Quantum-optical master equations: An interaction picture

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Quantum-optical master equations — exemplified by the Jaynes-Cummings model with damping—are turned into numerical partial differential equations of first order for phase-space functions, which are generalizations of the Wigner function and its relatives. The time dependence of these phase-space functions originates solely in the atom-photon interaction; all other time dependences, in particular the dissipative contribution of the photon damping, are accounted for by the time-dependent operator bases to which the phase-space functions refer. The judicious choice of operator basis also effects the absence of second-order derivatives in the partial differential equation. Our first-order equations are hyperbolic and can be integrated conveniently along their characteristics. As an illustrative application we study how the Jaynes-Cummings revivals are affected by photon damping. We show how to handle squeezed reservoirs and how to apply the method to laser cooling.

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I. INTRODUCTION

In recent years, we have been witnessing renewed interest in methods by which quantum-optical master equations are treated numerically. This interest was triggered by a common dissatisfaction with the standard approaches that use either rather large number-state matrices or rather inconvenient coupled Fokker-Planck equations. One new method renounces a direct solution of the master equation in favor of simulating the actual density operator by an ensemble of wave functions that follow a pseudounitary evolution and suffer state reductions at random instants [1]. Another new development employs the so-called "damping bases" for an algebraic frontal attack on the master equation [2]; the damping bases diagonalize the nonunitary, dissipative part of the master equation and so provide for a substantial simplification as compared to the formulation in terms of number-state matrices.

In the present contribution, we introduce an approach that combines the damping-bases strategy with the Fokker-Planck formalism, with the intention to utilize the advantages of both while avoiding their drawbacks. We find a set of first-order partial differential equations for phase-space functions that are generalizations of the familiar ones — the Wigner function, Glauber's Q and P functions, and the like. Second-order derivatives are absent, in marked contrast to the Fokker-Planck equations. This is achieved by employing time-dependent operator bases in the definition of our phase-space functions. Except for the atom-photon interaction, all dynamical changes are accounted for by this time dependence of the basis. Consequently, our phase-space functions evolve only as a result of the interaction. In other words, we are proposing a luxurious interaction picture.

Here is a brief outline of the paper. The ground is prepared in Sec. II where we take a look at explicit operator solutions for noninteracting systems. Then, in Sec. III, our first-order partial differential equations are derived and discussed. The numerical methods being used are presented in Sec. IV. Section V reports an application to Jaynes-Cummings revivals in the presence of photon damping. We close with a summary. Two appendices deal with extensions of the formalism; in Appendix A we show how to handle squeezed reservoirs; in Appendix B we report the equations of motion relevant for the treatment of laser cooling.

II. NONINTERACTING SYSTEMS

A. Photons

We describe the method in the exemplary context of a single photon mode; the generalization to more than one kind of photons is immediate. The photon state is specified by the density operator $\rho(t) = \rho(a^\dagger, a, t)$ which is a function of the dynamical variables $a^\dagger$ and a (the "ladder operators"). Its evolution is governed by the quantum master equation

$$\frac{\partial}{\partial t}\rho = \mathcal{L}_{\rm ph}\rho = i\omega[\rho, a^\dagger a] + z[a^\dagger, \rho] + z^*[\rho, a] - \frac{A}{2}(\nu+1)\left(a^\dagger a\rho - 2a\rho a^\dagger + \rho a^\dagger a\right) - \frac{A}{2}\nu\left(aa^\dagger\rho - 2a^\dagger\rho a + \rho aa^\dagger\right) . \quad (2.1)$$

The symbol ω denotes the natural frequency of the photon mode; A is the rate with which the state would re-

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lax toward its thermal equilibrium state (characterized by ν, the number of thermal photons) if there were no "injected signal" whose strength is measured by the numerical function $z(t)$. For the sake of simplicity, we shall be content with dynamics of this restricted form; in Appendix A we show how to deal with squeezed reservoirs and phase-sensitive unitary contributions.

The fundamental observation is this. Operators of the structure

$$\mathsf{B}\big(\kappa(t),\alpha^*(t),\alpha'(t)\big) \equiv \frac{1}{\kappa(t)} : \exp\Big(-\frac{1}{\kappa(t)}\big[a^\dagger-\alpha^*(t)\big]\big[a-\alpha'(t)\big]\Big) : \tag{2.2}$$

are solutions of the master equation (2.1) provided that the numerical functions $\kappa(t)$, $\alpha^*(t)$, and $\alpha'(t)$ obey

$$\begin{aligned}\frac{d}{dt}\kappa &= -A(\kappa-\nu-1)\,,\\ \frac{d}{dt}\alpha^* &= \big(i\omega-\tfrac{1}{2}A\big)\,\alpha^*+z^*\,,\\ \frac{d}{dt}\alpha' &= \big(-i\omega-\tfrac{1}{2}A\big)\,\alpha'+z\,,\end{aligned} \tag{2.3}$$

whereby $\alpha^*(t)$ need not equal the complex conjugate of $\alpha'(t)$, although this is the usual situation. [Of course, $z^*(t)$ must always be the complex conjugate of $z(t)$.] As usual, the pair of colons in (2.2) indicates normal ordering, all $a^\dagger$ operators to the left of all factors of a. Our assertion, viz. that B of (2.2) with (2.3) solves (2.1), is demonstrated with the aid of the identities

$$\begin{aligned}\frac{\partial}{\partial t}\mathsf{B} &= -\kappa^{-2}\left[\kappa\mathsf{B}-(a^\dagger-\alpha^*)\mathsf{B}(a-\alpha')\right]\frac{d\kappa}{dt}\\ &\quad+\kappa^{-1}(a^\dagger-\alpha^*)\mathsf{B}\frac{d\alpha'}{dt}+\kappa^{-1}\frac{d\alpha^*}{dt}\mathsf{B}(a-\alpha')\,,\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}[a^\dagger,\mathsf{B}] &= \kappa^{-1}(a^\dagger-\alpha^*)\mathsf{B}\,,\\ [\mathsf{B},a] &= \kappa^{-1}\mathsf{B}(a-\alpha')\,,\\ [\mathsf{B},a^\dagger a] &= \kappa^{-1}\alpha^*\mathsf{B}(a-\alpha')-\kappa^{-1}(a^\dagger-\alpha^*)\mathsf{B}\alpha'\end{aligned} \tag{2.5}$$

as well as

$$\begin{aligned}&a^\dagger a\mathsf{B}-2a\mathsf{B}a^\dagger+\mathsf{B}a^\dagger a\\ &\quad=2\kappa^{-1}(1-\kappa)\mathsf{B}-2\kappa^{-2}(1-\kappa)(a^\dagger-\alpha^*)\mathsf{B}(a-\alpha')\\ &\qquad+\kappa^{-1}(a^\dagger-\alpha^*)\mathsf{B}\alpha'+\kappa^{-1}\alpha^*\mathsf{B}(a-\alpha')\,,\\ &aa^\dagger\mathsf{B}-2a^\dagger\mathsf{B}a+\mathsf{B}aa^\dagger\\ &\quad=2\mathsf{B}-2\kappa^{-1}(a^\dagger-\alpha^*)\mathsf{B}(a-\alpha')\\ &\qquad-\kappa^{-1}(a^\dagger-\alpha^*)\mathsf{B}\alpha'-\kappa^{-1}\alpha^*\mathsf{B}(a-\alpha')\,.\end{aligned} \tag{2.6}$$

These are systematically derived by repeated applications of the elementary rules

$$[a,F]=\frac{\partial F}{\partial a^\dagger}\quad,\quad[F,a^\dagger]=\frac{\partial F}{\partial a}\,, \tag{2.7}$$

valid for any operator function $F(a^\dagger,a)$. Further, we recall that the initial state $\rho(0)=\rho(t=0)$ can be written as a linear superposition of B's according to

$$\rho(0)=\int\frac{\mathcal{D}(a^{\dagger'},a'')}{2\pi}\,f_0(a^{\dagger'},a'')\mathsf{B}(\kappa_0,a^{\dagger'},a'')\,, \tag{2.8}$$

where the phase-space function $f_0(a^{\dagger'},a'')$ is given by [3]

$$f_0(a^{\dagger'},a'')=\mathrm{tr}\{\rho(0)\mathsf{B}(1-\kappa_0,a^{\dagger'},a'')\}\,. \tag{2.9}$$

For any fixed value of a'', $f_0(a^{\dagger'},a'')$ is an analytic function of its complex variable $a^{\dagger'}$; likewise, f_0 is analytic in a''. We are employing the notational convention introduced by Dirac who denotes eigenvalues by attaching primes to the symbols of the corresponding operators. Thus, in writing $a^{\dagger'}$ and a'' we emphasize that these complex numbers are essentially eigenvalues of $a^\dagger$ and a, respectively.

In general terms, the differential $\mathcal{D}(a^{\dagger'},a'')$ in (2.8) symbolizes the injunction to integrate the complex variables $a^{\dagger'}$ and a'' over orthogonal contours. Since this generality is of little use in the present context, we shall only employ the standard parametrization

$$\begin{aligned}a^{\dagger'} &= a''^*=q-ip=re^{-i\varphi}\,,\\ \mathcal{D}(a^{\dagger'},a'') &= 2\,dp\,dq=2\,dr\,r\,d\varphi\,,\end{aligned} \tag{2.10}$$

where the cartesian phase-space variables p and q range from $-\infty$ to $+\infty$, whereas the polar radius $r=\sqrt{p^2+q^2}$ is limited to positive values and the polar angle φ covers any interval of length 2π. With (2.10), a'' equals the complex conjugate of $a^{\dagger'}$, as already indicated; therefore, we shall write a' rather than a'' from here on. As a further consequence of the parametrization (2.10), Eq. (2.8) associates a real phase-space function $f_0(a^{\dagger'},a')$ with the Hermitean $\rho(0)$.

Before proceeding we note that the trace in (2.9) is certainly well defined for $0\le\kappa_0\le\frac{1}{2}$. In particular, $f_0(a^{\dagger'},a')$ is the Wigner function of $\rho(0)$ when $\kappa_0=\frac{1}{2}$; $f_0(a^{\dagger'},a')$ becomes the Q function in the limit $\kappa_0\to 0$; if it exists, the P function is available for $\kappa_0\to 1$ [4].

Upon combining the observation, that B of (2.2) solves the master equation (2.1), with Eq. (2.8) we find the general solution of (2.1):

$$\rho(t)=\int\frac{\mathcal{D}(a^{\dagger'},a')}{2\pi}\,f_0(a^{\dagger'},a')\mathsf{B}\big(\kappa(t),\alpha^*(t),\alpha'(t)\big)\,, \tag{2.11}$$

where $\kappa(0)=\kappa_0$, $\alpha^*(0)=a^{\dagger'}$, and $\alpha'(0)=a'$ are the initial values supplementing Eqs. (2.3). Since $a'^*=a^{\dagger'}$ holds, we have $\alpha'^*(0)=\alpha^*(0)$ so that Eqs. (2.3) imply $\alpha'^*(t)=\alpha^*(t)$, and we shall drop the prime on α accordingly. Thus

$$\begin{aligned}\alpha^*(t) &= e^{i\omega t}e^{-At/2}a^{\dagger'}\\ &\quad+\int_0^t dt'\,e^{i\omega(t-t')}e^{-A(t-t')/2}z^*(t')\,,\\ \alpha'(t)\equiv\alpha(t) &= e^{-i\omega t}e^{-At/2}a'\\ &\quad+\int_0^t dt'\,e^{-i\omega(t-t')}e^{-A(t-t')/2}z(t')\,,\end{aligned} \tag{2.12}$$

and

$$\kappa(t) = \kappa_\infty - (\kappa_\infty - \kappa_0)\, e^{-At} \tag{2.13}$$

with

$$\kappa_\infty = \kappa(t \to \infty) = \nu + 1\,. \tag{2.14}$$

For the following it is essential that the time dependence is wholly contained in the operator basis $\mathsf{B}\big(\kappa(t), \alpha^*(t), \alpha(t)\big)$; the phase-space function $f_0(a^{\dagger'}, a')$ does not evolve.

Expectation values can then be calculated as phase-space integrals. Elementary examples are

$$\begin{aligned}
\mathrm{tr}\,\{\rho(t)\} &= \int \frac{\mathcal{D}(a^{\dagger'}, a')}{2\pi}\, f_0(a^{\dagger'}, a') = \mathrm{tr}\,\{\rho(0)\}\,, \\
\mathrm{tr}\,\{a^\dagger \rho(t)\} &= \int \frac{\mathcal{D}(a^{\dagger'}, a')}{2\pi}\, \alpha^*(t) f_0(a^{\dagger'}, a')\,, \\
\mathrm{tr}\,\{a\rho(t)\} &= \int \frac{\mathcal{D}(a^{\dagger'}, a')}{2\pi}\, \alpha(t) f_0(a^{\dagger'}, a')\,, \\
\mathrm{tr}\,\{a^\dagger a \rho(t)\} &= \int \frac{\mathcal{D}(a^{\dagger'}, a')}{2\pi}\, [\alpha^*(t)\alpha(t) + \kappa(t) - 1] \\
&\quad \times f_0(a^{\dagger'}, a')\,.
\end{aligned} \tag{2.15}$$

Some remarks concerning the connection between the time-dependent operator basis of (2.2) and the damping basis of [2] are in order. In its dependence on α^* and α, $\mathsf{B}(\kappa, \alpha^*, \alpha)$ is a generating function for the elements $\rho_n^{(k)}$ of the damping basis. The explicit relation is

$$\mathsf{B}(\kappa, \alpha^*, \alpha) = \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} b_n^{(k)} \rho_n^{(k)}, \tag{2.16}$$

with

$$\begin{aligned}
b_n^{(k)} = b_n^{(-k)^*} &= \frac{n!}{(n+k)!} \left(\frac{\kappa}{\kappa_\infty} - 1\right)^n \\
&\quad \times L_n^{(k)} \left(\frac{\alpha^* \alpha}{\kappa - \kappa_\infty}\right) \alpha^k
\end{aligned} \tag{2.17}$$

for $k \geq 0$. If there is no injected signal, $z(t) = 0$ in (2.12), then the argument of the generalized Laguerre polynomial is time independent, and $b_n^{(k)}$ is proportional to $\exp[-ik\omega t - (n + |k|/2)t]$, as it must be. For further details concerning the damping basis we refer the reader to Ref. [2].

B. Atoms

For the exemplary discussion of the atomic degree of freedom we shall be content with simple circumstances, too. Therefore, we deal with a single two-level atom only. Now $\rho(t) = \rho(\sigma_+, \sigma_-, t)$ denotes the state of this atom; it is a function of the dynamical variables σ_+ and σ_-,

$$\sigma_\pm = \sigma_x \pm i\sigma_y = (\sigma_\mp)^\dagger\,, \tag{2.18}$$

where σ_x, σ_y, and also $\sigma_z = (\sigma_+\sigma_- - \sigma_-\sigma_+)/4$ are analogs of Pauli's spin operators. As the quantum master equation for the evolution of $\rho(t)$ we take

$$\begin{aligned}
\frac{\partial}{\partial t}\rho = \mathcal{L}_{\mathrm{at}}\rho &= i\frac{\Omega}{2}\,[\rho, \sigma_z] \\
&\quad -\frac{B}{8}(1-s)\,(\sigma_+\sigma_-\rho - 2\sigma_-\rho\sigma_+ + \rho\sigma_+\sigma_-) \\
&\quad -\frac{B}{8}s\,(\sigma_-\sigma_+\rho - 2\sigma_+\rho\sigma_- + \rho\sigma_-\sigma_+) \\
&\quad -\frac{2C-B}{4}\,(\rho - \sigma_z\rho\sigma_z)\,.
\end{aligned} \tag{2.19}$$

Here, Ω is the natural frequency of the atomic transition; B is the rate at which $\left\langle \frac{1}{2}\,(1+\sigma_z)\right\rangle$, the population of the upper atomic level, relaxes toward its equilibrium value s (limited to the range $0 \leq s \leq 1$); and C is the rate at which the polarization $\langle\sigma_+\rangle$ decays. The restriction $C \geq B/2$ applies.

The four time-dependent operators

$$\begin{aligned}
\mathsf{b}_1 &= \tfrac{1}{2} + \tfrac{1}{2}(2s-1)\sigma_z\,, \\
\mathsf{b}_2 &= \tfrac{1}{2}e^{-Bt}\sigma_z\,, \\
\mathsf{b}_3 &= \tfrac{1}{2}e^{-i\Omega t}e^{-Ct}\sigma_+\,, \\
\mathsf{b}_4 &= \tfrac{1}{2}e^{i\Omega t}e^{-Ct}\sigma_- = \mathsf{b}_3^\dagger\,,
\end{aligned} \tag{2.20}$$

are the fundamental solutions of (2.19). They include the steady state of (2.19), b_1, which is actually constant in time. We write the general solution of (2.19) as a linear superposition of $\mathsf{b}_1, \ldots, \mathsf{b}_4$:

$$\rho(t) = \sum_{j=1}^{4} f_j \mathsf{b}_j(t)\,. \tag{2.21}$$

The constant coefficients $f_1, \ldots, f_4$ are determined by

$$\begin{aligned}
f_1 &= \mathrm{tr}\,\{\rho(0)\} = 1\,, \\
f_2 &= \mathrm{tr}\,\{\rho(0)\,(\sigma_z - 2s + 1)\}\,, \\
f_3 &= \tfrac{1}{2}\mathrm{tr}\,\{\rho(0)\sigma_-\}\,, \\
f_4 &= \tfrac{1}{2}\mathrm{tr}\,\{\rho(0)\sigma_+\} = f_3^*\,.
\end{aligned} \tag{2.22}$$

Note that b_1 and b_2 are Hermitean, whereas b_3 and b_4 are Hermitean conjugates of each other; accordingly, f_1 ($\equiv 1$) and f_2 are real, and f_3 is the complex conjugate of f_4. The basic expectation values are here

$$\begin{aligned}
\mathrm{tr}\,\{\rho(t)\} &= f_1 = \mathrm{tr}\,\{\rho(0)\}\,, \\
\mathrm{tr}\,\{\rho(t)\sigma_z\} &= (2s-1)f_1 + e^{-Bt}f_2\,, \\
\mathrm{tr}\,\{\rho(t)\sigma_-\} &= 2e^{-i\Omega t}e^{-Ct}f_3\,, \\
\mathrm{tr}\,\{\rho(t)\sigma_+\} &= 2e^{i\Omega t}e^{-Ct}f_4\,.
\end{aligned} \tag{2.23}$$

Clearly, Eqs. (2.20), (2.21), (2.22), and (2.23) are the analogs of (2.2), (2.11), (2.9), and (2.15), respectively. Again we emphasize that the time dependence in (2.21) is entirely in the basis operators $\mathsf{b}_1 \ldots \mathsf{b}_4$, not in the coefficients $f_1, \ldots f_4$.

C. Photons and atoms

If we now consider a single photon mode and a two-level atom jointly — without any interaction at the moment — the total state $\rho(t) = \rho(a^\dagger, a, \sigma_+, \sigma_-, t)$ obeys

$$\frac{\partial}{\partial t}\rho = \mathcal{L}_{\rm ph}\rho + \mathcal{L}_{\rm at}\rho\,. \tag{2.24}$$

The general solution is obtained by merging (2.21) with (2.11) into

$$\rho(t) = \int \frac{\mathcal{D}(a^{\dagger\prime},a')}{2\pi} \sum_{j=1}^{4} f_j(a^{\dagger\prime},a')\mathsf{b}_j(t) \times \mathsf{B}\big(\kappa(t),\alpha^*(t),\alpha(t)\big). \tag{2.25}$$

According to Eqs. (2.9) and (2.22) the four time-independent phase-space functions are related to the initial state $\rho(0)$ by

$$\begin{aligned} f_1(a^{\dagger\prime},a') &= \mathrm{tr}\{\rho(0)\mathsf{B}(1-\kappa_0,a^{\dagger\prime},a')\}\,,\\ f_2(a^{\dagger\prime},a') &= \mathrm{tr}\{\rho(0)\,(\sigma_z - 2s + 1)\,\mathsf{B}(1-\kappa_0,a^{\dagger\prime},a')\}\,,\\ f_3(a^{\dagger\prime},a') &= \tfrac{1}{2}\mathrm{tr}\{\rho(0)\sigma_-\mathsf{B}(1-\kappa_0,a^{\dagger\prime},a')\}\,,\\ f_4(a^{\dagger\prime},a') &= \tfrac{1}{2}\mathrm{tr}\{\rho(0)\sigma_+\mathsf{B}(1-\kappa_0,a^{\dagger\prime},a')\}\,, \end{aligned} \tag{2.26}$$

where the trace "tr" is over both the photon and the atom degrees of freedom.

Equations (2.15) and (2.23) tell us how to compute expectation values. The example

$$\mathrm{tr}\left\{a^\dagger\sigma_-\rho(t)\right\} = 2e^{-i\Omega t}e^{-Ct}\int\frac{\mathcal{D}(a^{\dagger\prime},a')}{2\pi}\alpha^*(t)f_3(a^{\dagger\prime},a') \tag{2.27}$$

may serve as an illustration.

III. INTERACTING SYSTEMS

A. Equations of motion

We are now prepared to introduce an interaction between the two-level atom and the single photon mode. The master equation (2.24) is supplemented by an interaction term,

$$\frac{\partial}{\partial t}\rho = \mathcal{L}_{\rm ph}\rho + \mathcal{L}_{\rm at}\rho + \mathcal{L}_{\rm int}\rho\,, \tag{3.1}$$

for which we choose the Jaynes-Cummings coupling

$$\mathcal{L}_{\rm int}\rho = -\tfrac{i}{2}g[\rho,a^\dagger\sigma_-] - \tfrac{i}{2}g^*[\rho,a\sigma_+]\,. \tag{3.2}$$

The coupling strength $g(t)$ may be a rather arbitrary complex function of time; its magnitude is the instantaneous Rabi frequency.

We write the general solution of (3.1) in the form (2.25) except that we now allow for a time dependence in the phase-space functions $f_j = f_j(a^{\dagger\prime},a',t)$ to account for the atom-photon interaction. Inasmuch as the factors $\mathsf{b}_j(t)\mathsf{B}\big(\kappa(t),\alpha^*(t),\alpha(t)\big)$ take care of the $\mathcal{L}_{\rm ph}$ and $\mathcal{L}_{\rm at}$ contributions to $\partial\rho/\partial t$, the time dependence of the f_j's has to reflect the effect of $\mathcal{L}_{\rm int}$,

$$\int\frac{\mathcal{D}(a^{\dagger\prime},a')}{2\pi}\sum_j\frac{\partial f_j}{\partial t}\mathsf{b}_j\mathsf{B} = \int\frac{\mathcal{D}(a^{\dagger\prime},a')}{2\pi}\sum_j f_j\mathcal{L}_{\rm int}\mathsf{b}_j\mathsf{B}\,. \tag{3.3}$$

Indeed, the phase-space functions $f_1,\ldots,f_4$ change only as a result of the interaction. In this sense, we have formulated a genuine *interaction picture* for the quantum master equation (3.1). This effort is rewarded with a set of first-order differential equations for the f_j's, which we derive next.

To turn the operator equation (3.3) into numerical statements about the f_j's, one must express $\mathcal{L}_{\rm int}\mathsf{b}_j\mathsf{B}$ in terms of the basic products $\mathsf{b}_j\mathsf{B}$ themselves. This is achieved in a few steps. First, we utilize relations like

$$\mathsf{b}_1\sigma_- = 2(1-s)e^{-i\Omega t}e^{Ct}\mathsf{b}_4 \tag{3.4}$$

for the identification of the factors multiplying $\mathsf{b}_1,\ldots,\mathsf{b}_4$ on the right-hand side of (3.3). This produces

$$\begin{aligned} &\int\frac{\mathcal{D}(a^{\dagger\prime},a')}{2\pi}\frac{\partial f_1}{\partial t}\mathsf{B}\\ &\quad= -ige^{-i\Omega t}e^{-Ct}\int\frac{\mathcal{D}(a^{\dagger\prime},a')}{2\pi}\left(\mathsf{B}a^\dagger - a^\dagger\mathsf{B}\right)f_3\\ &\qquad+ig^*e^{i\Omega t}e^{-Ct}\int\frac{\mathcal{D}(a^{\dagger\prime},a')}{2\pi}\left(a\mathsf{B}-\mathsf{B}a\right)f_4 \end{aligned} \tag{3.5}$$

and corresponding equations for $\partial f_2/\partial t$, $\partial f_3/\partial t$, and $\partial f_4/\partial t$. Second, the extra factors of a and $a^\dagger$ are eliminated with the aid of identities like

$$\begin{aligned} a^\dagger\mathsf{B} &= \left(\alpha^* + \kappa\frac{\partial}{\partial\alpha}\right)\mathsf{B} = \left(\alpha^* + \kappa e^{i\omega t}e^{At/2}\frac{\partial}{\partial a'}\right)\mathsf{B}\,,\\ a\mathsf{B} &= \left(\alpha - (1-\kappa)\frac{\partial}{\partial\alpha^*}\right)\mathsf{B}\\ &= \left(\alpha - (1-\kappa)e^{-i\omega t}e^{At/2}\frac{\partial}{\partial a^{\dagger\prime}}\right)\mathsf{B}\,. \end{aligned} \tag{3.6}$$

Finally, partial integrations produce equivalent replacements as exemplified by

$$-\frac{\partial\mathsf{B}}{\partial a^{\dagger\prime}}f_4 \to \mathsf{B}\frac{\partial f_4}{\partial a^{\dagger\prime}}\,. \tag{3.7}$$

The resulting *equations of motion* are compactly presented as

$$\frac{\partial}{\partial t}\mathbf{f} = \mathbf{G}^{(a)}\frac{\partial}{\partial a'}\mathbf{f} + \mathbf{G}^{(a^\dagger)}\frac{\partial}{\partial a^{\dagger\prime}}\mathbf{f} + \mathbf{H}\mathbf{f}\,, \tag{3.8}$$

where $\mathbf{f}(a^{\dagger\prime},a',t)$ is the four-entry column

$$\mathbf{f} = \begin{pmatrix} f_1\\ f_2\\ f_3\\ f_4 \end{pmatrix} \tag{3.9}$$

and the 4×4 matrices $\mathbf{G}^{(a)}$, $\mathbf{G}^{(a^\dagger)}$, and $\mathbf{H}$ are given by

$$\mathbf{G}^{(a)} = \begin{pmatrix} 0 & 0 & G_{13}^{(a)} & 0 \\ 0 & 0 & G_{23}^{(a)} & 0 \\ 0 & 0 & 0 & 0 \\ G_{41}^{(a)} & G_{42}^{(a)} & 0 & 0 \end{pmatrix},$$

$$\mathbf{G}^{(a^\dagger)} = \begin{pmatrix} 0 & 0 & 0 & G_{14}^{(a^\dagger)} \\ 0 & 0 & 0 & G_{24}^{(a^\dagger)} \\ G_{31}^{(a^\dagger)} & G_{32}^{(a^\dagger)} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.10}$$

with

$$\begin{aligned} G_{13}^{(a)} &= G_{14}^{(a^\dagger)^*} = -ige^{-i\Delta t}e^{(A/2-C)t}, \\ G_{23}^{(a)} &= G_{24}^{(a^\dagger)^*} = -2ige^{-i\Delta t}e^{(A/2+B-C)t}(1-s-\kappa), \\ G_{41}^{(a)} &= G_{31}^{(a^\dagger)^*} = -ige^{-i\Delta t}e^{(A/2+C)t}(1-s-\kappa+2s\kappa), \\ G_{42}^{(a)} &= G_{32}^{(a^\dagger)^*} = \tfrac{i}{2}ge^{-i\Delta t}e^{(A/2-B+C)t}(1-2\kappa), \end{aligned} \tag{3.11}$$

where $\Delta \equiv \Omega - \omega$ denotes the detuning between the atom and the photon, as well as

$$\mathbf{H} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & H_{23} & H_{24} \\ H_{31} & H_{32} & 0 & 0 \\ H_{41} & H_{42} & 0 & 0 \end{pmatrix}, \tag{3.12}$$

with

$$\begin{aligned} H_{24} &= H_{23}^* = 2ig^*e^{i\Omega t}e^{(B-C)t}\alpha, \\ H_{31} &= H_{41}^* = (1-2s)ig^*e^{i\Omega t}e^{Ct}\alpha, \\ H_{32} &= H_{42}^* = -ig^*e^{i\Omega t}e^{-(B-C)t}\alpha. \end{aligned} \tag{3.13}$$

Please note that the matrices $\mathbf{G}^{(a)}$, $\mathbf{G}^{(a^\dagger)}$, and $\mathbf{H}$ have many null entries. In particular, there are no $\partial f_4/\partial a'$ and no $\partial f_3/\partial a^{\dagger'}$ terms in (3.8), and the equation for $\partial f_1/\partial t$ does not contain an inhomogeneous contribution.

For the numerical treatment of (3.1) the polar coordinates of (2.10) are particularly useful. We rewrite the derivatives in (3.1) in accordance with

$$\begin{aligned} \frac{\partial}{\partial a'} &= \tfrac{1}{2}e^{-i\varphi}\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial \varphi}\right), \\ \frac{\partial}{\partial a^{\dagger'}} &= \tfrac{1}{2}e^{i\varphi}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \varphi}\right), \end{aligned} \tag{3.14}$$

and switch from $\mathbf{f}(a^{\dagger'}, a', t)$ to the column $\tilde{\mathbf{f}}(r, \varphi, t)$, defined by

$$\tilde{\mathbf{f}} = \begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \\ \tilde{f}_3 \\ \tilde{f}_4 \end{pmatrix} \equiv \begin{pmatrix} f_1 \\ f_2 \\ e^{-i\varphi}f_3 \\ e^{i\varphi}f_4 \end{pmatrix} = \begin{pmatrix} \tilde{f}_1^* \\ \tilde{f}_2^* \\ \tilde{f}_4^* \\ \tilde{f}_3^* \end{pmatrix}, \tag{3.15}$$

which is more fitting here. This leads us to

$$\frac{\partial}{\partial t}\tilde{\mathbf{f}} = \mathbf{G}^{(r)}\frac{\partial}{\partial r}\tilde{\mathbf{f}} + \mathbf{G}^{(\varphi)}\frac{\partial}{\partial \varphi}\tilde{\mathbf{f}} + \tilde{\mathbf{H}}\tilde{\mathbf{f}}, \tag{3.16}$$

with

$$\begin{aligned} \mathbf{G}^{(r)} &= \frac{1}{2}(\mathbf{G}^{(a^\dagger)} + \mathbf{G}^{(a)}), \\ \mathbf{G}^{(\varphi)} &= \frac{i}{2r}(\mathbf{G}^{(a^\dagger)} - \mathbf{G}^{(a)}), \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} \tilde{\mathbf{H}} = \frac{1}{2r}&\begin{pmatrix} 0 & 0 & G_{13}^{(a)} & G_{14}^{(a^\dagger)} \\ 0 & 0 & G_{23}^{(a)} & G_{24}^{(a^\dagger)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & e^{i\varphi}H_{23} & e^{-i\varphi}H_{24} \\ e^{-i\varphi}H_{31} & e^{-i\varphi}H_{32} & 0 & 0 \\ e^{i\varphi}H_{41} & e^{i\varphi}H_{42} & 0 & 0 \end{pmatrix}. \end{aligned} \tag{3.18}$$

Incidentally, we remark that (3.16) is hyperbolic if $0 < \kappa < 1$, as will become clear in the next section where we find the eigenvalues of $\mathbf{G}^{(r)}$.

Please note that the matrices $\mathbf{G}^{(r)}$ and $r\mathbf{G}^{(\varphi)}$ neither depend on φ nor on r. The φ dependence of $\tilde{\mathbf{H}}$ becomes more transparent as soon as we combine (2.12) with (2.10) to arrive at

$$\alpha(t) = e^{-i\omega t}e^{-At/2}e^{i\varphi}\left[r + e^{-i\varphi}Z(t)\right], \tag{3.19}$$

where

$$Z(t) = \int_0^t dt'\, e^{i\omega t'}e^{At'/2}z(t') \tag{3.20}$$

measures the effect of the injected signal. Then

$$\tilde{H}_{31} = (1-2s)ig^*(t)e^{i\Delta t}e^{-(A/2-C)t}\left[r + e^{-i\varphi}Z(t)\right], \tag{3.21}$$

for example, shows that $\tilde{\mathbf{H}}$ is independent of φ, too, unless $Z(t) \neq 0$. In other words, the injected signal introduces a φ dependence into $\tilde{\mathbf{H}}$; no injected signal — no φ dependence.

We exhibit the φ dependence of $\tilde{\mathbf{H}}$ by decomposing it according to

$$\begin{aligned} \tilde{\mathbf{H}} = \tilde{\mathbf{H}}^{(0)} &+ ig^*(t)Z(t)e^{i\Delta t}e^{-At/2}e^{-i\varphi}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2e^{(B-C)t} \\ (1-2s)e^{Ct} & -e^{-(B-C)t} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &- ig(t)Z^*(t)e^{-i\Delta t}e^{-At/2}e^{i\varphi}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2e^{(B-C)t} & 0 \\ 0 & 0 & 0 & 0 \\ (1-2s)e^{Ct} & -e^{-(B-C)t} & 0 & 0 \end{pmatrix} \\ &\equiv \tilde{\mathbf{H}}^{(0)} + e^{-i\varphi}\mathbf{Z}^{(+)} + e^{i\varphi}\mathbf{Z}^{(-)}, \end{aligned} \tag{3.22}$$

where the nonvanishing matrix elements of $\tilde{\mathbf{H}}^{(0)}$ are

$$\begin{aligned}
\tilde{H}^{(0)}_{13} &= \tilde{H}^{(0)^*}_{14} = \frac{1}{2r}G^{(a)}_{13}\,,\\
\tilde{H}^{(0)}_{23} &= \tilde{H}^{(0)^*}_{24} = \frac{1}{2r}G^{(a)}_{23} - 2ige^{-i\Delta t}e^{-(A/2-B+C)t}r\,,\\
\tilde{H}^{(0)}_{31} &= \tilde{H}^{(0)^*}_{41} = (1-2s)ig^*e^{i\Delta t}e^{-(A/2-C)t}r\,,
\end{aligned} \tag{3.23}$$

and

$$\tilde{H}^{(0)}_{32} = \tilde{H}^{(0)^*}_{42} = -ig^*e^{i\Delta t}e^{-(A/2+B-C)t}r\,. \tag{3.24}$$

In $\mathbf{G}^{(\varphi)}$ and $\tilde{\mathbf{H}}^{(0)}$ we encounter matrix elements that are proportional to $1/r$. At first sight, these appear to cause a problem at $r=0$. A closer look, however, will convince us that these terms are harmless. We proceed from noting that $\tilde{\mathbf{f}}$ is periodic in φ by construction. When inserting its Fourier series,

$$\tilde{\mathbf{f}}(r,\varphi,t) = \sum_{k=-\infty}^{\infty}\tilde{\mathbf{f}}_k(r,t)e^{ik\varphi}\,, \tag{3.25}$$

into (3.16) we find

$$\begin{aligned}
\frac{\partial}{\partial t}\tilde{\mathbf{f}}_k = \mathbf{G}^{(r)}\frac{\partial}{\partial r}\tilde{\mathbf{f}}_k + (ik\mathbf{G}^{(\varphi)} + \tilde{\mathbf{H}}^{(0)})\tilde{\mathbf{f}}_k\\
+\mathbf{Z}^{(+)}\tilde{\mathbf{f}}_{k+1} + \mathbf{Z}^{(-)}\tilde{\mathbf{f}}_{k-1}\,,
\end{aligned} \tag{3.26}$$

where neighboring k values are coupled to each other only when there is an injected signal. The critical terms are contained in the second summand on the right-hand side. Explicitly they are

$$\frac{1}{2r}\begin{pmatrix} 0 & 0 & (1+k)G^{(a)}_{13} & (1-k)G^{(a^\dagger)}_{14}\\ 0 & 0 & (1+k)G^{(a)}_{23} & (1-k)G^{(a^\dagger)}_{24}\\ -kG^{(a)^\dagger}_{31} & -kG^{(a^\dagger)}_{32} & 0 & 0\\ kG^{(a)}_{41} & kG^{(a)}_{42} & 0 & 0 \end{pmatrix}\begin{pmatrix}\tilde{f}_{k1}\\ \tilde{f}_{k2}\\ \tilde{f}_{k3}\\ \tilde{f}_{k4}\end{pmatrix}\,. \tag{3.27}$$

The analyticity of the original $f_j(a^{\dagger'},a',t)$ in both $a^{\dagger'}$ and a' implies that the power series of $\tilde{f}_{kj}(r,t)$ starts with $r^{|k|}$ for $j=1$ and $j=2$, $r^{|k+1|}$ for $j=3$, and $r^{|k-1|}$ for $j=4$ [recall the extra factors of $e^{i\varphi}$ and $e^{-i\varphi}$ in (3.15)]. Consequently, either there is a power of r available in $\tilde{f}_{kj}$, or the factors k, $k+1$,and $k-1$ supply the necessary cancellation. Indeed, the $1/r$ terms are not problematic.

While we are at it, let us mention that all the functions $\tilde{f}_{kj}(r,t)$ are even or odd in r. Specifically,

$$\begin{aligned}
\tilde{f}_{k1}(-r,t) &= (-1)^k\tilde{f}_{k1}(r,t)\,,\\
\tilde{f}_{k2}(-r,t) &= (-1)^k\tilde{f}_{k2}(r,t)\,,\\
\tilde{f}_{k3}(-r,t) &= -(-1)^k\tilde{f}_{k3}(r,t)\,,\\
\tilde{f}_{k4}(-r,t) &= -(-1)^k\tilde{f}_{k4}(r,t)\,,
\end{aligned} \tag{3.28}$$

which can be regarded either as another consequence of the analytic properties of the $f_j(a^{\dagger'},a',t)$ functions, or as following from the invariance of the polar parametrization in (2.10) under the replacements $r\to -r$, $\varphi\to\varphi+\pi$. So we can extend the r range to negative values in a quite natural way. This is of some importance for the numerical treatment.

B. Characteristics

The time-dependent 4×4 matrix $\mathbf{G}^{(r)}$ of (3.17) can be diagonalized, provided $0<\kappa(t)<1$,

$$\mathbf{R}\mathbf{G}^{(r)} = \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&v&0\\ 0&0&0&-v\end{pmatrix}\mathbf{R} \equiv \mathbf{V}\mathbf{R}\,, \tag{3.29}$$

where the nonzero eigenvalues $\pm v(t)$ are given by

$$\begin{aligned}
v(t) &= |g(t)|e^{At/2}\sqrt{\kappa(t)\big(1-\kappa(t)\big)} \qquad (3.30)\\
&= |g(t)|\sqrt{\kappa_\infty-\kappa_0}\sqrt{\frac{\kappa(t)\big(1-\kappa(t)\big)}{\kappa_\infty-\kappa(t)}}
\end{aligned}$$

which — quite remarkably — does not involve any of the atomic parameters Ω, B, C, and s, nor the natural frequency ω of the photon mode. For $0<\kappa(t)<1$, $v(t)$ is positive, and all eigenvalues of $\mathbf{G}^{(r)}$ are real. As stated above, in this regime (3.16) is hyperbolic. The matrix $\mathbf{R}$,

$$\mathbf{R} = \begin{pmatrix} (1-s-\kappa) & -\frac{1}{2}e^{-Bt} & 0 & 0\\ 0 & 0 & e^{-i\Delta t}g/|g| & e^{i\Delta t}g^*/|g|\\ (1-s-\kappa+2s\kappa)e^{Ct} & (\kappa-\frac{1}{2})e^{(C-B)t} & -ie^{-i\Delta t}e^{-At/2}v/g^* & ie^{i\Delta t}e^{-At/2}v/g\\ (1-s-\kappa+2s\kappa)e^{Ct} & (\kappa-\frac{1}{2})e^{(C-B)t} & ie^{-i\Delta t}e^{-At/2}v/g^* & -ie^{i\Delta t}e^{-At/2}v/g \end{pmatrix}\,, \tag{3.31}$$

consists of the eigenrows of $\mathbf{G}^{(r)}$. Up to elementary transformations, $\mathbf{R}$ is unique. We note that the expressions $g/|g|$ and v/g remain meaningful for $g \to 0$, provided the complex phase of g behaves reasonably in this limit.

The diagonalization (3.29) enables us to decouple the derivatives in (3.26). This produces

$$\left(\frac{\partial}{\partial t} - \mathbf{V}\frac{\partial}{\partial r}\right)\mathbf{x}_k = (ik\mathbf{\Phi} + \mathbf{M})\mathbf{x}_k + \mathbf{Y}^{(+)}\mathbf{x}_{k+1} + \mathbf{Y}^{(-)}\mathbf{x}_{k-1}\,, \tag{3.32}$$

where the transformed quantities are

$$\begin{aligned} \mathbf{x}_k &= \mathbf{R}\tilde{\mathbf{f}}_k\,, \\ \mathbf{\Phi} &= \mathbf{R}\mathbf{G}^{(\varphi)}\mathbf{R}^{-1}\,, \\ \mathbf{M} &= \mathbf{R}\tilde{\mathbf{H}}^{(0)}\mathbf{R}^{-1} + \frac{\partial \mathbf{R}}{\partial t}\mathbf{R}^{-1}\,, \end{aligned} \tag{3.33}$$

and

$$\mathbf{Y}^{(\pm)} = \mathbf{R}\mathbf{Z}^{(\pm)}\mathbf{R}^{-1}\,. \tag{3.34}$$

Again, different k values are not coupled, unless there is an injected signal.

Rather than (3.26) we can transform (3.16) to find the corresponding equation of motion for

$$\mathbf{x}(r,\varphi,t) = \sum_{k=-\infty}^{\infty} \mathbf{x}_k(r,t)e^{ik\varphi}\,. \tag{3.35}$$

It reads

$$\left(\frac{\partial}{\partial t} - \mathbf{V}\frac{\partial}{\partial r}\right)\mathbf{x} = \mathbf{\Phi}\frac{\partial}{\partial \varphi}\mathbf{x} + \mathbf{M}\mathbf{x} + e^{-i\varphi}\mathbf{Y}^{(+)}\mathbf{x} + e^{i\varphi}\mathbf{Y}^{(-)}\mathbf{x}\,. \tag{3.36}$$

Since the matrices $\mathbf{G}^{(r)}$ and $\mathbf{G}^{(\varphi)}$ cannot be diagonalized simultaneously without destroying the unit matrix multiplying $\partial/\partial t$, we have to be content with the partial diagonalization achieved in (3.36).

The curves

$$\tau \mapsto \begin{cases} (\tau, r_0, \varphi_0) \qquad \text{(two-fold)}\,, \\ \left(\tau, r_0 - \int_0^\tau d\tau'\, v(\tau'), \varphi_0\right)\,, \\ \left(\tau, r_0 + \int_0^\tau d\tau'\, v(\tau'), \varphi_0\right)\,, \end{cases} \tag{3.37}$$

in the three-dimensional τ, r, φ space are called (bi)characteristics. Please note that the shape of these characteristics does not depend on the starting point $(0, r_0, \varphi_0)$. For illustration, typical characteristics are drawn in Fig. 1.

The differential operators on the left-hand sides of (3.32) and (3.36) can be regarded as differentiations with respect to τ along the characteristics. Consequently, the partial differential equation can be integrated along these curves similar to an ordinary differential equation. This is the main advantage of our approach over the common one that leads to coupled second-order equations of the Fokker-Planck type.

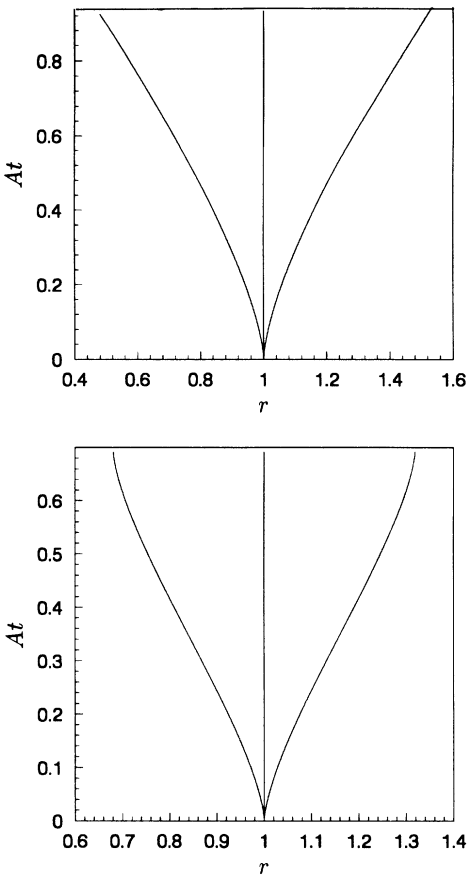

FIG. 1. The characteristics in a plane with fixed φ_0, going through the point $(t_0 = 0, r_0 = 1, \varphi_0)$, are plotted. They start with $\kappa_0 = 0$ and end with $\kappa_\infty = 1$ (left-hand side) or $\kappa_\infty = 2$ (right-hand side), respectively. The top graph could be continued to $t \to \infty$ in contrast to the bottom graph, where the characteristics end at $At = -\ln(1/2) = 0.69\ldots$.

In the theory of partial differential equations [5] the term "characteristic" originally denotes rather an entire hypersurface in the definition space than a single curve. The specific meaning of these surfaces is that a general initial value problem (Cauchy problem) is improperly posed if the initial values are given on a characteristic, because the underlying partial differential equation relates the values on a characteristic surface to each other; it does not determine neighboring values outside the surface. This can be easily seen for Eqs. (3.32) and (3.36).

IV. NUMERICAL METHODS

Explicit finite-difference methods for a direct numerical integration of (3.32) or (3.36) are not satisfactory with respect to their stability. Integration along the characteristics (3.37), as anticipated above, is to be preferred. In contrast to a true ordinary differential equation, solutions with different initial conditions are coupled because several characteristics intersect. If the values of

$\mathbf{x}_k$ are given in r direction on a grid with fixed spacing, the width of the time steps has to be adjusted, so that the characteristics connect grid points at different times. For the integration we use either a two-step predictor-corrector scheme, see Fig. 2, or a generalized four-step Runge-Kutta scheme with intermediate integration points lying on intersections of characteristics through other grid points, see Fig. 3. The two-step predictor-corrector scheme with j corrector steps is given by

$$\mathbf{x}_k^{(1)}(t+h) = \mathbf{x}_k(t) + hy\big(\mathbf{x}_k(t)\big), \tag{4.1}$$

$$\mathbf{x}_k^{(j+1)}(t+h) = \mathbf{x}_k(t) + h\frac{y\big(\mathbf{x}_k(t)\big) + y\big(\mathbf{x}_k^{(j)}(t+h)\big)}{2},$$

with y symbolizing the right-hand side of (3.32). The integration along the characteristics is not indicated explicitly. The size h of the time step is determined by the course of the characteristics (cf. Fig. 2). Two or three corrector steps are most favorable.

A generalized four-step Runge-Kutta scheme can be described by the table

$$\begin{array}{c|cccc} 0 & & & & \\ \alpha_2 & \beta_{21} & & & \\ \alpha_3 & \beta_{31} & \beta_{32} & & \\ \alpha_4 & \beta_{41} & \beta_{42} & \beta_{43} & \\ \hline & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \end{array} \tag{4.2}$$

with

$$\mathbf{x}_k(t+h) = \mathbf{x}_k(t) + h(\gamma_1 y_1 + \gamma_2 y_2 + \gamma_3 y_3 + \gamma_4 y_4) \tag{4.3}$$

and

$$\begin{aligned} y_1 &= y(t, \mathbf{x}_k), \\ y_2 &= y(t+\alpha_2 h, \mathbf{x}_k + h\beta_{21} y_1), \\ y_3 &= y(t+\alpha_3 h, \mathbf{x}_k + h[\beta_{31} y_1 + \beta_{32} y_2]), \\ y_4 &= y(t+\alpha_4 h, \mathbf{x}_k + h[\beta_{41} y_1 + \beta_{42} y_2 + \beta_{43} y_3]). \end{aligned} \tag{4.4}$$

We use the unique table

$$\begin{array}{c|cccc} 0 & & & & \\ \alpha & \alpha & & & \\ \beta & -\frac{\beta(4\alpha^2-3\alpha+\beta)}{2\alpha(1-2\alpha)} & \frac{\beta(\beta-\alpha)}{2\alpha(1-2\alpha)} & & \\ 1 & 1-\beta_{42}-\beta_{43} & \frac{(1-\alpha)(\alpha-4\beta^2+5\beta-2)}{2\alpha(\beta-\alpha)(6\alpha\beta-4\alpha-4\beta+3)} & \frac{(1-\beta)(1-\alpha)(1-2\alpha)}{\beta(\beta-\alpha)(6\alpha\beta-4\alpha-4\beta+3)} & \\ \hline & \frac{6\alpha\beta-2\alpha-2\beta+1}{12\alpha\beta} & \frac{2\beta-1}{12\alpha(\beta-\alpha)(1-\alpha)} & \frac{1-2\alpha}{12\beta(\beta-\alpha)(1-\beta)} & \frac{6\alpha\beta-4\alpha-4\beta+3}{12(1-\alpha)(1-\beta)} \end{array} \tag{4.5}$$

with

$$\begin{gathered} \alpha \neq \beta, \\ \alpha \neq 1/2, \\ 6\alpha\beta - 4\alpha - 4\beta + 3 \neq 0, \end{gathered} \tag{4.6}$$

that approximates the solution of (3.32) up to fourth order with arbitrary α and β. The coefficients α and β as well as h are determined by the course of the characteristics (cf. Fig. 3). These are calculated by numerical integration of the ordinary differential equations

$$\frac{dt}{dr} = \pm\frac{1}{v(t)} = \pm\frac{1}{\sqrt{|g(t)|^2 e^{At}\kappa(t)\,[1-\kappa(t)]}}, \tag{4.7}$$

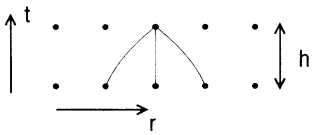

FIG. 2. Integration along characteristics with constant φ_0 using a two-step scheme is indicated. The step size in t direction is adjusted with respect to the course of the characteristics.

with the aid of the classical four-step Runge-Kutta scheme. The restrictions for α and β in (4.6) have no practical importance, because slightly curved characteristics, compared to the size of h, lead to values of $\alpha \approx \frac{1}{3}$ and $\beta \approx \frac{2}{3}$.

If $\nu > 0$, the parameter $\kappa(t)$ will reach $\kappa = 1$ at a certain time and then leave the region of hyperbolicity. Furthermore, the numerical properties of the schemes used are most favorable for $\kappa(t)$ around $\kappa = \frac{1}{2}$. For these reasons it could become necessary to restart the dynamics

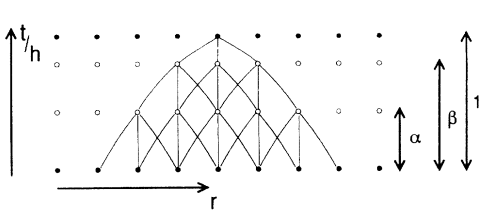

FIG. 3. Several grid points are involved in one integration step along the characteristics using a four-step Runge-Kutta scheme. In t direction two intermediate grids are required. The corresponding time steps are given in units of the total step size h.

at some time t_R with a new initial value for κ. Then the phase-space function $\mathbf{x}(a', a^{\dagger'}, t_R)$ with $\kappa_1 = \kappa(t_R)$ has to be transformed into a new initial function $\mathbf{x}_2(a', a^{\dagger'}, t_R)$ with $\kappa_2 < \kappa_1$. In the standard parametrization (2.10) this is achieved by

$$\mathbf{x}_2(a'', a''^*, t_R)$$

$$= \int \frac{\mathcal{D}(a'^*, a')}{2\pi} \mathbf{x}(a', a'^*, t_R) \times \frac{1}{\kappa_1 - \kappa_2} \exp\left(-\frac{1}{\kappa_1 - \kappa_2}|\alpha(t_R) - a''|^2\right), \tag{4.8}$$

a simple Gaussian integral transform.

V. JAYNES-CUMMINGS REVIVALS

In this section, we study the influence of damping on the so called Jaynes-Cummings revivals using the method developed above. If there is no injected signal $z(t)$, it is sufficient to assume the superposition

$$\rho_F = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\varphi_0 \, |\alpha_0\rangle\langle\alpha_0| \tag{5.1}$$

of coherent states $|\alpha_0\rangle$ with $\alpha_0 = r_0 e^{i\varphi_0}$ for the initial state of the field. This is equivalent to disregarding the off-diagonal elements of some $|\alpha_0\rangle\langle\alpha_0|$ expanded in a Fock basis. This simplification is possible because of the decoupling in (3.26) for $\mathbf{Z}^{(\pm)} = \mathbf{0}$. The full information about the atomic inversion and the photon number is contained in $\tilde{\mathbf{f}}_0$; the $k \neq 0$ terms are not needed.

All results shown in Figs. 4–8 refer to the case of zero temperature ($\nu = 0$) and no explicit atomic damping

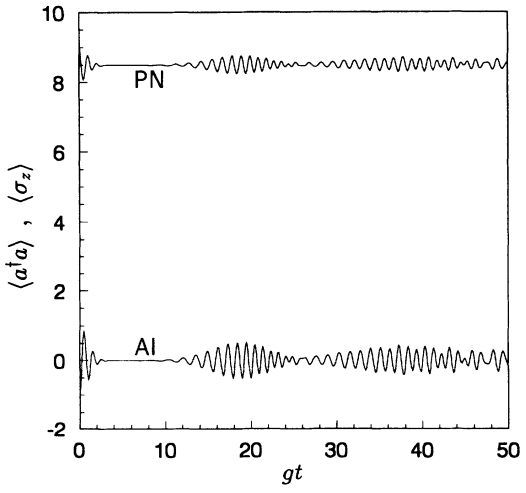

FIG. 4. As a reference, the pure Jaynes-Cummings dynamics without damping ($A = 0$) is plotted. The calculation was performed using the Wigner function ($\kappa = \frac{1}{2}$). Here and in Figs. 5–8, curve PN shows the photon number $\langle a^\dagger a\rangle$, and curve AI the atomic inversion $\langle \sigma_z\rangle$.

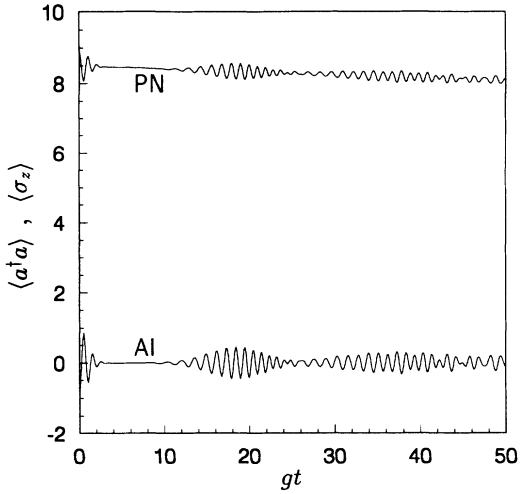

FIG. 5. On the time scale of the revivals a weak damping of $A/g = 0.001$ mainly leads to a slight decay of the mean photon number. The initial κ parameter of the phase-space functions is $\kappa_0 = 0.4$, a value only a little less than the one corresponding to the Wigner function.

($B = 0, C = 0$). The amplitude of the coherent initial state of the field is $r_0 = 3$, while the atom is in its ground-state initially. The coupling constant g is taken to be constant. The phase-space functions are discretized in r direction on the interval $[0; 5]$ with a step size of $\Delta r = 0.0025$.

Figure 4 shows the undamped revivals for comparison. In Figs. 5–8 the ratio A/g is increased by factors of 10 in three steps from $A/g = 10^{-3}$ to $A/g = 1$. For the very weak damping of $A/g = 10^{-3}$ the revivals are essentially unaffected, the main effect being a slow decay of the mean photon number. For $A/g = 10^{-2}$, we observe a much faster decay of the photon number; the revivals are still there but with a markedly reduced amplitude. A

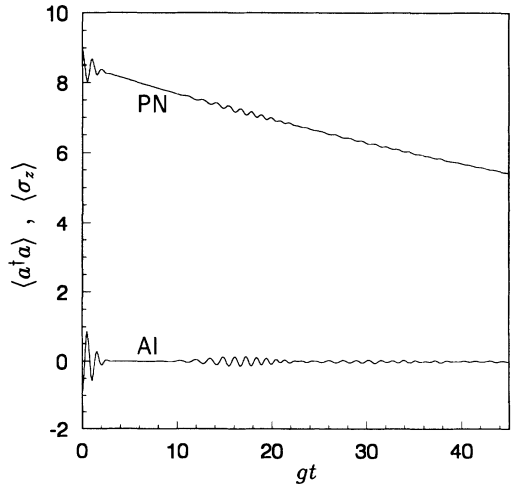

FIG. 6. A damping of $A/g = 0.01$ causes a strong reduction of the amplitude of the revivals in addition to the decay of the photon number. The atomic inversion is still oscillating around values near zero.

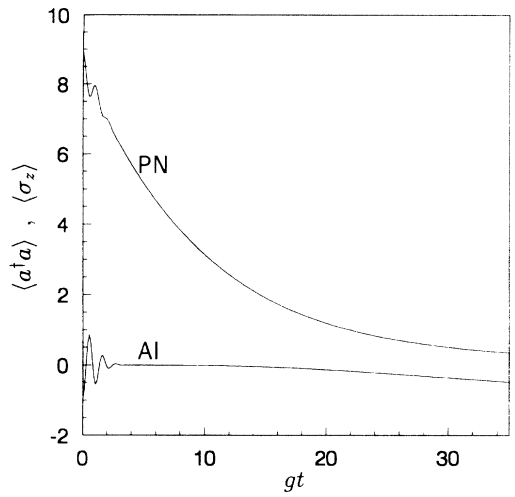

FIG. 7. With a damping of $A/g = 0.1$ revivals are no longer visible. In addition to the decay of the photon number, a decay of the atomic inversion can be seen.

damping of $A/g = 10^{-1}$ destroys the revivals completely; only the few initial Rabi oscillations survive. Finally, for the strong damping of $A/g = 1$, the mean photon number decreases monotonically and no longer exhibits Rabi oscillations at all; these are still present in the atomic inversion though. Please note the different time ranges in Figs. 5–8.

In addition to the numerical results reported in Figs. 5–8, we have calculated some of the long-time plots given by Eiselt and Risken [6] and found no disagreement [7].

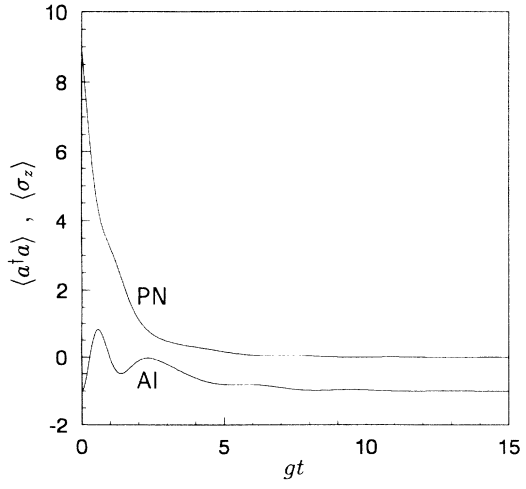

FIG. 8. The quite strong damping of $A/g = 1$ makes the initial Rabi oscillations of the photon number disappear, too. Both the photon number and the atomic inversion reach their stationary values quickly.

VI. SUMMARY

We have used the explicit dynamical operator solutions for a damped photon mode and a damped (or pumped) two-level atom to introduce a genuine interaction picture for the Jaynes-Cummings model with dissipation. The numerical phase-space functions that parametrize the time-dependent density operator are appropriate generalizations of the familiar Wigner function and its relatives. These phase-space functions obey a hyperbolic first-order partial differential equation; their time-dependence originates solely in the atom-photon coupling whose strength is measured by the Rabi frequency. The numerical treatment is based upon integrations along the characteristics of the hyperbolic equation of motion. The generalizations to more than one photon mode or atoms with more than two levels are immediate.

As an application we have looked at Jaynes-Cummings revivals in the presence of photon damping. We find that even for a relatively short photon lifetime (about 30 periods of the vacuum Rabi oscillation in Fig. 6) the first revival is still noticeable. Further applications of the interaction-picture formalism to laser cooling and atom diffraction are hinted at in Appendix B. Results will be reported in due course.

ACKNOWLEDGMENTS

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APPENDIX A: SQUEEZED RESERVOIRS

Photon master equations that are generalizations of (2.1) appear in some problems. An example is given by Eq. (1) of Ref. [8] which has the form

$$\frac{\partial}{\partial t} = \mathcal{L}_{\rm ph}\rho + \frac{d}{2}[\rho, a^2] + \frac{d^*}{2}[a^{\dagger 2}, \rho] - \frac{D}{2}(a^2\rho - 2a\rho a + \rho a^2) - \frac{D^*}{2}(a^{\dagger 2}\rho - 2a^\dagger \rho a^\dagger + \rho a^{\dagger 2}) \tag{A1}$$

with $\mathcal{L}_{\rm ph}$ as in (2.1). In the jargon of quantum optics,

TABLE I. Identification of the parameters used in this paper with those of Ref. [8].

This paper	Ref. [8]
$A\nu$	$\alpha\,\mathrm{Re}\,[\rho_{aa}\mathcal{L}_1 + \rho_{bb}\mathcal{L}_2]$
$A(\nu+1)$	$\gamma + \alpha\,\mathrm{Re}\,[\rho_{bb}\mathcal{L}_1 + \rho_{cc}\mathcal{L}_2]$
D^*	$\frac{1}{2}\alpha\rho_{ca}\mathcal{L}_0\,(\mathcal{L}_1 + \mathcal{L}_2)$
ω	$\Omega - \nu + \frac{1}{2}\alpha\,\mathrm{Im}\,[(\rho_{aa} - \rho_{bb})\mathcal{L}_1 + (\rho_{bb} - \rho_{cc})\mathcal{L}_2]$
d^*	$\frac{1}{2}\alpha\rho_{ca}\mathcal{L}_0\,(\mathcal{L}_1 - \mathcal{L}_2)$
z^*	$is\,(\rho_{ba}\mathcal{L}_1 + \rho_{cb}\mathcal{L}_2)$

the terms proportional to D and D^* describe a "squeezed reservoir," those proportional to d and d^* are colloquially associated with "injected coherence." The restriction $|D| \leq A\sqrt{\nu(\nu+1)}$ applies; otherwise, (A1) would not conserve the positivity of ρ. For the record, Table I reports the relation between the parameters in Ref. [8] and the ones used here.

The formalism presented in the present paper can be extended and then used to handle (A1) as well. The extension consists of introducing another parameter, $\lambda(t)$, into the fundamental solution (2.2). This generalization reads

$$\mathsf{B}\big(\kappa(t),\lambda(t),\alpha^*(t),\alpha'(t)\big) = \frac{1}{\sqrt{\kappa^2-\lambda^*\lambda}} : \exp\left(\frac{1}{\kappa^2-\lambda^*\lambda}\left[\frac{\lambda}{2}(a^\dagger-\alpha^*)^2 - \kappa(a^\dagger-\alpha^*)(a-\alpha') + \frac{\lambda^*}{2}(a-\alpha')^2\right]\right) : , \tag{A2}$$

which equals (2.2) for $\lambda \equiv 0$. The time dependences of κ, λ, λ^*, α, and α^* are here determined by

$$\frac{d}{dt}\begin{pmatrix}\lambda\\ \kappa\\ \lambda^*\end{pmatrix} = \begin{pmatrix}-(A+2i\omega) & 2d & 0\\ d^* & -A & d\\ 0 & 2d^* & -(A-2i\omega)\end{pmatrix}\begin{pmatrix}\lambda\\ \kappa\\ \lambda^*\end{pmatrix} + \begin{pmatrix}-D-d\\ A(\nu+1)\\ -D^*-d^*\end{pmatrix} \tag{A3}$$

and

$$\frac{d}{dt}\begin{pmatrix}\alpha^*\\ \alpha'\end{pmatrix} = \begin{pmatrix}-(A/2-i\omega) & d^*\\ d & -(A/2+i\omega)\end{pmatrix}\begin{pmatrix}\alpha^*\\ \alpha'\end{pmatrix} + \begin{pmatrix}z^*\\ z\end{pmatrix}, \tag{A4}$$

which replace Eqs. (2.3). Rather than Eqs. (2.11) and (2.9) we now have more generally

$$\rho(t) = \int \frac{\mathcal{D}(a^{\dagger\prime},a')}{2\pi} f_0(a^{\dagger\prime},a')\mathsf{B}\big(\kappa(t),\lambda(t),\alpha^*(t),\alpha'(t)\big) \tag{A5}$$

and

$$f_0(a^{\dagger\prime},a'') = \mathrm{tr}\{\rho(0)\mathsf{B}(1-\kappa_0,-\lambda_0,a^{\dagger\prime},a'')\}, \tag{A6}$$

with $\kappa_0 = \kappa(0)$, $\lambda_0 = \lambda(0)$, $a^{\dagger\prime} = \alpha^*(0)$, and $a'' = \alpha(0)$. All other equations have to be modified accordingly. We note that the trace in (A6) is certainly well defined if the inequalities $0 \leq \kappa_0 \leq \frac{1}{2} - |\lambda_0|$ are obeyed.

APPENDIX B: LASER COOLING

The interaction-picture formalism can also be applied to quantum master equations of quite a different structure. Consider, for instance, the one-dimensional laser-cooling model recently used by Mølmer, Castin, and Dalibard [9] for a comparison of the "Monte Carlo wave-function method" with a direct numerical integration. Here, the state $\rho(p,z,\sigma_+,\sigma_-,t)$ is a function both of the dynamical variables σ_+, σ_- for the internal two-level degree of freedom and of the momentum p and position z, which are the dynamical variables for the center-of-mass motion along the z axis.

The master equation is

$$\frac{\partial}{\partial t}\rho = \mathcal{L}_0\rho + \mathcal{L}_1\rho, \tag{B1}$$

where

$$\begin{aligned}\mathcal{L}_0\rho = {}& \frac{i}{\hbar}\frac{1}{2m}[\rho,p^2] + \frac{i}{4}\Omega[\rho,\sigma_+\sigma_-]\\ & -\frac{\Gamma}{8}(\sigma_+\sigma_-\rho + \rho\sigma_+\sigma_-)\\ & +\frac{\Gamma}{4}\int dk'\,N(k')e^{-ik'z}\sigma_-\rho\sigma_+e^{ik'z}\end{aligned} \tag{B2}$$

accounts for the free unitary evolution and the nonunitary spontaneous transition to the ground state, and

$$\mathcal{L}_1\rho = -\frac{i}{2}ge^{i\omega t}[\rho,\sigma_-\cos(kz)] - \frac{i}{2}g^*e^{-i\omega t}[\rho,\sigma_+\cos(kz)] \tag{B3}$$

represents the unitary coupling to the standing laser wave. Except for notational changes, this is the master equation of Ref. [9]. In (B2) and (B3), m is the mass of the atom; $\hbar\Omega$ is the energetic separation of the atomic levels; Γ is the rate at which spontaneous transitions occur; k and ω specify the spatial and temporal periodicity of the laser wave; g is the (complex) coupling strength; and the numerical function

$$dk'\,N(k') = \begin{cases} dk'\,\frac{3}{8k}[1+(k'/k)^2] & \text{for } k'^2 < k^2,\\ 0 & \text{for } k'^2 > k^2,\end{cases} \tag{B4}$$

is the probability to find the z component of the momentum of the spontaneously emitted photon in the range $\hbar k' \cdots \hbar(k'+dk')$. Incidentally, we note that the master equation (B1) with (B2) and (B3) appears also in the context of atom diffraction by a standing light wave [10]; the major difference is an essential time dependence of g, whereas g is constant in the context of laser cooling.

For $g \equiv 0$, the fundamental solutions of (B1) are

$$\rho_1(s,\kappa,t) = \frac{1}{4}\exp\left(\frac{i}{\hbar}(s+\hbar\kappa t/m)\,p - i\kappa z\right) \times \left[e^{-\Gamma t}\sigma_+\sigma_- + u(s,\kappa,t)\sigma_-\sigma_+\right],$$

$$\rho_2(s,\kappa,t) = \frac{1}{4}\exp\left(\frac{i}{\hbar}(s+\hbar\kappa t/m)\,p - i\kappa z\right)\sigma_-\sigma_+,$$

$$\rho_3(s,\kappa,t) = \frac{1}{2}\exp\left(\frac{i}{\hbar}(s+\hbar\kappa t/m)\,p - i\kappa z\right) \times e^{-\Gamma t/2}e^{-i\Omega t}\sigma_+,$$

$$\rho_4(s,\kappa,t) = \frac{1}{2}\exp\left(\frac{i}{\hbar}(s+\hbar\kappa t/m)\,p - i\kappa z\right) \times e^{-\Gamma t/2}e^{i\Omega t}\sigma_-, \tag{B5}$$

where the distance s and the wave number κ are numerical parameters that are real but otherwise unrestricted, and the real function

$$u(s,\kappa,t) = \int_0^t dt'\,\Gamma e^{-\Gamma t'} \times \int dk'\,N(k')\exp\left(ik'\,(s+\hbar\kappa t'/m)\right) \tag{B6}$$

is an integral transform of $N(k')$.

For the general solution of (B1) we thus make the ansatz

$$\rho(t) = \int \frac{ds\,d\kappa}{2\pi}\sum_{j=1}^{4} f_j(s,\kappa,t)\rho_j(s,\kappa,t), \tag{B7}$$

wherein the initial phase-space functions $f_j(s,\kappa,t=0)$ are given by expectation values at $t=0$,

$$f_j(s,\kappa,0) = \mathrm{tr}\{[\rho_j(s,\kappa,0)]^\dagger\rho(0)\}. \tag{B8}$$

With (B7) the master equation (B1) appears as

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_0\right)\rho(t) = \int \frac{ds\,d\kappa}{2\pi}\sum_j \frac{\partial f_j}{\partial t}\rho_j = \int \frac{ds\,d\kappa}{2\pi} f_j\mathcal{L}_1\rho_j. \tag{B9}$$

Upon expressing $\mathcal{L}_1\rho_j$ in terms of the ρ_j's themselves, one finds the differential equations obeyed by the f_j's. When we combine the f_j's into a four-entry column $\mathbf{f}$, as in (3.9), this equation of motion reads

$$\frac{\partial}{\partial t}\mathbf{f}(s,\kappa,t) = \mathbf{E}^{(+)}(s,\kappa,t)\mathbf{f}\,(s-\hbar kt/m,\kappa+k,t) + \mathbf{E}^{(-)}(s,\kappa,t)\mathbf{f}\,(s+\hbar kt/m,\kappa-k,t). \tag{B10}$$

The nonzero entries of the 4×4 matrices

$$\mathbf{E}^{(\pm)} = \begin{pmatrix} 0 & 0 & E_{13}^{(\pm)} & E_{14}^{(\pm)} \\ 0 & 0 & E_{23}^{(\pm)} & E_{24}^{(\pm)} \\ E_{31}^{(\pm)} & E_{32}^{(\pm)} & 0 & 0 \\ E_{41}^{(\pm)} & E_{42}^{(\pm)} & 0 & 0 \end{pmatrix} \tag{B11}$$

are

$$E_{13}^{(\pm)} = (E_{14}^{(\pm)})^* = E_{42}^{(\pm)} = (E_{32}^{(\pm)})^* = -\frac{i}{2}ge^{-i\Delta t}e^{\Gamma t/2}e^{\pm i\theta},$$

$$E_{23}^{(\pm)} = (E_{24}^{(\pm)})^* = \frac{i}{2}ge^{-i\Delta t}[e^{-\Gamma t/2}e^{\mp i\theta} + e^{\Gamma t/2}e^{\pm i\theta}u\,(s,\kappa,t)],$$

$$E_{41}^{(\pm)} = (E_{31}^{(\pm)})^* = \frac{i}{2}ge^{-i\Delta t}[e^{-\Gamma t/2}e^{\mp i\theta} - e^{\Gamma t/2}e^{\pm i\theta}u\,(s\mp\hbar kt/m,\kappa\pm k,t)], \tag{B12}$$

where $\Delta = \Omega - \omega$ is the detuning between the laser and the atomic transition, and

$$\theta(s,\kappa,t) \equiv \frac{k}{2}(s+\hbar\kappa t/m) \tag{B13}$$

is a s, κ, and t dependent phase.

As soon as the f_j's have been computed in accordance with (B10) and (B8), time-dependent expectation values can be evaluated. With respect to laser cooling, the most interesting ones are $\langle p\rangle(t)$ and $\langle p^2\rangle(t)$. They are available as the $n=1,2$ cases of

$$\langle p^n\rangle(t) = \left(i\hbar\frac{\partial}{\partial s}\right)^n \times\{[e^{-\Gamma t} + u(s,\kappa,t)]f_1(s,\kappa,t) + f_2(s,\kappa,t)\}\big|_{\substack{s=0\\ \kappa=0}}, \tag{B14}$$

valid for $n=0,1,2,\ldots$; this involves

$$u(s,\kappa=0,t) = \frac{3}{2}\left[\frac{\sin(ks)}{ks}+\frac{\cos(ks)}{(ks)^2}-\frac{\sin(ks)}{(ks)^3}\right]\left(1-e^{-\Gamma t}\right). \tag{B15}$$

The sum $e^{-\Gamma t}+u(s=0,\kappa=0,t)=1$ is time independent; in conjunction with

$$(E_{13}^{(\pm)}+E_{23}^{(\pm)})|_{\substack{s=0\\ \kappa=0}} = (E_{14}^{(\pm)}+E_{24}^{(\pm)})^*|_{\substack{s=0\\ \kappa=0}} = 0 \tag{B16}$$

this property ensures that the trace of $\rho(t)$ is conserved, as it must be.

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