k=\frac{|q|-1}{2}}^{\infty} \frac{(|2k-1|!!)^2}{2^{2k+1}\left(k+\frac{1-q}{2}\right)!\left(k+\frac{1+q}{2}\right)!} \ . \qquad (4.20)$$

Comparison of these formulas with Eqs. (4.11)–(4.14) shows that the results of the two methods coincide due to the relation

$$\mathcal{A}(q) = \frac{2}{\pi q^2} \ , \qquad (4.21)$$

which was confirmed by numerical calculation of the complicated expression (4.20). We find that the calculational method based on the property of the antinormal ordering [Eq. (3.26)] is much simpler than the straightforward calculation used to find Eqs. (4.18)–(4.21). As follows from Eqs. (2.17), (2.18), and (2.23) the knowledge of a function $G(\theta)$ enables us to find immediately the Ψ-space operator, which corresponds in the weak limit to the given SG operator. Hence by using Eqs. (4.7) and (4.8), we get the PB operators that tend in the weak limit to the antinormally ordered CN phase operators:

$$\hat{\phi}_\theta(S) = \sum_{m=0}^{s}\left(\frac{\pi}{2} - \left|\theta_m - \frac{\pi}{2}\right|\right)|\theta_m\rangle\langle\theta_m|, \quad \theta_0 = -\frac{\pi}{2} \ , \qquad (4.22)$$

$$\hat{\phi}_\theta(C) = \sum_{m=0}^{s}|\theta_m||\theta_m\rangle\langle\theta_m|, \quad \theta_0 = -\pi \ . \qquad (4.23)$$

According to the above formalism,

$$\lim_{s\to\infty_{\rm weak}} \hat{\phi}_\theta(S) = \overset{*}{*}\hat{\phi}_S\overset{*}{*} \ , \qquad (4.24)$$

$$\lim_{s\to\infty_{\rm weak}} \hat{\phi}_\theta(C) = \overset{*}{*}\hat{\phi}_C\overset{*}{*} \ . \qquad (4.25)$$

It is possible to use the present methods for calculating more complicated cases. For example, a direct calculation of ${}^*_*[\hat{\phi}_C^n]^*_*$ by a straightforward use of the Taylor expansion in Eq. (4.2) will be very complicated since it needs to take a power of infinite series. But it follows from Theorem 1 and Eq. (3.24) that

$$ {}^*_*[\hat{\phi}_C^n]^*_* = {}^*_*[({}^*_*\hat{\phi}_C{}^*_*)^n]^*_* \ . \tag{4.26}$$

Substituting (4.8) for ${}^*_*\hat{\phi}_C{}^*_*$, we get

$$ {}^*_*[\hat{\phi}_C^n]^*_* = {}^*_* \left[\int_{-\pi}^{\pi} d\theta |\theta| |\theta\rangle\rangle\langle\langle\theta| \right]^n {}^*_* \ , \tag{4.27}$$

and by using Eq. (3.26), we finally find

$$ {}^*_*[\hat{\phi}_C^n]^*_* = \int_{-\pi}^{\pi} d\theta |\theta|^n |\theta\rangle\rangle\langle\langle\theta| \ . \tag{4.28}$$

It is obvious now that

$$ \lim_{s\to\infty_{\rm weak}} \hat{\phi}_\theta^n(C) = {}^*_*[\hat{\phi}_C^n]^*_* \ , \tag{4.29}$$

where $\hat{\phi}_\theta(C)$ is given by Eq. (4.23) and in the PB description it is easy to find

$$ \hat{\phi}_\theta^n(C) = \sum_{m=0}^{s} |\theta_m|^n |\theta_m\rangle\langle\theta_m|, \quad \theta_0 = -\pi \ . \tag{4.30}$$

For even powers of n,

$$ \hat{\phi}_\theta^n(C) = \hat{\phi}_\theta^n, \quad \theta_0 = -\pi \ . \tag{4.31}$$

We find here the interesting result that the weak limits of even powers of the PB phase operator $\hat{\phi}_\theta$ ($\theta_0 = -\pi$) are equal to the antinormally ordered corresponding powers of the CN operator $\hat{\phi}_C$. By using Theorem 1 and Eqs. (3.24) and (3.26), we are able to represent in the form (2.23) any antinormally ordered well behaved function of the CN phase operators. For example,

$$ {}^*_*\hat{U}_C{}^*_* = {}^*_* e^{i\hat{\phi}_C} {}^*_* = \sum_{n=0}^{\infty} \frac{i^n}{n!} {}^*_*[\hat{\phi}_C^n]^*_* = \sum_{n=0}^{\infty} \frac{i^n}{n!} {}^*_*[({}^*_*\hat{\phi}_C{}^*_*)^n]^*_* = {}^*_* e^{i {}^*_*\hat{\phi}_C{}^*_*} {}^*_* \ . \tag{4.32}$$

By using Eq. (4.28), we find

$$ {}^*_*\hat{U}_C{}^*_* = \int_{-\pi}^{\pi} d\theta\, e^{i|\theta|} |\theta\rangle\rangle\langle\langle\theta| \ . \tag{4.33}$$

and similarly,

$$ {}^*_*\hat{U}_S{}^*_* = {}^*_* e^{i\hat{\phi}_S} {}^*_* = \int_{-\pi/2}^{3\pi/2} d\theta\, e^{i(\pi/2 - |\theta-\pi/2|)} |\theta\rangle\rangle\langle\langle\theta| \ . \tag{4.34}$$

One should take into account that the quantum phase formalism is related to the expansion into Fourier-like series. For an operator, which is an antinormally ordered function of $\hat{C}$ and/or $\hat{S}$, it is possible to obtain the Fourier-like series by Taylor expansion of this function (as was made above for ${}^*_*\hat{\phi}_C{}^*_*$ and ${}^*_*\hat{\phi}_S{}^*_*$). The use of the desirable properties of the antinormal ordering [especially Eq. (3.26)] for these cases greatly simplifies such a procedure. In the examples with ${}^*_*\hat{\phi}_C{}^*_*$ and ${}^*_*\hat{\phi}_S{}^*_*$ the use of Eq. (3.26) was very effective but not obligatory. However, often a straightforward calculation of the Taylor expansion without the use of Eq. (3.26) does not lead to corresponding Fourier-like series. If we take as an example the operator ${}^*_*(1/i)\ln(\widehat{e^{i\phi}_{\rm SG}}){}^*_*$ the Taylor expansion of ln gives meaningless results unless Eq. (3.26) is used:

$$ {}^*_*\frac{1}{i}\ln(\widehat{e^{i\phi}_{\rm SG}}){}^*_* = {}^*_*\frac{1}{i}\ln\left[\int_{\theta_0}^{\theta_0+2\pi} d\theta\, e^{i\theta}|\theta\rangle\rangle\langle\langle\theta|\right]{}^*_* = \frac{1}{i}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} {}^*_*\left[\int_{\theta_0}^{\theta_0+2\pi} d\theta\, e^{i\theta}|\theta\rangle\rangle\langle\langle\theta| - 1\right]^n {}^*_* \ . \tag{4.35}$$

It is now possible to use the completeness of the states $|\theta\rangle\rangle$ [2]:

$$ \int_{\theta_0}^{\theta_0+2\pi} d\theta |\theta\rangle\rangle\langle\langle\theta| = 1 \ . \tag{4.36}$$

And by using Eq. (3.26), we find

$$ \begin{aligned} {}^*_*\frac{1}{i}\ln(\widehat{e^{i\phi}_{\rm SG}}){}^*_* &= \frac{1}{i}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int_{\theta_0}^{\theta_0+2\pi} d\theta (e^{i\theta}-1)^n |\theta\rangle\rangle\langle\langle\theta| \\ &= \int_{\theta_0}^{\theta_0+2\pi} d\theta \frac{1}{i}\ln(e^{i\theta})|\theta\rangle\rangle\langle\langle\theta| = \int_{\theta_0}^{\theta_0+2\pi} d\theta\, \theta |\theta\rangle\rangle\langle\langle\theta| \ . \end{aligned} \tag{4.37}$$

So, we get the weak limit of the PB phase operator:

$$ \lim_{s\to\infty_{\rm weak}} \hat{\phi}_\theta = \hat{\phi}_{\rm SG} = {}^*_*\frac{1}{i}\ln(\widehat{e^{i\phi}_{\rm SG}}){}^*_* = \int_{\theta_0}^{\theta_0+2\pi} d\theta\, \theta|\theta\rangle\rangle\langle\langle\theta| \ . \tag{4.38}$$

The corresponding Fourier-like series is

$$ \hat{\phi}_{\rm SG} = \sum_{k=-\infty}^{\infty} \tilde{\theta}_k (\widehat{e^{-i\phi}_{\rm SG}})^k \ , \tag{4.39}$$

where

$$\tilde{\theta}_k = \frac{1}{2\pi}\int_{\theta_0}^{\theta_0+2\pi} d\theta\, \theta e^{ik\theta} = \begin{cases} \theta_0+\pi, & k=0 \\ \frac{1}{ik}e^{ik\theta_0}, & k\neq 0\,. \end{cases} \quad (4.40)$$

We find here the SG counterpart of the PB Hermitian phase operator. But if we do not use Eq. (3.26) and carry out a straightforward calculation, we will get

$$\,^*_*\frac{1}{i}\ln(\widehat{e^{i\phi}_{\rm SG}})^*_* = \frac{1}{i}\sum_{p=0}^{\infty} R_p(\widehat{e^{i\phi}_{\rm SG}})^p\,, \quad (4.41)$$

where the coefficients

$$R_p = \frac{(-1)^{p+1}}{p!}\sum_{k=p+\delta_{p,0}}^{\infty}\frac{(k-1)!}{(k-p)!} \quad (4.42)$$

are divergent for any non-negative integer p. So, this expansion is meaningless.

Applications of the phase probability function $Q(\theta)$. The phase probability function $Q(\theta)$ was described in Eqs. (3.7) and (3.11) and it was shown that it is useful for investigating phase properties of different physical states of light. An important advantage in such formalism is the ability, by using the antinormal ordering, to work directly in the infinite-dimensional Hilbert space $\mathcal{H}$, which is natural for the description of the electromagnetic field. As recognized by Pegg and Barnett [5,6] the number states are the states of random phase. Indeed, we get from Eq. (3.11) for any number state $Q(\theta)=1/2\pi$, which exactly coincides with the corresponding result for the uniform classical distribution. As an interesting example, we discuss the phase properties of a coherent state $|\alpha\rangle$, where $\alpha=|\alpha|e^{i\chi}$. By a straightforward calculation, we get from Eq. (3.11)

$$Q(\theta,\alpha) = \frac{1}{2\pi}e^{-|\alpha|^2}\sum_{k=-\infty}^{\infty}|\alpha|^{|k|}e^{ik(\theta-\chi)}\Psi_{|k|}(|\alpha|)\,, \quad (4.43)$$

where

$$\Psi_q(|\alpha|) = \sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{\sqrt{n!(n+q)!}},\quad q\geq 0\,. \quad (4.44)$$

In the classical limit of a large mean photon number [13,2], we get for any non-negative integer q,

$$\lim_{|\alpha|^2\to\infty}\Psi_q(|\alpha|) = \frac{1}{|\alpha|^q}e^{|\alpha|^2}\,. \quad (4.45)$$

Therefore,

$$\lim_{|\alpha|^2\to\infty}Q(\theta,\alpha) = \delta(\theta-\chi)\,. \quad (4.46)$$

So, in this limit we get, as can be expected, a classical wave with a perfectly defined phase. We find from Eqs. (3.8) and (4.43) that the expectation value of a phase operator is given by

$$\langle\alpha|\hat{G}_{\rm SG}|\alpha\rangle = e^{-|\alpha|^2}\sum_{k=-\infty}^{\infty}|\alpha|^{|k|}\tilde{G}_k e^{-ik\chi}\Psi_{|k|}(|\alpha|)\,. \quad (4.47)$$

Considering $\hat{G}_{\rm SG} = \,^*_*\hat{C}^n\,^*_*$, we get

$$\langle\alpha|^*_*\hat{C}^n\,^*_*|\alpha\rangle = \frac{e^{-|\alpha|^2}}{2^n}\sum_{p=0}^{n}\frac{n!}{p!(n-p)!}|\alpha|^{|n-2p|}\Psi_{|n-2p|} \times e^{-i(n-2p)\chi}\,. \quad (4.48)$$

We substitute

$$q = n-2p\,, \quad (4.49)$$

and then find

$$\langle\alpha|^*_*\hat{C}^n\,^*_*|\alpha\rangle = \frac{e^{-|\alpha|^2}}{2^{n-1}}\sum_q^n\frac{n!|\alpha|^q\Psi_q(|\alpha|)}{\left(\frac{n-q}{2}\right)!\left(\frac{n+q}{2}\right)!(1+\delta_{q,0})}\cos(q\chi)\,, \quad (4.50)$$

where

$$q = \begin{cases} 0,2,\ldots,n, & n-\text{even} \\ 1,3,\ldots,n, & n-\text{odd}\,. \end{cases}$$

Our results obtained by the antinormal ordering are essentially different from the corresponding formulas of CN [13,2] when the mean photon number $|\alpha|^2$ is sufficiently small. But the present theory implies that if we use the PB formalism and then take the infinite-dimensional limit [6,14,15] we will get the same results as derived here. The use of the $Q(\theta)$ function and the antinormal ordering enables us to get rid of the infinite-dimensional limiting procedure. Such a technique may be considered as a simplification that will be useful in various physical problems that are related to quantum phase.

V. CONCLUSIONS

We have discussed the relations between the PB quantum phase formalism defined in the finite-dimensional Hilbert space Ψ and the antinormal ordering of the SG phase operators on the infinite-dimensional Hilbert space $\mathcal{H}$. In addition to that we have suggested the physical explanation of the antinormal ordering, which is based on the idea of defining the vacuum as a random-phase state. We have shown that phase properties of light can be conveniently investigated by using the $|\theta\rangle\rangle$ states since the antinormal ordering effectively eliminates the problem related to nonorthogonality of these states. We have developed a technique that essentially simplifies the calculations with the antinormal ordering. We have derived the Parseval-like identity and have shown the relation between this identity and the algebraic properties of the SG phase operators under the antinormal ordering. This relation is analogous to that used in the theory of ordinary Fourier series. We have described various applications of the present formalism. Especially we find the PB operators, which tend in the weak limit to the antinormally ordered CN phase operators, and have analyzed the properties of these CN operators and their functions under the antinormal ordering. Finally, we have investigated by using the antinormal ordering formalism the phase properties of coherent states. This example demonstrates the applicability of the $Q(\theta)$ function for different physical phase-related problems.

ACKNOWLEDGMENTS

We thank Dr. J. A. Vaccaro for sending us information on his recent work. C. Brif thanks the Venezuela Technion Society for financial support. This research was supported by the Fund for the Promotion of Research at the Technion.

APPENDIX A: PROOF OF THEOREM 1

In this proof we use the properties of the weak limit [7,11]. We define $F(e^{i\hat{\phi}_\theta}, e^{-i\hat{\phi}_\theta})$, $G(e^{i\hat{\phi}_\theta}, e^{-i\hat{\phi}_\theta})$, and $H(e^{i\hat{\phi}_\theta}, e^{-i\hat{\phi}_\theta})$ as polynomials in the PB unitary operators. They have the same functional form [see Eq. (3.20)] as the polynomials defined in Sec. III, $F(\widehat{e^{i\phi}_{\rm SG}}, \widehat{e^{-i\phi}_{\rm SG}})$, $G(\widehat{e^{i\phi}_{\rm SG}}, \widehat{e^{-i\phi}_{\rm SG}})$, and $H(\widehat{e^{i\phi}_{\rm SG}}, \widehat{e^{-i\phi}_{\rm SG}})$, respectively, but differ in their arguments:

$$H(e^{i\hat{\phi}_\theta}, e^{-i\hat{\phi}_\theta}) = F(e^{i\hat{\phi}_\theta}, e^{-i\hat{\phi}_\theta}) G(e^{i\hat{\phi}_\theta}, e^{-i\hat{\phi}_\theta}) \ .$$

By using the definitions of the weak limit and the antinormal ordering, one obtains [11]

$$\lim_{s\to\infty_{\rm weak}} G(e^{i\hat{\phi}_\theta}, e^{-i\hat{\phi}_\theta}) = {}^*_*G(\widehat{e^{i\phi}_{\rm SG}}, \widehat{e^{-i\phi}_{\rm SG}})^*_* \ ,$$

and similarly for F and H. Now we use this result and Eq. (2.24) to calculate

$$\begin{aligned}
{}^*_*H(\widehat{e^{i\phi}_{\rm SG}}, \widehat{e^{-i\phi}_{\rm SG}})^*_* &= {}^*_*F(\widehat{e^{i\phi}_{\rm SG}}, \widehat{e^{-i\phi}_{\rm SG}})G(\widehat{e^{i\phi}_{\rm SG}}, \widehat{e^{-i\phi}_{\rm SG}})^*_* \\
&= \lim_{s\to\infty_{\rm weak}} H(e^{i\hat{\phi}_\theta}, e^{-i\hat{\phi}_\theta}) = \lim_{s\to\infty_{\rm weak}} [F(e^{i\hat{\phi}_\theta}, e^{-i\hat{\phi}_\theta})G(e^{i\hat{\phi}_\theta}, e^{-i\hat{\phi}_\theta})] \\
&= {}^*_*\left(\lim_{s\to\infty_{\rm weak}} F(e^{i\hat{\phi}_\theta}, e^{-i\hat{\phi}_\theta}) \lim_{s\to\infty_{\rm weak}} G(e^{i\hat{\phi}_\theta}, e^{-i\hat{\phi}_\theta}) \right)^*_* \\
&= {}^*_*[{}^*_*F(\widehat{e^{i\phi}_{\rm SG}}, \widehat{e^{-i\phi}_{\rm SG}})^*_*][{}^*_*G(\widehat{e^{i\phi}_{\rm SG}}, \widehat{e^{-i\phi}_{\rm SG}})^*_*]^*_* \ ,
\end{aligned}$$

and we get the desired result.

APPENDIX B: PROOF OF THEOREM 2

We use the form (2.23) for an operator in the SG description, the form (3.30) for its Hermitian conjugate, Fourier series for corresponding functions, and property (2.25) of the antinormal ordering. Then we find

$$\begin{aligned}
\langle n|{}^*_*\hat{F}^\dagger_{\rm SG}\hat{G}_{\rm SG}{}^*_*|n\rangle &= \int_{\theta_0}^{\theta_0+2\pi} d\theta\, F^*(\theta)G(\theta)|\langle\langle\theta|n\rangle|^2 \\
&= \frac{1}{2\pi}\int_{\theta_0}^{\theta_0+2\pi} d\theta\, F^*(\theta)G(\theta) \\
&= \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \tilde{F}^*_k\tilde{G}_l \frac{1}{2\pi}\int_{\theta_0}^{\theta_0+2\pi} d\theta\, e^{i(k-l)\theta} = \sum_{k=-\infty}^{\infty} \tilde{F}^*_k\tilde{G}_k \ .
\end{aligned}$$

and we get the result of Theorem 2 [Eq. (3.31)]. Similarly, we treat the case of nondiagonal matrix elements [Eq. (3.32)]:

$$\begin{aligned}
\langle n|{}^*_*\hat{F}^\dagger_{\rm SG}\hat{G}_{\rm SG}{}^*_*|n'\rangle &= \int_{\theta_0}^{\theta_0+2\pi} d\theta\, F^*(\theta)G(\theta)\langle n|\theta\rangle\rangle\langle\langle\theta|n'\rangle \\
&= \frac{1}{2\pi}\int_{\theta_0}^{\theta_0+2\pi} d\theta\, F^*(\theta)G(\theta)e^{i(n-n')\theta} \\
&= \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \tilde{F}^*_k\tilde{G}_l \frac{1}{2\pi}\int_{\theta_0}^{\theta_0+2\pi} d\theta\, e^{i(k-l+n-n')\theta} = \sum_{k=-\infty}^{\infty} \tilde{F}^*_k\tilde{G}_{k+n-n'} \ .
\end{aligned}$$

[1] L. Susskind and J. Glogower, Physics **1**, 49 (1964).
[2] P. Carruthers and M. M. Nieto, Rev. Mod. Phys. **40**, 411 (1968).
[3] W. H. Louisell, Phys. Lett. **7**, 60 (1963).
[4] D. T. Pegg and S. M. Barnett, Europhys. Lett. **6**, 483 (1988).
[5] S. M. Barnett and D. T. Pegg, J. Mod. Opt. **36**, 7 (1989).
[6] D. T. Pegg and S. M. Barnett, Phys. Rev. A **39**, 1665

(1989).
[7] J. A. Vaccaro and D. T. Pegg, Phys. Scr. **T48**, 22 (1993).
[8] A. Lukš and V. Peřinová, Czech. J. Phys. **41**, 1205 (1991).
[9] A. Lukš and V. Peřinová, Phys. Scr. **T48**, 94 (1993).
[10] Y. Ben-Aryeh (unpublished).
[11] J. A. Vaccaro (unpublished).
[12] R. V. Churchill, *Fourier Series and Boundary Value Problems*, 4th ed. (McGraw-Hill, New York, 1987).
[13] P. Carruthers and M. M. Nieto, Phys. Rev. Lett. **14**, 387 (1965).
[14] R. Lynch, Phys. Rev. A **41**, 2841 (1990).
[15] C. C. Gerry and K. E. Urbanski, Phys. Rev. A **42**, 662 (1991).