Radiation in spherically symmetric structures. I. The coupled-amplitude equations for vector spherical waves

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A scalar Green's-function technique is used to derive coupled-amplitude equations for electromagnetic waves propagating in a three-dimensional structure with a radially varying refractive index. The vector nature of the problem is discussed and a method is outlined for reducing the vector wave equations to characteristic scalar equations. These scalar equations are then solved via an exact coupled-amplitude formalism, and closed-form solutions are compared with numerical results for the particular case of a spherical Bragg region. The derivation of the coupled-amplitude equations for vector spherical waves is a significant portion of the calculation, described in the companion paper [Sullivan and Hall, following paper, Phys. Rev. A **50**, 2708 (1994)], of the radiative effects due to the presence of a spherical Bragg structure. Furthermore, the formulation is a powerful complement to previously developed Debye potential and transfer-matrix methods.

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I. INTRODUCTION

Coupled-amplitude equations are commonly encountered in the analysis of problems involving wave propagation. Equation systems of this type have been used extensively in optics, for example, to characterize integrated optical couplers [1,2], distributed-feedback laser cavities [3], and grating-surface interactions [4]. A recent paper by Hall [5] demonstrated a Green's-function method for extracting coupled-amplitude equations from a scalar, second-order differential equation common in form to the time-independent Schrödinger equation and Maxwell's time-independent wave equation. The latter is most generally a vector wave equation, but reduces to a scalar equation often enough to make that case interesting. The earlier paper considered, as a specific example, the coupling of oppositely propagating scalar waves in one-, two-, and three-dimensional, linearly or radially periodic structures. The analysis demonstrated an intuitively appealing dimensional scaling: the coupling process evolving from contradirectional plane waves to circular waves to spherical waves as the dimensionality increased from one to two to three, respectively.

Though restricted to scalar-wave propagation, the earlier paper suggested that problems concerning vector-field propagation can sometimes be treated by a similar approach. In this paper, we extend the scalar theory to examine the vector-wave problem in a three-dimensional system. For continuity with Ref. [5], we consider the coupling of oppositely propagating vector spherical waves in a spherical structure with a radially varying refractive index. We choose this particular problem to illustrate the technique for a specific reason: the calculation proves useful in treating the vector fields generated by a real source placed within a radially periodic, spherical cavity. The properties of an elementary source, such as an atomic dipole, within what might be called a Bragg cavity have attracted interest recently in connection with studies of systems that exhibit photonic band gaps [6,7]. A full treatment of the coupling between an internal source and the fields within a spherical Bragg cavity is deferred to the following paper. Here we focus on how the coupled-amplitude equations for vector waves are developed and solved to obtain the fields within such a cavity.

The structure of this paper is as follows. We first present a calculation detailing the derivation of scalar wave equations for the characteristic field components from the vector wave equations. These scalar equations are used to formulate coupled-amplitude equations for the vector waves via a scalar Green's-function technique. The solution to the coupled-amplitude equations is discussed for the particular idealization of a spherical Bragg structure, and cases which can be solved in a closed-form, analytic fashion are examined. The construction of the transverse field components from the radial portions is also outlined to allow for a complete field determination. The theory of vector spherical harmonics is used for this task [8].

II. COUPLED-AMPLITUDE EQUATIONS

The formulation of a field solution within a spherically symmetric geometry naturally begins with a derivation of the pertinent vector wave equations. Assuming $\exp(-i\omega t)$ harmonic time dependence and linear media, Maxwell's equations in a source-free region take the form

$$\vec{\nabla}\cdot\vec{D}=0 , \tag{1a}$$

$$\vec{\nabla}\cdot\vec{B}=0 , \tag{1b}$$

$$\vec{\nabla}\times\frac{1}{\varepsilon(r)}\vec{D}=i\omega\vec{B} , \tag{1c}$$

$$\vec{\nabla}\times\vec{B}=i\mu\omega\vec{D} , \tag{1d}$$

where $\varepsilon(r)$ is the radially varying permittivity and the medium is assumed to be nonmagnetic so that the permeability μ is constant. The permittivity $\varepsilon(r)$ can be defined in terms of a radially varying refractive index such that

$$\varepsilon(r)=\varepsilon_0 n^2(r)=\varepsilon_0[n_0^2+\Delta n^2(r)] \ , \tag{2}$$

where ε_0 is the permittivity of free space, n_0 is the constant background refractive index, and $\Delta n^2(r)$ provides the radial dependence of the permittivity function. A typical geometry of practical interest, specifically the spherical Bragg structure, is depicted in Fig. 1, where the permittivity variation is oscillatory in nature in the region of interest ($r_1<r<r_1+L$). The radially varying function $\Delta n^2(r)$ is zero outside this region of interest resulting in a permittivity which is everywhere continuous.

Taking the curl of Eqs. (1c) and (1d) and simplifying the forms, vector wave equations can be derived

$$\vec{\nabla}\times\vec{\nabla}\times\vec{B}-\mu\varepsilon(r)\omega^2\vec{B}=\frac{1}{\varepsilon(r)}\frac{d\varepsilon}{dr}[\hat{r}\times(\vec{\nabla}\times\vec{B})] \ , \tag{3a}$$

$$\vec{\nabla}\times\vec{\nabla}\times\frac{1}{\varepsilon(r)}\vec{D}-\mu\omega^2\vec{D}=0 \ . \tag{3b}$$

Applying vector identities and noting that for a well-behaved vector field $\vec{V}$ it can be shown that [9]

$$\vec{r}\cdot(\nabla^2\vec{V})=\nabla^2(\vec{r}\cdot\vec{V})-2(\vec{\nabla}\cdot\vec{V}) \ , \tag{4}$$

the vector wave equations reduce to the characteristic scalar wave equations

$$[\nabla^2+n_0^2k_0^2][\vec{r}\cdot\vec{B}(\vec{r})]=-\Delta n^2(r)k_0^2[(\vec{r}\cdot\vec{B}(\vec{r})] \ , \tag{5a}$$

$$[\nabla^2+n_0^2k_0^2][(\vec{r}\cdot\vec{D}(\vec{r})]$$
$$=\left\{-\Delta n^2(r)k_0^2+\frac{d\varepsilon}{dr}\left[\frac{1}{\varepsilon r}+\frac{1}{\varepsilon}\frac{\partial}{\partial r}\right]\right\}[\vec{r}\cdot\vec{D}(\vec{r})] \ , \tag{5b}$$

where k_0 is the free space wave number. These scalar equations are significant for several reasons. Implicit in the form of Eqs. (5), the field components which are transverse to the radius vector decouple at a spherical interface. Thus we can separate the field into components: (i) transverse magnetic with respect to the radius vector (TM case)

$$\vec{r}\cdot\vec{B}(\vec{r})=0 \ , \quad \vec{r}\cdot\vec{D}(\vec{r}) \tag{6a}$$

and (ii) transverse electric with respect to the radius vector (TE case)

$$\vec{r}\cdot\vec{D}(\vec{r})=0 \ , \quad \vec{r}\cdot\vec{B}(\vec{r}) \ . \tag{6b}$$

These characteristic components are proportional to the conventional Debye potentials employed in the literature [9,10]. Furthermore, these components are continuous functions as physically they represent the normal field components with continuity required by Maxwell's equations. The problem of vector waves propagating in a three-dimensional, radially varying refractive index structure is now essentially scalar in nature. Thus we can apply the scalar Green's-function technique of Ref. [5] to form the coupled-amplitude equations and proceed to solve these equations for a specified permittivity function. In general, these equations are solved numerically, but certain limits exist in which closed-form solutions are derivable. Linked to this development is that, at some point, the remaining field components can be extracted to yield the total field. This calculation will be demonstrated by the definition and application of vector spherical harmonic functions.

To derive the coupled-amplitude equations for this geometry, begin with a generalization of Eqs. (5) such that

$$[\nabla^2+k^2](\vec{r}\cdot\vec{V})=-4\pi f(\vec{r}) \ , \tag{7a}$$

where the wave number $k=n_0k_0$. In Eq. (7a), the driving function $f(\vec{r})$ is generally defined as

$$f(\vec{r})=O(r)(\vec{r}\cdot\vec{V}) \ , \tag{7b}$$

where $O(r)$, which can be identified by comparing Eqs. (5) and (7), is either a function of r (TE case) or a differential operator which acts only upon the radial variable (TM case). Regardless of the specific form of the driving function, $f(\vec{r})$ vanishes in the absence of a refractive index variation. For $f(\vec{r})=0$, the general solution to Eq. (7a) is given by

$$(\vec{r}\cdot\vec{V})=\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}[A_{lm}^{(1)}h_l^{(1)}(kr)+A_{lm}^{(2)}h_l^{(2)}(kr)]Y_{lm}(\theta,\varphi) \ , \tag{8}$$

where the coefficients $A_{lm}^{(i)}$ are constants determined by the boundary or initial conditions. Note that the solution of Eq. (8) is simply a linear superposition of inward and outward traveling spherical waves as described by the spherical Hankel functions. The angular dependence of the solution is provided by the spherical harmonic functions $Y_{lm}(\theta,\varphi)$. The nature of the solution suggests a similar approach for $f(\vec{r})\neq 0$, i.e., the development of coupled-amplitude equations where the physically intui-

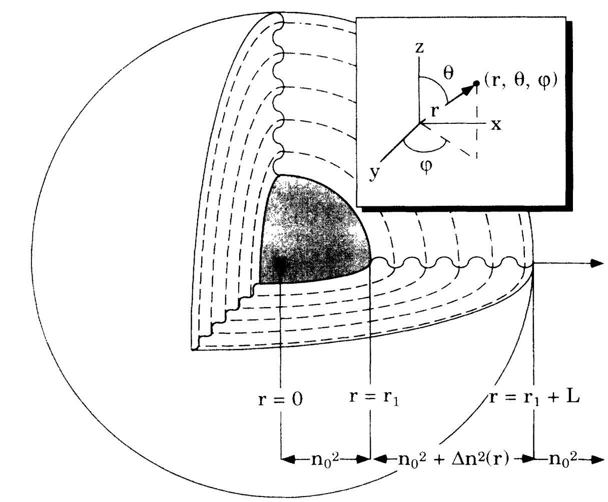

FIG. 1. A spherical Bragg structure. For this case, the permittivity varies sinusoidally with a specified amplitude within the region of examination ($r_1<r<r_1+L$) and is everywhere continuous.

tive concepts of contradirectional coupling and distributed feedback are modeled.

The particular solution to Eq. (7a) when the refractive index variation is nonzero is given by the integral expression

$$(\vec{r}\cdot\vec{V})=\int\int\int G(\vec{r},\vec{r}\,')f(\vec{r}\,')d^3r' \;, \tag{9}$$

where the Green's function $G(\vec{r},\vec{r}\,')$ is defined as a solution to the equation

$$[\nabla^2+k^2]G(\vec{r},\vec{r}\,')=-4\pi\delta(\vec{r}-\vec{r}\,') \;. \tag{10}$$

In three dimensions, Eq. (10) can be solved to yield the familiar form of the scalar Green's function

$$G(\vec{r},\vec{r}\,')=\frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} \;. \tag{11}$$

Note that Eq. (11) satisfies the Sommerfeld radiation condition of purely outgoing waves at infinity. To express Eq. (11) in a form similar to the homogeneous solution, a spherical wave expansion is employed for the Green's function [8] such that

$$\begin{aligned} G(\vec{r},\vec{r}\,')&=\frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} \\ &=i4\pi k\sum_{l=0}^{\infty}\sum_{m=-l}^{+l} j_l(kr_<)h_l^{(1)}(kr_>) \\ &\quad\times Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi) \;, \end{aligned} \tag{12}$$

where $r_>$ $(r_<)$ is the greater (lesser) of r and r'.

The procedure to obtain the coupled-amplitude equations is analogous to the method of Ref. [5]. Substituting Eq. (12) into Eq. (9) and accounting for the proper interpretation of the Green's function according to the integration variable, the particular field solution assumes the form

$$\begin{aligned}(\vec{r}\cdot\vec{V})=i4\pi k\sum_l\sum_m\Bigg[&\left\{\int_0^{2\pi}\int_0^{\pi}\int_0^{r} j_l(kr')Y_{lm}^*(\theta',\varphi')f(\vec{r}\,')[r'^2\sin\theta']dr'd\theta'd\varphi'\right\}h_l^{(1)}(kr) \\ &+\left\{\int_0^{2\pi}\int_0^{\pi}\int_r^{\infty} h_l^{(1)}(kr')Y_{lm}^*(\theta',\varphi')f(\vec{r}\,')[r'^2\sin\theta']dr'd\theta'd\varphi'\right\}j_l(kr)\Bigg]Y_{lm}(\theta,\varphi) \;.\end{aligned} \tag{13}$$

Applying the relation

$$j_l(kr)=\tfrac{1}{2}[h_l^{(1)}(kr)+h_l^{(2)}(kr)] \;, \tag{14}$$

Eq. (13) yields the result

$$(\vec{r}\cdot\vec{V})=\sum_l\sum_m[A_{lm}^{(1)}(r)h_l^{(1)}(kr)+A_{lm}^{(2)}(r)h_l^{(2)}(kr)]Y_{lm}(\theta,\varphi) \;, \tag{15}$$

where the field-amplitude coefficients are radially varying functions defined by the following integral relations:

$$\begin{aligned}A_{lm}^{(1)}(r)=i2\pi k\Bigg\{&\int_0^{2\pi}\int_0^{\pi}\int_0^{r}[h_l^{(1)}(kr')+h_l^{(2)}(kr')]Y_{lm}^*(\theta',\varphi')f(\vec{r}\,')[r'^2\sin\theta']dr'd\theta'd\varphi' \\ &+\int_0^{2\pi}\int_0^{\pi}\int_r^{\infty}h_l^{(1)}(kr')Y_{lm}^*(\theta',\varphi')f(\vec{r}\,')[r'^2\sin\theta']dr'd\theta'd\varphi'\Bigg\} \;,\end{aligned} \tag{16a}$$

$$A_{lm}^{(2)}(r)=i2\pi k\left\{\int_0^{2\pi}\int_0^{\pi}\int_r^{\infty}h_l^{(1)}(kr')Y_{lm}^*(\theta',\varphi')f(\vec{r}\,')[r'^2\sin\theta']dr'd\theta'd\varphi'\right\} \;. \tag{16b}$$

A more explicit form for the driving function $f(\vec{r}\,')$ can be inserted in Eqs. (16a) and (16b) by combining Eqs. (7b) and (15). Finally, utilizing the orthogonality relation for the spherical harmonic functions

$$\int_0^{2\pi}\int_0^{\pi}Y_{lm}^*(\theta'\varphi')Y_{l'm'}(\theta',\varphi')\sin\theta'd\theta'd\varphi'=\delta_{ll'}\delta_{mm'} \tag{17}$$

and differentiating Eqs. (16a) and (16b) with respect to the radial variable, the coupled-amplitude equations are derived in the following form:

$$\begin{aligned}\frac{dA_{lm}^{(1)}(r)}{dr}&=i2\pi kr^2h_l^{(2)}(kr) \\ &\quad\times[O(r)\{A_{lm}^{(1)}(r)h_l^{(1)}(kr) \\ &\qquad+A_{lm}^{(2)}(r)h_l^{(2)}(kr)\}] \;,\end{aligned} \tag{18a}$$

$$\begin{aligned}\frac{dA_{lm}^{(2)}(r)}{dr}&=-i2\pi kr^2h_l^{(1)}(kr) \\ &\quad\times[O(r)\{A_{lm}^{(1)}(r)h_l^{(1)}(kr) \\ &\qquad+A_{lm}^{(2)}(r)h_l^{(2)}(kr)\}] \;.\end{aligned} \tag{18b}$$

Thus a solution to the above coupled-amplitude equations combined with Eq. (15) fully specifies the characteristic scalar component for a particular choice of the polarization.

For the TE case $[\vec{V}(\vec{r})\equiv\vec{B}(\vec{r})]$, the operator $O(r)$ is simply a function of r, which is proportional to the variation in the permittivity. The solution to this problem, analytically, corresponds to the scalar-waves case examined in Ref. [5]. For the TM case $[\vec{V}(\vec{r})\equiv\vec{D}(\vec{r})]$, however, $O(r)$ is a differential operator which acts solely on the radial variable. Thus Eqs. (18) adopt the general form

$$\frac{dA_{lm}^{(1)}(r)}{dr}=a_1+a_2\frac{dA_{lm}^{(1)}(r)}{dr}+a_3\frac{dA_{lm}^{(2)}(r)}{dr}, \tag{19a}$$

$$\frac{dA_{lm}^{(2)}(r)}{dr}=b_1+b_2\frac{dA_{lm}^{(1)}(r)}{dr}+b_3\frac{dA_{lm}^{(2)}(r)}{dr}, \tag{19b}$$

where the a_i and b_i expressions do not involve either of the two amplitude derivatives. By substituting the explicit form of the operator $O(r)$ for the TM case in Eqs. (18), expressions for a_i and b_i are derived. The following identities can be analytically proven for this case:

$$a_1b_3=a_3b_1\ , \tag{20a}$$

$$a_1b_2=a_2b_1\ , \tag{20b}$$

$$a_2=-b_3\ , \tag{20c}$$

$$(1-a_2)(1-b_3)-a_3b_2=1\ . \tag{20d}$$

Equations (19) can be solved in terms of a_i and b_i to yield

$$\frac{dA_{lm}^{(1)}(r)}{dr}=\frac{a_1(1-b_3)+a_3b_1}{(1-a_2)(1-b_3)-a_3b_2}, \tag{21a}$$

$$\frac{dA_{lm}^{(2)}(r)}{dr}=\frac{b_1(1-a_2)+b_2a_1}{(1-a_2)(1-b_3)-a_3b_2}. \tag{21b}$$

Using the identities of Eqs. (20) in Eqs. (21), the coupled-amplitude equations for the TM case reduce to

$$\frac{dA_{lm}^{(1)}(r)}{dr}=a_1\ , \tag{22a}$$

$$\frac{dA_{lm}^{(2)}(r)}{dr}=b_1\ . \tag{22b}$$

For completeness, the explicit forms of a_1 and b_1 are provided below:

$$a_1=\frac{i}{2}h_l^{(2)}(kr)\left[\left\{\Delta n^2(r)k_0^2-\frac{d\varepsilon}{dr}\frac{1}{\varepsilon r}\right\}[A_{lm}^{(1)}(r)h_l^{(1)}(kr)+A_{lm}^{(2)}(r)h_l^{(2)}(kr)]-\frac{d\varepsilon}{dr}\frac{1}{\varepsilon}\left[A_{lm}^{(1)}(r)\frac{\partial}{\partial r}h_l^{(1)}(kr)+A_{lm}^{(2)}(r)\frac{\partial}{\partial r}h_l^{(2)}(kr)\right]\right]kr^2\ , \tag{23a}$$

$$b_1=-\frac{h_l^{(1)}(kr)}{h_l^{(2)}(kr)}a_1\ . \tag{23b}$$

The coupled-wave equations [Eqs. (18) and (22)] are very useful for an exact description of the propagation of spherical waves in a spherically symmetric structure. For example, the waves generated by a given source can be described by an expansion in terms of spherical waves of specified harmonic and azimuthal orders. The interaction of these waves with the spherically symmetric structure can then be numerically calculated from the coupled-wave equations. This calculation is useful for determining both the nature of the waves leaving the structure and of the waves reflected back onto the source position. The latter calculation is performed in our companion paper to examine the classical effects of a spherical, periodic structure of finite size on the radiative properties of a dipole source.

III. ANALYTICAL AND NUMERICAL RESULTS

The amplitude reflection coefficients can now be determined for both the TE and TM cases. The problem reduces to solving the coupled-amplitude equations developed above for a particular choice of boundary conditions and permittivity function. For demonstration, consider a spherical Bragg structure with a permittivity function defined such that

$$\varepsilon(r)=\varepsilon_0\left[n_0^2+Q\sin\left[\frac{2\pi}{\Lambda}(r-r_1)\right]\right],\quad r_1<r<r_1+L\ , \tag{24}$$

where Q is a constant proportional to the amplitude of modulation in the permittivity, r_1 denotes the beginning of the index-modulated region, and Λ is the period of oscillation. This is the case depicted in Fig. 1. Assuming that the interaction length of the Bragg structure L is an integer number of sinusoidal periods, the boundary conditions for this case assume the form

$$A_{lm}^{(1)}(r_1)\ \text{specified}\ , \tag{25a}$$

$$A_{lm}^{(2)}(r_1+L)=0\ . \tag{25b}$$

Physically, the boundary conditions correspond to an outward traveling wave of arbitrary amplitude incident at $r=r_1$ and the requirement that no waves are approaching from infinity, respectively. The amplitude reflection coefficient ρ_l for the lth harmonic is then defined as

$$\rho_l=\frac{A_{lm}^{(2)}(r_1)h_l^{(2)}(kr_1)}{A_{lm}^{(1)}(r_1)h_l^{(1)}(kr_1)}\ , \tag{26}$$

where, due to the form of the coupled-amplitude equations, ρ_l has no dependence upon the azimuthal order integer m. Note that, in general, a phase term may be included in the argument of the sinusoid function of Eq. (24). In this case, the reflection coefficient of Eq. (26), for a finite structure, would not account for the possibility of additional reflections due to refractive index discontinuities at $r=r_1$ and r_1+L. Thus a dynamic boundary condition would have to be incorporated as discussed by Chinn for the linear grating case [11]. An alternative method of formulating the problem is to use transfer-matrix theory for propagation of the radial fields through discrete spherical layers by partitioning the permittivity function [12].

Certain limits exist in which the coupled-amplitude equations can be treated in an approximate analytical fashion. Specifically, the $l=0$ case can be treated through the application of a synchronous (nearly Bragg-matched) approximation. This approximation, discussed in Ref. [5] and elsewhere, involves the discarding of terms for which the (first-order) Bragg condition is not nearly satisfied. The TE case, which corresponds to the scalar-wave problem examined in Ref. [5], results in the following analytical forms for the amplitude coefficients subject to the boundary conditions of Eqs. (25):

$$[A_{00}^{(1)}(r)]^{\rm TE}=\left\{\frac{\alpha\cosh\{\alpha[(r-r_1)-L]\}+i\delta\sinh\{\alpha[(r-r_1)-L]\}}{\alpha\cosh(\alpha L)-i\delta\sinh(\alpha L)}\right\}e^{-i\delta(r-r_1)}A_{00}^{(1)}(r_1)\ , \tag{27a}$$

$$[A_{00}^{(2)}(r)]^{\rm TE}=\left\{\frac{-\kappa\sinh\{\alpha[(r-r_1)-L]\}}{\alpha\cosh(\alpha L)-i\delta\sinh(\alpha L)}\right\}e^{i\delta(r+r_1)}e^{i(2\pi/\Lambda)r_1}A_{00}^{(1)}(r_1)\ , \tag{27b}$$

where the notation of the earlier paper has been used:

$$\kappa=\frac{k_0^2Q}{4k}\ , \tag{28a}$$

$$2\delta=2k-\left[\frac{2\pi}{\Lambda}\right]\ , \tag{28b}$$

$$\alpha^2+\delta^2=\kappa^2\ , \tag{28c}$$

$$k=n_0k_0\ . \tag{28d}$$

Thus, substituting Eqs. (27) into Eq. (26), the amplitude reflection coefficient for this case is given by

$$(\rho_0)^{\rm TE}=\frac{-\kappa\sinh(\alpha L)}{\alpha\cosh(\alpha L)-i\delta\sinh(\alpha L)}\xrightarrow{\delta\to 0}-\tanh(\kappa L)\ , \tag{29}$$

where the second expression is the result in the exact Bragg matching ($\delta=0$) limit. Equation (29) is identical to the results of solving the TE reflection problems for a one-dimensional periodic region as well as for a cylindrically periodic region, for which the zeroth-order Hankel functions are well approximated by their asymptotic forms (see Ref. [5]). This equivalence is an example of the dimensional scaling which can occur in the treatment of wave coupling. The $l=0$ spherical Hankel functions are exactly traveling spherical waves which interact with a spherically symmetric, periodic structure. This interaction is identical to that of traveling plane (circular) waves with a one- (two-) dimensional periodic structure of the same period and modulation. For the TM case, in the weak grating, nearly Bragg-matched limit, the amplitude reflection coefficient is approximately given by Eq. (29) with an additional π phase shift, i.e., $(\rho_0)^{\rm TM}\approx-(\rho_0)^{\rm TE}$.

The remaining case that can be treated analytically in a closed-form manner occurs when the asymptotic expansions for the spherical Hankel functions are valid. That this case can also be solved analytically is not surprising: in the asymptotic limit, the spherical Hankel functions exactly approach traveling spherical waves with an order-dependent phase shift. Thus, in this limit, there exists a great similarity to the $l=0$ case. The asymptotic expansions for the spherical Hankel functions are given by [13]

$$h_l^{(1)}(kr)=(-i)e^{-il(\pi/2)}\frac{e^{ikr}}{kr}\ , \tag{30a}$$

$$h_l^{(2)}(kr)=ie^{il(\pi/2)}\frac{e^{-ikr}}{kr}\ , \tag{30b}$$

which are valid for $(kr)\gg l(l+1)/2$. Solving the coupled-amplitude equations within the synchronous approximation, the results for the amplitude reflection coefficients are identical to the $l=0$ results. The order-dependent phase shift of Eqs. (30) is evident in the amplitude coefficients which are related to the $l=0$ results by the expressions

$$A_{lm}^{(1)}(r)=A_{00}^{(1)}(r)e^{il\pi/2}\ , \tag{31a}$$

$$A_{lm}^{(2)}(r)=A_{00}^{(2)}(r)e^{-il\pi/2}\ . \tag{31b}$$

Equations (31) hold for both TE and TM polarized wave amplitudes. As expected, the amplitude reflection coefficient, which includes the ratio of spherical Hankel functions exactly canceling the order-dependent phase shift from the ratio of the amplitude coefficients, is translationally invariant in the asymptotic limit. Excluding the $l=0$ and asymptotic-limit cases, an analytic solution to the coupled-amplitude equations may not be possible due to the increasingly complicated dependence of the spherical Hankel functions on the radial variable. In these cases, a numerical solution to the coupled-amplitude equations is necessary. The numerical results also allow for a comparison with the analytic solutions detailed above to examine the accuracy of the synchronous approximation.

A numerical solution to the coupled-amplitude equations formally involves the solution to a two-point boundary-value problem. There exist several methods of solution for this problem, most notably shooting and relaxation methods. Due to the nonlinearity in the radial coordinate of the coupled-amplitude equations, a determination of the numerical solution may take multiple iterations. For the calculations presented in this paper, we have implemented a shooting method which incorporates multidimensional Newton-Raphson and fourth-order Runge-Kutta routines [14]. The numerical generation of the functions necessary to encode the coupled-amplitude equations is thoroughly discussed in the literature [13–15].

For comparison, the approximate analytical results and the exact numerical results for the TE, $l=0$ case are plotted in Fig. 2 for a coupling strength product of $\kappa L=2$. In terms of characteristic exponential decay and growth, the analytical and numerical results are nearly identical. The absence of oscillatory behavior in the approximate solution is a consequence of the synchronous approximation where harmonic terms which are not nearly phase matched are discarded. The figure also illustrates the nature of the coupling process: a distributed-feedback mechanism in which energy is transferred from the outward- to the inward-traveling spherical wave. As noted in our companion paper, the $l=0$ term of the field expansion does not exist for a dipole source. However, the same graphical behavior depicted in Fig. 2 is observed for the case in which the asymptotic expansions for the spherical Hankel functions are valid. This equivalence is understood in terms of Eqs. (31), where the amplitude coefficients for the asymptotic case are related to the $l=0$ amplitude coefficients by an order-dependent phase shift. Thus the moduli of the field-amplitude coefficients, as depicted in Fig. 2, are equivalent for these two cases.

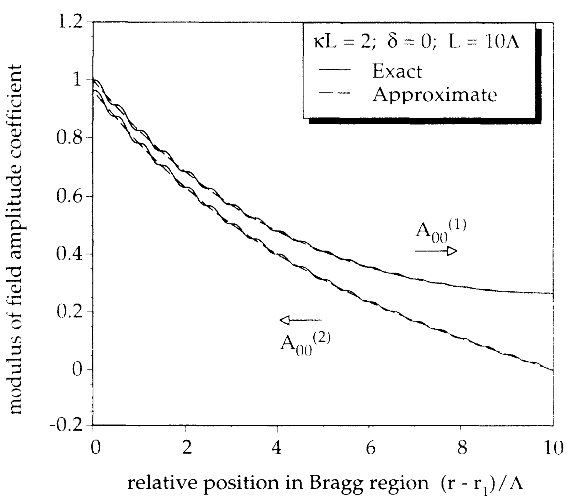

FIG. 2. Modulus of the field amplitude coefficient $[A_{00}^{(i)}(r)]^{\rm TE}$ vs relative position in the Bragg region for the TE case of a spherical Bragg structure with a coupling product of $\kappa L=2$, an interaction length of $L=10\Lambda$, and perfect Bragg matching ($\delta=0$). "Approximate" refers to the synchronous approximation solution [Eqs. (27)] and "exact" denotes a numerical solution to the coupled-amplitude equation system [Eqs. (18)].

IV. TRANSVERSE FIELD COMPONENTS

With the radial portion of the characteristic fields determined by Eqs. (15), (18), and (22), it is desirable to construct the remaining transverse field components to fully specify the electromagnetic fields. For this task, an application of vector spherical harmonics [8] proves sufficient. Using Eq. (1d), the radial component of the electric displacement field can be related to the magnetic field such that

$$(\vec{r}\cdot\vec{D})=\frac{i}{\mu\omega}(\vec{r}\times\vec{\nabla})\cdot\vec{B}=-\frac{1}{\mu\omega}\vec{L}\cdot\vec{B}\ , \tag{32}$$

where the differential operator $\vec{L}$ has been introduced with the definition

$$\vec{L}=\frac{1}{i}(\vec{r}\times\vec{\nabla})\ . \tag{33}$$

The $\vec{L}$ operator acts only on angular variables and is independent of r; it is also proportional to the familiar orbital angular momentum operator of quantum mechanics. Furthermore, as in quantum mechanics, this operator acting on a spherical harmonic function transforms the azimuthal integer value m without changing the l value.

Using the radial component field expansion of Eq. (15), Eq. (32) can be rewritten

$$\begin{aligned}\vec{L}\cdot\vec{B}&=-\mu\omega(\vec{r}\cdot\vec{D})\\&=-\mu\omega\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\{[A_{lm}^{(1)}(r)]^{\rm TM}h_l^{(1)}(kr)\\&\qquad+[A_{lm}^{(2)}(r)]^{\rm TM}h_l^{(2)}(kr)\}Y_{lm}(\theta,\varphi)\ .\end{aligned} \tag{34}$$

Since $\vec{L}$ operates only on angular variables, the TM portion of the magnetic field assumes the form

$$\begin{aligned}\vec{B}^{\rm TM}=-\mu\omega\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}c_l\{[A_{lm}^{(1)}(r)]^{\rm TM}h_l^{(1)}(kr)\\+[A_{lm}^{(2)}(r)]^{\rm TM}h_l^{(2)}(kr)\}\\\times\vec{L}Y_{lm}(\theta k,\varphi)\ ,\end{aligned} \tag{35}$$

where the coefficient c_l is a function solely of the integer l and not the azimuthal integer. The vector spherical harmonics $\vec{L}Y_{lm}(\theta,\varphi)$ appear in the field expansion and provide the angular dependence for the transverse field components. Operating with $\vec{L}$ on each side of Eq. (35), using the identity

$$L^2Y_{lm}(\theta,\varphi)=l(l+1)Y_{lm}(\theta,\varphi)\ , \tag{36}$$

and comparing forms with Eq. (34), the coefficient c_l is determined such that

$$c_l = \frac{1}{l(l+1)} \ . \tag{37}$$

Thus, combining Eqs. (35) and (37), the magnetic field for the TM case is given by the relation

$$\vec{B}^{TM} = -\mu\omega \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \frac{1}{l(l+1)} \{ [A_{lm}^{(1)}(r)]^{TM} h_l^{(1)}(kr) + [A_{lm}^{(2)}(r)]^{TM} h_l^{(2)}(kr) \} \times \vec{L} Y_{lm}(\theta,\varphi) \ . \tag{38}$$

Note that a coefficient similar in form to c_l is often used in the definition of a normalized vector spherical harmonic function [8,13]. The corresponding electric displacement field is calculated through the application of Eq. (1d). Employing the method above, the electric displacement field for the TE case can be similarly determined to yield

$$\vec{D}^{TE} = \varepsilon(r)\omega \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \frac{1}{l(l+1)} \{ [A_{lm}^{(1)}(r)]^{TE} h_l^{(1)}(kr) + [A_{lm}^{(2)}(r)]^{TE} h_l^{(2)}(kr) \} \times \vec{L} Y_{lm}(\theta,\varphi) \ . \tag{39}$$

The corresponding magnetic field is calculated through the application of Eq. (1c). Finally, the total field is given by a linear combination of the TE and TM field solutions.

V. CONCLUDING REMARKS

We have described a method to derive the coupled-amplitude equations for vector waves in a three-dimensional, radially varying refractive index structure. This method demonstrates another example of a growing list of problems whose details can be elucidated by an application of a scalar Green's-function technique. Specifically, Green's-function methods have recently been practically applied to investigate coupled-mode theory in a circularly symmetric distributed-feedback laser [16,17] and the inhibition of radiation in a cylindrical Bragg resonator [18].

We have discussed the solution to the derived coupled-amplitude equations for the particular choice of a spherical Bragg structure. This discussion includes cases in which approximate analytical solutions can be determined, while noting the similarity of these results to those of the familiar linear Bragg geometry. For completeness, the characteristic scalar wave equations were derived and a method for construction of the remaining field components was provided. It must be stressed that the formulation comprising the main body of this paper is numerically equivalent to previously developed theories, including transfer matrix and Debye potential methods. However, our formulation has a benefit inherent to coupled-amplitude formalisms: it provides intuition regarding the dynamic nature of the coupling which has made both coupled-amplitude and coupled-mode formalisms so prevalent in modeling physical systems.

Finally, the calculation presented here can be considered an intermediate step toward the completion of certain cavity calculations. Specifically, the coupling of a source field to the cavity modes of a radially varying refractive index structure can be investigated. For a spherical Bragg structure, this investigation can be directed towards a numerical characterization of the radiative lifetime and resonance frequency shifts of an embedded source as detailed in our companion paper. This calculation will quantify, for a finite-length structure, the degree to which a spherically symmetric system behaves as a photonic band-gap structure.

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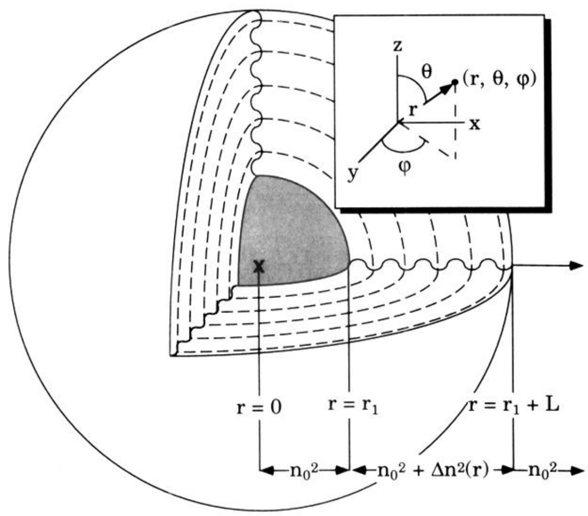

FIG. 1. A spherical Bragg structure. For this case, the permittivity varies sinusoidally with a specified amplitude within the region of examination $(r_1 < r < r_1 + L)$ and is everywhere continuous.