Ramsey fringes in atomic interferometry: Measurability of the influence of space-time curvature

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The influence of space-time curvature on quantum matter which can be theoretically described by covariant wave equations has not been experimentally established yet. In this paper we analyze in detail the suitability of the Ramsey atom beam interferometer for the measurement of the phase shift caused by the Riemannian curvature of the Earth or alternatively of two parallel oriented lead blocks. It appears that for the lead blocks the detection should be possible with realistic modifications of existing devices within the near future. For the Earth's gravitational field the experimental difficulties are too big. The paper is divided into two parts. The first one is concerned with the derivation of general relativistic correction terms to the Pauli equation starting from the fully covariant Dirac equation and their physical interpretation. The inertial effects of acceleration and rotation are included. The calculation makes use of Fermi coordinates. In the second part we calculate the shift of the Ramsey fringes for the two different sources of curvature and examine various possibilities to enlarge the sensitivity of the apparatus to space-time curvature. Since the two parts may be more or less interesting for physicists with different research fields they are written in such a way that each one may be read without much reference to the other one.

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I. INTRODUCTION

The connection between general relativity and quantum theory has been the subject of intense theoretical investigations for decades. Nevertheless, due to the smallness of the influence of gravity on quantum systems in the laboratory, there is presently still a huge gap between the top theoretical level of quantized gravity on one hand and the level of empirical verification on the other. The latter is the subject of this paper. If we neglect any quantum effects from gravity itself as they are predicted by, e.g., quantum gravity or superstring theory, we have to formulate quantum theory and the laws of physics in general for arbitrarily moving observers in a given curved space-time and to look for empirical implications. Based on the results of special relativity this formulation is done by means of several theoretical principles such as Einstein's equivalence principle, for example. The results are empirically well confirmed for classical matter such as test particles and light rays. For quantum systems, on the other hand, special relativistic effects of course are demonstrated up to extremely high energies. But up to recent years there were practically no experiments establishing how quantum systems react on gravity and inertia in genuine quantum effects.

This situation changed when matter wave interferometry with electrons, neutrons, and especially atoms took advantage of technological progress in the development of, e.g., single crystals or lasers. By the study of the induced phase shift of the fringe pattern it was possible to establish the influence of noninertial motion and of the homogeneous Earth gravitational field on quantum systems: Colella *et al.* [1] measured the effect of the Earth's acceleration on neutrons. In contrast, Bonse and Wroblewski [2] have demonstrated the influence of the constant acceleration of the reference frame, thus showing the equality of inertial and gravitational mass for neutrons (restricted version of the equivalence principle). For atomic beam interference the influence of the homogeneous gravitational acceleration has been shown by Kasevich and Chu [3] and Shimizu *et al.* [4]. The Sagnac effect for matter waves representing the influence of the rotation of the reference frame has been measured for neutrons and the rotating Earth by Werner *et al.* [5], for atomic beams on a turntable by Riehle *et al.* [6], and for electrons by Hasselbach and Nicklaus [7].

Turning to the theoretical discussion we mention that the measurability of the corresponding effects for atomic beam interferometers has been discussed by Clauser [8] and for the Ramsey interferometer by Bordé [9]. Mashhoon [10], Silverman [11], and Audretsch and Lämmerzahl [12] have pointed out that in addition to the Sagnac effect a spin-rotation effect may be measurable for neutrons and atoms, respectively. Audretsch *et al.* [13] have shown that Lorentz invariance may be tested with atomic beam interferometry.

In this paper we discuss the next step in this context and turn to the deepest feature of gravity, to space-time curvature. Our main aim is to demonstrate the measurability of the influence of curvature on the dynamics of a quantum system under laboratory conditions, a question also addressed by Anandan [14] in the context of neutron interferometry. Typical results could be a shift of energy levels or a phase shift in an interference experiment. In the laboratory the first effect is hopelessly small whereas the curvature caused shift of the optically

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induced Ramsey fringes in atomic interferometry seems to be in the range of today's equipment. This will be shown below. To prevent misunderstandings we point out that presently quantum systems are not able to serve as probes which are capable of discriminating better between alternative space-time theories of gravity [such as parametrized post-Newtonian (PPN) parameters] than astrophysical "test particle" systems such as the binary pulsar do. The conceptual importance of the interference experiment must rather be seen in the fact that it will be possible to demonstrate that and how Einstein's curvature tensor becomes effective within a quantum system under laboratory conditions. This would be a very important experimental investigation of the connection between quantum theory, which is typically valid on microscopic scales, and general relativity, which describes well our world on very large scales. Seen from another point of view, this would be an experiment which enlarges the range of validity of general relativity to much smaller scales and especially to scales where the classical description of matter breaks down.

Experimental quantum optics is a rapidly evolving field with quick progress in the development of new methods of measurement and the improvement of sensitivities. It therefore seems to be most promising to examine also future possibilities offered by this field of physics. Accordingly, it is the second aim of this paper to give a short and transparent derivation of the respective influences of the curvature on different levels of approximation starting from the Hamiltonian operator of the complete theory of quantum mechanics in curved space-time based on the general relativistic Dirac equation. The underlying physical principles, the different approximation steps, and the theoretical status of the experiment will thereby become clear. This theoretical part is also intended to be a contribution to *quantum optics in curved space-time*, which may be regarded as a special domain of quantum optics under non-Minkowskian conditions.

In addition to curvature we include below inertial influences from the beginning as in Audretsch *et al.* [15]. The corresponding single effects obtained on the final level of approximation are not new. This unifying approach seems nevertheless to be justified because other authors specialize on acceleration or rotation or curvature only. An alternative approach based on the quantization of the Hamiltonian of a point particle in a weak gravitational field [16,17] is less general because it cannot handle spin effects.

The paper is organized as follows. In Sec. II we derive the Dirac Hamiltonian with an electromagnetic potential for stationary space-times and rewrite it with reference to the Fermi coordinates attributed to an accelerated rotating observer. The influence of inertia and curvature on the energy levels of a hydrogen atom is worked out in Sec. III. In Sec. IV we go to the nonrelativistic limit and discuss the resulting correction terms in the Pauli equation. For weak gravitational fields the curvature terms are related to the Newtonian potential in Sec. V. This serves in Sec. VI as a foundation for the discussion of the theoretical relevance of the proposed experiments. In Sec. VII we turn to the experiments and work out the influence of acceleration (as deviation from free fall), rotation, and space-time curvature on the fringes of a Ramsey interferometer. The resulting phase shifts are given for different orientations of the interferometer. Finally, in Sec. VIII the orders of magnitude of the phase shifts are discussed for existing experimental setups and modifications of the Ramsey interferometers which should be technically possible in the near future. For the gravitational field we thereby refer to the field of the Earth and alternatively to the field of laboratory sized lead blocks. Details of the calculations are presented in the four Appendixes.

Due to the twofold aim of the paper its sections may be more or less interesting for physicists with different research fields. We have taken this into account. Those who are interested in the derivation of the correction terms to the Pauli equation and not in interferometry should focus on Secs. II–V and the Appendixes. Those who would like to see the experimental implications may skip these sections and start reading with Sec. VI.

We use natural units ($\hbar = c = 1$) unless otherwise stated. Further conventions can be found in Appendix A.

II. DIRAC HAMILTONIAN AND FERMI COORDINATES

In a first step we derive the Dirac Hamiltonian in a stationary curved space-time in a rigorous way and introduce Fermi coordinates as a local coordinate system. We thereby generalize the work of Parker [18] to an observer in arbitrary motion including rotation and acceleration.

A key assumption of this approach is that hydrogen-like atoms can be modeled by an electron in the given Coulomb field of the nucleus. In another approach Fischbach, Freeman, and Cheng [19] have treated the hydrogen atom as a two-body system in the gravitational field of the Earth and have found that there are differing correction terms depending on whether one uses center of mass or center of energy coordinates. Especially they obtained correction terms to the Hamiltonian which do not vanish even in the limit of very high proton to electron mass ratio (Table I of Ref. [19]). For the purpose of interferometry this causes no difficulties since these terms act on the internal degrees of freedom which can be neglected in the phase shift (see below).

Since the calculations in Secs. II and III are close to those of Parker we keep the presentation concise and refer to Ref. [18] and to our Appendix A for more details.

We rewrite the Dirac equation in an electromagnetic field with four-vector potential A_μ in curved space-time in the form

$$i\partial_0\psi = H\psi \tag{1}$$

with

$$H = -i(g^{00})^{-1}e^{\underline{\alpha}0}e^{\underline{\beta}i}\gamma_{\underline{\alpha}}\gamma_{\underline{\beta}}(\partial_i - \Gamma_i - iqA_i) + i\Gamma_0 - qA_0 - i(g^{00})^{-1}me^{\underline{\alpha}0}\gamma_{\underline{\alpha}}\,. \tag{2}$$

Because of

$$(\psi, H\psi) - (H\psi, \psi) = i \int d^3x\, \psi^+ \gamma^{\underline{0}} \frac{\partial}{\partial \tau}(\sqrt{-g}\; e^0{}_{\underline{\mu}} \gamma^{\underline{\mu}})\psi \,, \tag{3}$$

the operator H is in general not Hermitian with respect to the conserved scalar product (A2) given in Appendix A. It can therefore be interpreted as the Hamiltonian only in certain space-time geometries, especially when the metric is stationary. This reflects one of the numerous conceptual problems which arise for quantum theory in curved space-times (compare Ref. [20]). In the following we assume stationarity. H of Eq. (2) is then Hermitian. It is then interpreted as the Hamiltonian operator representing the observable total energy of the massive quantum object.

This can be seen more clearly if we apply the method of Lämmerzahl [21] and refer to the energy-momentum tensor $T_{\mu\nu}$ of the Dirac field:

$$T_{\mu\nu} = -\frac{1}{\sqrt{-g}} \frac{\delta \mathcal{L}}{\delta e^\nu_{\underline{\alpha}}} e_{\alpha\mu} \tag{4}$$

($\mathcal{L}$ is the Lagrangian). Let ξ^μ be the timelike Killing vector of the stationary space-time and assume that the Maxwell field is also stationary. Then the integration of $T_{\mu\nu}\xi^\mu$ over a spacelike hypersurface gives a conserved quantity which can be interpreted as the energy of the Dirac field which includes the gravitational potential energy. The Hamiltonian operator is defined by setting its expectation value equal to the conserved energy

$$E = \int_\Sigma T_{\mu\nu}\xi^\mu d\Sigma^\nu =: (\psi, H\psi) \,. \tag{5}$$

The result is

$$\begin{aligned} H\psi = {} & \xi^\mu n_\mu \{ i\gamma^\nu n_\nu m\psi - i(\gamma^\nu\gamma^\sigma - g^{\nu\sigma}) n_\sigma \hat{D}_\nu \psi \} \\ & + i(\delta_\nu{}^\mu + n_\nu n^\mu)\xi^\nu \hat{D}_\mu \psi \\ & + \frac{i}{4}(\gamma^\mu\gamma^\nu - g^{\mu\nu})\,(D_\mu \xi_\nu)\psi - q\Phi\psi \,, \end{aligned} \tag{6}$$

where n_μ is the normal vector of the hypersurface, $\hat{D}_\mu = D_\mu - iqA_\mu$, and Φ is defined by

$$\partial_\mu \Phi = F_{\mu\nu}\xi^\nu \,. \tag{7}$$

Choosing the hypersurface $x^0 =$ constant, i.e., $n_\mu = (1,0,0,0)/\sqrt{-g^{00}}$, and using the fact that $\xi^\mu = (1,0,0,0)$ is a Killing vector if $\partial_0 g_{\mu\nu} = 0$ holds, we can easily prove the agreement of this definition with the one of Eq. (1), provided the Maxwell field is stationary $\partial_0 A^\mu = 0$. The stationarity condition on the Maxwell field prevents us from running into trouble with gauge invariance as indicated by Yang [22]. Consider first the case of flat space-time. The only term that is not gauge invariant is then the one proportional to A_0 [in Eq. (6) this corresponds to the Φ term]. The stationarity condition restricts the gauge transformations to those of the form $A'_\mu = A_\mu + \partial_\mu \chi$ with $\chi = g(\vec{x}) + kx^0$ for some constant k. A gauge transformation leads therefore only to the addition of a constant k to the Hamiltonian which does not affect the eigenfunctions of it. In Eq. (6) the situation is even simpler since Φ is determined by Eq. (7), which is manifestly gauge invariant. The remaining freedom to add a constant to Φ corresponds to the effect of allowed gauge transformations in flat space.

Turning to Fermi coordinates we attribute to our quantum system an observer moving along the world line $P_0(\tau)$ with four-velocity $u^\alpha(\tau)$. τ is its proper time along the world line. The experimental setup (e.g., the interferometer) is assumed to remain fixed with regard to a *comoving orthonormal tetrad* $e^\alpha_{\underline{\mu}}$ representing a frame of reference, which is introduced as follows: $e^\alpha_{\underline{0}}$ is identical to u^α and the orthonormal spatial triad $e^\alpha_{\underline{i}}(\tau)$ $(i = 1,2,3)$ rotates together with the setup with proper angular velocity $\vec{\omega}$. Nongravitational forces lead to a proper three-acceleration $\vec{a}$ causing a deviation from the trajectory of the free fall ($a^\alpha = \nabla_u u^\alpha$). The gravitational field is represented by the Riemann curvature tensor $R_{\mu\nu\rho\sigma}(\tau)$ along the world line.

Fermi coordinates are the local coordinate system which is designed to analyze experiments in the proper frame of reference of an accelerated rotating observer in curved space-time. They are introduced as follows. The time coordinate on $P_0(\tau)$ is the proper time of the observer. The spatial coordinate lines are constructed by sending out geodesics orthogonal to the observers world line corotating with the spatial triad $e^\alpha_{\underline{i}}(\tau)$. Each event near the observers world line is intersected by one spacelike geodesic with tangent vector n^α orthogonal to $e^\alpha_{\underline{0}}$ in the point $P_0(\tau)$ with observer time τ. Let s be the distance of the event measured along this geodesic. The Fermi coordinates of the event are then given by $x^0 = \tau$ and $x^k = sn^\alpha e^{\underline{k}}_\alpha$. At the observers world line they represent a rectangular grid attached to the experimental setup, which is accelerated with $\vec{a}$ and rotating with $\vec{\omega}$ (compare Ref. [23]).

The metric in this coordinate system is Minkowskian on the whole world line $P_0(\tau)$ and can be calculated up to a given order in the spatial coordinates x^l. To second order it is given by [24, 25]

$$\begin{aligned} g_{00} &= -(1 + \vec{a}\cdot\vec{x})^2 + (\vec{\omega}\times\vec{x})^2 - R_{\underline{0l0m}} x^l x^m + O((x^l)^3)\,, \\ g_{0i} &= \varepsilon_{\underline{ijk}}\omega^{\underline{j}} x^k - \tfrac{2}{3} R_{\underline{0lim}} x^l x^m + O((x^l)^3)\,, \\ g_{ij} &= \delta_{\underline{ij}} - \tfrac{1}{3} R_{\underline{iljm}} x^l x^m + O((x^l)^3)\,, \end{aligned} \tag{8}$$

where $\vec{a} = \{a_{\underline{i}}\}$, $\vec{\omega} = \{\omega_{\underline{i}}\}$, and the curvature tensor $R_{\underline{\mu\nu\rho\sigma}}$ are space-time scalars obtained by projection with respect to the tetrad on the observer's world line. They may depend on x^0. The use of Eq. (8) represents an approximation which is good as long as the characteristic dimension s of the quantum system is small compared to the characteristic lengths attributed to inertia and curvature:

$$s \ll \min \left\{ \frac{1}{|\vec{a}|}\,,\; \frac{1}{|\vec{\omega}|}\,,\; \frac{1}{|R_{\underline{\mu\nu\rho\sigma}}|^{1/2}}\,,\; \frac{|R_{\underline{\mu\nu\rho\sigma}}|}{|\partial_\lambda R_{\underline{\mu\nu\rho\sigma}}|} \right\} . \tag{9}$$

The expressions for the curvature given in Appendix C show that this can be very well fulfilled for quantum systems up to extremely strong curvature.

We rewrite Eq. (2) with reference to Fermi coordinates using the results of Appendix A and obtain for the Hamiltonian operator

$$H = -im\{[1 + \vec{a}\cdot\vec{x} + \tfrac{1}{2}R_{\underline{0}l\underline{0}m}x^l x^m]\gamma_{\underline{0}} + \tfrac{1}{6}R_{\underline{0}l\underline{i}m}x^l x^m \gamma_{\underline{i}}\} - \vec{\omega}\cdot\vec{J}_0 + q\vec{x}\cdot(\vec{A}\times\vec{\omega}) - qA_0$$
$$+\{\alpha_{\underline{i}} + \vec{a}\cdot\vec{x}\alpha_{\underline{i}} + \tfrac{1}{2}[R_{\underline{0}l\underline{i}m} + R_{\underline{0}l\underline{0}m}\alpha_{\underline{i}} + \tfrac{1}{3}R_{\underline{i}l\underline{j}m}\alpha_{\underline{j}} + \tfrac{1}{3}R_{\underline{0}l\underline{j}m}\gamma_{\underline{j}}\gamma_{\underline{i}}]x^l x^m\}(-i\partial_i - qA_i)$$
$$-\tfrac{i}{2}a_{\underline{i}}\alpha_{\underline{i}} + \tfrac{i}{4}\gamma_{\underline{i}}\gamma_{\underline{j}}R_{\underline{0}\underline{i}m\underline{j}}x^m + \tfrac{i}{4}\alpha_{\underline{j}}[R_{\underline{j}m} - R_{\underline{0}\underline{j}\underline{0}m}]x^m\,, \tag{10}$$

where $\vec{J_0} = -i(\vec{x}\times\vec{\nabla}) + \vec{\Sigma}/2$ is the total angular momentum in absence of an electromagnetic field. We have used $R_{\underline{mijk}}\varepsilon^{\underline{ijk}} = 0$ and

$$\gamma_{\underline{i}}\gamma_{\underline{j}}\gamma_{\underline{k}} = \delta_{ij}\gamma_{\underline{k}} - \delta_{ik}\gamma_{\underline{j}} + \delta_{jk}\gamma_{\underline{i}} + i\varepsilon_{ijk}\gamma_{\underline{5}}\gamma_{\underline{0}}\,. \tag{11}$$

III. ENERGY SHIFT OF THE HYDROGEN ATOM

Once the complete Hamiltonian is given one can use time independent perturbation theory for degenerate states to calculate the influence of $\vec{a}$, $\vec{\omega}$, and $R_{\underline{\mu\nu\rho\sigma}}$ on the energy levels of the hydrogen atom. The knowledge of these energy corrections is important also in connection with Ramsey interferometry since we have to answer the question whether the corresponding modification of the internal degrees of freedom has to be taken into account or not. They are given to lowest order by the eigenvalues of the matrix $\langle\alpha|H_{\rm pert}|\beta\rangle$, where $H_{\rm pert}$ is the difference between Hamiltonian in Fermi coordinates and the Hamiltonian H_0 in Minkowski space

$$H_0 = -i\alpha_{\underline{i}}\partial_i + \frac{qZe}{r} - im\gamma_{\underline{0}} \tag{12}$$

and $|\alpha\rangle$ and $|\beta\rangle$ are two degenerate states of a hydrogen-like atom written as Dirac spinors. We assume that the multiplicity of the unperturbed energy level is two.

To calculate the shift of the energy for the ground state of the hydrogen atom we have to insert the field of a point charge resting at the origin of the Fermi coordinate system into the Hamiltonian. In flat space this is the ordinary Coulomb field, but the presence of curvature and the noninertial motion of the atom lead to corrections to the Coulomb potential which have to be included in the energy calculation. In Appendix B we give a derivation of these correction terms. Generalizing the calculation of Parker [18] we find, for the energy shift of the ground state,

$$\Delta E = \frac{2\gamma+1}{6m}\left\{\vec{\omega}^2 - \vec{a}^2 + \frac{1}{4}R + R_{\underline{00}}\left(\frac{3}{2} + \frac{\gamma(\gamma+1)}{2\zeta^2}\right)\right\} \pm \frac{1}{2}|\vec{\omega}|\,, \tag{13}$$

where $\zeta = -qZe$ and $\gamma = (1-\zeta^2)^{1/2}$.

Most of these energy shifts are far outside the measuring range of modern experiments. If one assumes Einstein's field equations, R and $R_{\underline{00}}$ vanish in a vacuum where the experiments should be performed. The terms quadratic in the rotation and acceleration give the contributions 8.4×10^{-37} eV$\times[\omega/(1\text{ Hz})]^2$ and 9×10^{-52} eV$\times(a/g)^2$, where $g = 9.81$ ms^{-2}. The last term, caused by rotation, has a magnitude of 6.6×10^{-16} eV$\times\omega/(1$ Hz). This may be big enough to be detected via an induced optical activity in atoms [26].

To return to interferometry, supposing one can manage that the time of flight of the atoms is very long, say, 1 s (1.5×10^{15} eV^{-1} in natural units), we can see that the energy shift of the ground state results only in a very tiny phase shift $\Delta\phi$ by setting approximately $\Delta\phi = \Delta Et$. Again, only the last rotational term may cause measurable effects. For Ramsey interferometry we may therefore refer to the unperturbed energy levels of the atoms.

IV. NONRELATIVISTIC LIMIT

In general the phase shift of an interferometric pattern caused by some external force increases with decreasing velocity of the particle beam. It is therefore adequate to derive a nonrelativistic approximation of the Hamiltonian which leads us at last to the modified Pauli equation. In a systematic way this is best done by use of the *Foldy-Wouthuysen transformation* FWT (see, e.g., Ref. [27]). The idea of this transformation is to construct a unitary transformation $\psi = \exp(-iS)\psi'$ with $(\phi, S\psi) = (S\phi, \psi)$ such that in the operator H' acting on ψ',

$$H' \approx H + i[S,H] - \frac{1}{2}[S,[S,H]] - \dot{S} - \frac{i}{2}[S,\dot{S}] + \cdots\,, \tag{14}$$

the odd operators, which couple the small components of the spinor to the large ones, are relativistically suppressed, i.e., are of higher order in $1/m$. Because of this intended separation of the large and small spinor components the calculations are usually done in the standard representation of the Clifford algebra in which, with our conventions for the sign of the metric, $\gamma_{\underline{0}} = i$ diag$(1,1,-1,-1)$ holds. It should be stressed that S must be Hermitian with respect to the scalar product (A2) in curved space-time in order to preserve the respective Hermiticity of transformed operators. In Fermi coordinates the scalar product is given by Eq. (A7).

We now proceed with the derivation of the FWT in Fermi coordinates in constructing the operator S. In Minkowski space it is chosen to be $\gamma_{\underline{j}}(-i\partial_j - qA_j)/2m$ in order to suppress the $\alpha_{\underline{i}}(-i\partial_i - qA_i)$ term. Its Hermitian generalization in Fermi coordinates is

$$S = \frac{1}{2m}[\gamma_{\underline{j}}(-i\partial_j - qA_j) + \tfrac{1}{6}R_{\underline{0}l\underline{i}m}x^l x^m\,\gamma_{\underline{0}}\gamma_{\underline{i}}\gamma_{\underline{j}}(-i\partial_j - qA_j) + \tfrac{i}{6}R_{\underline{iji}m}x^m\,\gamma_{\underline{j}}] \tag{15}$$

and leads, to lowest order in each perturbation, to

$$H' = -im(1 + \vec{a}\cdot\vec{x} + \tfrac{1}{2}R_{\underline{0}l\underline{0}m}x^l x^m)\gamma_{\underline{0}} - \tfrac{i}{6}mR_{\underline{0}l\underline{i}m}x^l x^m \times\gamma_{\underline{i}} - \tfrac{i}{2m}\gamma_{\underline{0}}(-i\partial_i - qA_i)(-i\partial_i - qA_i) - qA_0 -\vec{\omega}\cdot\vec{J}_0 + q\vec{x}\cdot(\vec{A}\times\vec{\omega}) + i\tfrac{q}{2m}\gamma_{\underline{0}}\vec{\Sigma}\cdot\text{rot}\vec{A} + \tfrac{q}{2m}\gamma_{\underline{j}}(\dot{A}_j - \partial_j A_0)\,. \tag{16}$$

It is remarkable that, though S does not depend on the acceleration, all odd terms containing $\vec{a}$ are removed to this order.

The Hamiltonian H' still contains two odd terms. The last one ($\propto q\gamma_j$) can be treated as in the textbooks [27]. We will omit this. The first one ($\propto R_{0lim}$) needs particular attention. It cannot be removed by a second FWT, because only the combination

$$-im(\gamma_0 + \tfrac{1}{6}R_{0lim}x^l x^m \gamma_i) \tag{17}$$

in H' is Hermitian. If a Hermitian operator S suppressing the R_{0lim} term would exist, it would produce a non-Hermitian Hamiltonian in contradiction to the scheme.

To solve the problem in a consistent way we propose the following approach: Although we refer to the quasi-Cartesian Fermi coordinates we still have to make use of the scalar product (A7), which does not agree with the nonrelativistic scalar product in Cartesian coordinates. To make the correspondence complete we look for an operator O, which transforms ψ' to $\tilde{\psi} = O^{-1}\psi'$ in such a way that the scalar product is changed to the one in flat space:

$$(\phi', \psi') = (O\tilde{\phi}, O\tilde{\psi}) = (\tilde{\phi}, \tilde{\psi})_0 \equiv \int d^3x\, \tilde{\phi}^+ \,\tilde{\psi}\,. \tag{18}$$

Given this condition the conserved scalar product remains conserved after the transformation. One can do this transformation formally for arbitrary coordinates and the general scalar product (A2), which may be written as

$$(\phi', \psi') = (\phi', T\psi')_0\,, \quad T := -\sqrt{-g}\, e^0{}_\mu \gamma^0 \gamma^\mu\,. \tag{19}$$

The condition (18) for O then takes the form

$$(\phi', \psi') = (O\tilde{\phi}, TO\tilde{\psi})_0 \stackrel{!}{=} (\tilde{\phi}, \tilde{\psi})_0\,, \tag{20}$$

where the last equality is the demand. This leads immediately to

$$O^{*T}TO = 1\,, \quad O^{*T}T = O^{-1}\,. \tag{21}$$

Accompanying the transformation of states with the corresponding transformation $A' \to \tilde{A} = O^{-1}A'O$ of operators we obtain

$$(\phi', A'\psi') = (\tilde{\phi}, O^{*T}TA'O\tilde{\psi})_0 = (\tilde{\phi}, \tilde{A}\tilde{\psi})_0\,. \tag{22}$$

Similarly one can show that

$$(A'\phi', \psi') = (\tilde{A}\tilde{\phi}, \tilde{\psi})_0\,. \tag{23}$$

Using these two equations (22) and (23) it is not difficult to see that if any operator A' is Hermitian with respect to the curved scalar product $(\phi', A'\psi') = (A'\phi', \psi')$, then so is $\tilde{A}$ with respect to $(,)_0$. In the same sense the unitarity of an operator is conserved under this transformation. Because of its properties the product changing transformation will be called *quasiunitary*. Of great importance for the consistency of the FWT is the fact that the ordering of the two transformations $\exp(iS)$ and O plays no role since

$$\begin{aligned} O^{-1}e^{iS}A'e^{-iS}O &= O^{-1}e^{iS}O\tilde{A}O^{-1}e^{-iS}O \\ &= e^{i\tilde{S}}\tilde{A}e^{-i\tilde{S}}\,. \end{aligned} \tag{24}$$

If we perform the quasiunitary transformation first we simply have to take the transformed operator $\tilde{S}$ to do the FWT. This sequence is in fact more convenient since the Hermiticity of $\tilde{S}$ with respect to $(,)_0$ is more easily checked than the one of S with respect to the curved scalar product.

For the general scalar product (A2) the operator O takes the form

$$O = P + Qe^{0i}\alpha_i\,, \tag{25}$$

where

$$P = \frac{1}{\sqrt{2g_\Sigma}}[\sqrt{g_\Sigma} - \sqrt{-g}\, e^{00}]^{1/2} \tag{26}$$

and

$$Q = \frac{-\sqrt{-g}}{\sqrt{2g_\Sigma}}[\sqrt{g_\Sigma} - \sqrt{-g}\, e^{00}]^{-1/2}\,. \tag{27}$$

g_Σ is the determinant of the spatial part of the metric. With respect to Fermi coordinates and the related approximation, the operator O becomes

$$O = 1 + \frac{1}{12}R_{ilim}x^l x^m + \frac{1}{12}R_{0lim}x^l x^m\, \alpha_i + O((x^l)^3)\,. \tag{28}$$

Application to H' of Eq. (16) removes the curvature induced odd term so that we finally find, for the Hamiltonian in the nonrelativistic limit, the result

$$\begin{aligned} \tilde{H} = {} & -im(1 + \vec{a}\cdot\vec{x} + \tfrac{1}{2}R_{0l0m}x^l x^m)\gamma_0 \\ & - \tfrac{i}{2m}\gamma_0(-i\partial_i - qA_i)(-i\partial_i - qA_i) - qA_0 \\ & -\vec{\omega}\cdot\vec{J}_0 + q\vec{\omega}\cdot(\vec{x}\times\vec{A}) + i\tfrac{q}{2m}\gamma_0\vec{\Sigma}\cdot\mathrm{rot}\vec{A} \\ & + \tfrac{q}{2m}\gamma_j(\dot{A}_j - \partial_j A_0)\,, \end{aligned} \tag{29}$$

where $\vec{J}_0 = -i(\vec{x}\times\vec{\nabla}) + \vec{\Sigma}/2$ is the total angular momentum in absence of an electromagnetic field. The correction terms in the Pauli equation are thus

$$m\vec{a}\cdot\vec{x} \tag{30}$$

for acceleration,

$$-\vec{\omega}\cdot\vec{J}_0 + q\vec{x}\cdot(\vec{A}\times\vec{\omega}) \tag{31}$$

for rotation, and

$$\frac{m}{2}R_{0l0m}x^l x^m \tag{32}$$

for curvature. The term proportional to the rotation and the electromagnetic field can be understood to arise from the minimal coupling of the spinor and the four-potential so that no problem with gauge invariance occurs. Each term is Hermitian with respect to the Schrödinger-type scalar product $(,)_0$ of Eq. (18) in Cartesian coordinates even if the term is time dependent. Due to the approximation (8) of the metric, which is correct only to second order in the spatial coordinates, we find no mixing between $\vec{a}$, $\vec{\omega}$, and $R_{\mu\nu\rho\sigma}$. Only $\vec{a}$ and $\vec{\omega}$ are mixed in the

g^{0i} components of the metric and this leads to a corresponding term in the vector potential (B7). A higher expansion of the metric [28] leads also to a mixing between $\vec{a}$, $\vec{\omega}$, and $R_{\underline{\mu\nu\rho\sigma}}$ and would presumably produce corresponding effects in the vector potential and in the Schrödinger equation.

Our results for acceleration and rotation are in agreement with previous calculations [29–31] and have an obvious physical interpretation (compare Ref. [12]). The curvature term confirms Parker's [18] approximative approach to the hydrogen atom and coincides with another derivation which, instead of starting from a covariant quantum theory, takes a classical point particle in a weak gravitational field as starting point [16, 17, 32].

V. WEAK GRAVITATIONAL FIELDS

For later use we give a physical interpretation of the curvature term in relating it to the Newtonian potential in the limit of weak gravitational fields (compare, e.g., Chap. 18 of [23]). To do so we turn in a last step to the linearized approximation of general relativity which is fulfilled in the solar system or for gravitational waves. In this case the metric is written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\ , \quad |h_{\mu\nu}| \ll 1\,. \tag{33}$$

Defining

$$\bar{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^{\rho}{}_{\rho} \tag{34}$$

and choosing harmonic coordinates so that $\bar{h}^{\mu\alpha}{}_{,\alpha} = 0$ the linearized field equations become

$$-\partial_0\partial_0\bar{h}_{\mu\nu} + \sum_{i=1}^{3}\partial_i\partial_i\bar{h}_{\mu\nu} = -2\kappa T_{\mu\nu}\,, \tag{35}$$

where $T_{\mu\nu}$ is the energy-momentum tensor of the matter producing the gravitational field and the constant κ is defined to be $\kappa = 8\pi G$, where G is Newton's constant. Static nearly Newtonian sources obey $T_{00} \gg |T_{0j}|$ and $T_{00} \gg |T_{jk}|$ with $T_{00} = \rho(\vec{x})$. In this case the solution of the field equation is $\bar{h}_{0j} = \bar{h}_{jk} = 0$ and

$$h_{00}(\vec{x}) = h_{i(i)}(\vec{x}) = \frac{1}{2}\bar{h}_{00}(\vec{x}) = 2G\int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d^3x'\,, \tag{36}$$

which corresponds exactly to the *Newtonian potential*: $\phi(\vec{x}) = -h_{00}/2$.

Let us consider an observer resting in this space $\dot{x}^i = 0$. Without loss of generality we can take $\vec{x} = 0$. From Eq. (C2) the observer's acceleration is found to be

$$a^0 = \ddot{x}^0 = 0\ ,\ a^i = \partial_i\phi(0)\,(\dot{x}^0)^2 = \partial_i\phi(0)\,. \tag{37}$$

We see in accordance with the interpretation given in Sec. II that $m\vec{a}$ is exactly the force needed to resist the Newtonian gravitational force, i.e., the negative of it. Note that, due to $\dot{x}^\mu\dot{x}_\mu = -1$, the equation

$$x^0(\tau) = \tau[1-\phi(0)] \tag{38}$$

holds. The components of the curvature tensor take the values

$$R_{\mu\nu\alpha\beta} = \frac{1}{2}[\partial_\nu\partial_\alpha h_{\mu\beta} + \partial_\mu\partial_\beta h_{\nu\alpha} - \partial_\mu\partial_\alpha h_{\nu\beta} - \partial_\nu\partial_\beta h_{\mu\alpha}] \tag{39}$$

so that

$$R_{0l0m} = \partial_l\partial_m\phi(\vec{x})\,. \tag{40}$$

These curvature components are gauge invariant to linear terms in $h_{\mu\nu}$ so that the components in the Fermi coordinate system agree with those of (40) (compare Ref. [42]):

$$R_{\mu\nu\alpha\beta} = R_{\underline{\mu\nu\alpha\beta}}\,. \tag{41}$$

This illuminates the physical meaning of the curvature term (32). For the general structure of Fermi coordinates in weak gravitational fields see Ref. [33].

One may look at these results also from a different point of view starting with the Newtonian potential. If we expand $m\phi(\vec{x})$ around $\vec{x} = 0$ we get

$$\begin{aligned} m\phi(\vec{x}) &= m\phi(0) + mx^l\partial_l\phi(0) \\ &\quad + \frac{m}{2}\partial_l\partial_m\phi(0)\,x^lx^m + O((x^l)^3) \\ &= m\phi(0) + ma^lx^l + \frac{m}{2}R_{0l0m}(0)\,x^lx^m \\ &\quad + O((x^l)^3)\,, \end{aligned} \tag{42}$$

so that, in this case, the acceleration term and the curvature term in the Hamiltonian are simply the first terms of the Taylor expansion of the Newtonian potential difference. The potential $\phi(0)$ on the world line of the observer is absorbed into the proper time τ via (38) and does therefore not appear if the Hamiltonian is written in Fermi coordinates.

VI. CURVATURE AND NEWTONIAN GRAVITY

The basic conceptual tool used above to mesh quantum mechanics with gravity was Einstein's equivalence principle, which demands that in a local inertial frame (local Lorentz frame) all laws of physics must take their special-relativistic form. This leads to the general relativistic Dirac equation which fixes the dynamics. Because operators and states are attributed to hypersurfaces of equal "time" additional considerations were necessary. Following a succession of approximations we obtained the Hamiltonian (29) for noninertial reference frames which includes the space-time curvature components $R_{\underline{0l0m}}$. For weak static gravitational fields $R_{\underline{0l0m}}$ may be related according to (40) to an inhomogeneous Newtonian gravitational potential $\phi(\vec{x})$.

Three approximation steps were imposed on us by the characteristic physical scales of quantum systems in noninertial frames in the solar system (in particular on and near to the Earth).

(i) The extension of the quantum system is small compared to the characteristic lengths of curvature, acceleration, and rotation [compare Eq. (9)]. A treatment in Fermi coordinates is therefore physically well justified.

(ii) In atomic beam spectroscopy atoms are very slow. The Foldy-Wouthuysen transformation provides also in curved space-time a systematic approach to the nonrelativistic limit.

(iii) The interferometer experiments will be done in a laboratory on Earth. Because of the precision which can be obtained today we will not be able to test the respective post-Newtonian terms which would correct relation (40). With the exact expressions for $\underline{R_{\mu\nu\rho\sigma}}$ provided in Appendix C it would be easy to work out these corrections for Einstein's theory.

In going to the correspondence limit of weak gravity and low velocities of the quantum objects, have we lost the concept of space-time curvature in this way? Equation (42) demonstrates that this is not the case. Space-time curvature manifests itself as inhomogeneous gravitation and vice versa. The trajectories of freely falling test particles are geodesics of space-time. Curvature shows up in the deviation of two nearby geodesics. This effect is physically equivalent to the relative acceleration of the test particles. The respective tide-producing gravitational forces are expressed in terms of the curvature tensor $R_{\mu\nu\rho\sigma}$. Space-time flatness would be equivalent to the absence of tidal gravitational forces. But there is relative acceleration of freely falling test particles in the limit of weak gravitational fields. Accordingly, there is Einstein curvature also in this case showing the same characteristic influence on matter as in strong gravitational fields. Its particular influence on quantum objects has been described above.

It has been pointed out by Misner *et al.* (p. 305 of Ref. [23]) that the deepest features of Newtonian gravity are (i) the equivalence principle and (ii) space-time curvature. They manifest themselves in the second and third terms of the Taylor expansion of the potential $\phi(\vec{x})$, as can be seen in (42).

It must be stressed that this Taylor expansion is tied to the traditional Newtonian picture of gravity. Seen in this way, curvature is simply the next Taylor coefficient of the potential. In general relativity, however, acceleration and curvature have a completely different origin. Acceleration is something that is bound to the motion of the observer [see Eq. (C2)] and is produced by any *nongravitational* forces. Accelerational effects arise if we describe nature in the natural frame of reference or the Fermi coordinates of the accelerated observer. In this frame freely falling, forceless objects (such as the apple) seem clearly to be accelerated because their speed relative to the accelerated observer is changing. Curvature, on the other hand, is *the* quantity that describes the deviation of space-time from a flat manifold. It has nothing to do with the observer or his motion in space. Since on large scales general relativity describes the world better than Newton's theory, this point of view is more appropriate. It follows that the measurement of the influence of tidal forces on quantum systems would be a qualitatively new contribution to the understanding of the microscopic world. The influence of the acceleration of an observer resting on the Earth has been tested for quantum mechanical systems, whereas the influence of space-time itself via curvature is only established on large scales, in a classical region, where the problems of the description of quantum matter in a curved space are absent.

The validity of the equivalence principle for quantum systems has already been tested; see Sec. I. To demonstrate that not only classical test particles but quantum systems too react in a measurable way on tidal forces and therefore on space-time curvature is the aim of the subsequent part of this paper.

The gravitational field available in a laboratory is the field of laboratory sized objects and the field of the Earth. To detect the influence of the Earth's field there are two possible types of motion of the interferometer. It may be at rest at the surface of the rotating Earth or freely falling either on a path towards the Earth or fixed to an orbiting satellite. The difference between the first and the last two cases is a Lorentz boost and possibly a rotation. To obtain the order of magnitude of the effects in question it is therefore sufficient to consider different orientations of the interferometer in a laboratory on Earth because the relative velocity of a flying laboratory is so small that the respective relativistic corrections caused by the boost will not show up. We now turn to the description of the interferometer.

VII. PHASE SHIFT OF THE RAMSEY FRINGES

With the results (30)–(32) of Sec. V we have the ability to examine the influence of acceleration, rotation, and space-time curvature on nonrelativistic experimental setups. With regard to a demonstration of the influence of curvature, the *Ramsey atom beam spectrometer* seems to be well suited. Most experiments done with this setup are concerned with high resolution spectroscopy. But recently Bordé [9] has pointed out that it can also be used as an atom interferometer. It is this aspect that we are interested in.

To describe the apparatus we fix the comoving observer tetrad to the interferometer so that the respective Fermi coordinates can be interpreted approximately as the ordinary Cartesian coordinate system (x^1, x^2, x^3) with range over the apparatus. The setup consists of an atomic beam which moves initially in the x^1 direction and four traveling laser waves parallel or antiparallel to the x^3 direction. The laser waves are tuned to be nearly in resonance with a particular transition between two states of the atoms. The first two laser beams are copropagating in the x^3 direction and the third and the fourth beam copropagate in the $-x^3$ direction (see Fig. 1). See Ref. [36] for more general laser configurations.

The lasers are assumed to be arranged in such a way that the time of flight T between the two lasers of a copropagating pair is the same for each pair. Between the two pairs the atoms move for a time T'. While passing the lasers, the atoms absorb or reemit photons. By the corresponding recoil the atomic wave function is coherently split and recombined so that the two pairs of interfering atomic beams of Fig. 1 are obtained. The outgoing wave functions $b^{(+1)}$ and $b^{(-1)}$ correspond to an excited part of the wave function. The index denotes the pair. The respective population can be read out by detection of its fluorescence radiation. It oscillates with

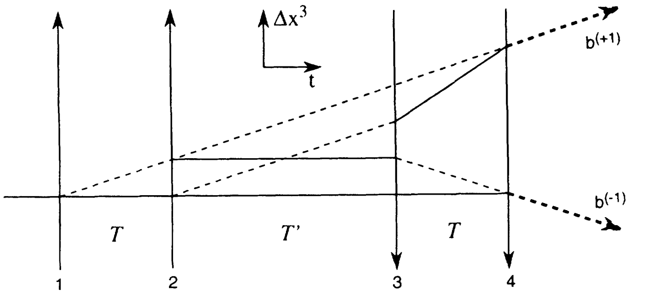

FIG. 1. Ramsey atom interferometer: Four traveling laser waves (1–4) running parallel or antiparallel to the x^3 direction split and recombine an atomic beam. The partial beams build up two interferometer geometries. Dashed lines correspond to excited atoms. The atoms are initially deexcited. They travel for the time T and T' between the lasers. Δx^3 represents the relative distance between the partial beams.

the laser detuning. The resulting oscillations are called *Ramsey fringes.* With regard to their resolution, note that the excited atoms (dashed lines in Fig. 1) may decay according to their lifetime τ_l. For further details of the experiment and its theory see Bordé *et al.* [34].

In a previous paper [35] we have calculated the first order contribution to the shift $\Delta\varphi$ of the Ramsey fringes due to a fairly general perturbation

$$H^{(1)} = \hat{H}(\vec{p})\ (x_1)^{N_1}(x_2)^{N_2}(x_3)^{N_3} \tag{43}$$

proportional to powers of the coordinates with N_i being integer. $\hat{H}(\vec{p})$ contains an arbitrary dependence on the center of mass momentum operator $\vec{p}$. In a paper to be published [36] higher orders of non-Minkowskian influences have been taken into account. It is important to note that these calculations are not made within the usual WKB approach. One does not deal with classical paths but follows a unitary time evolution of quantum states. Any bending of a beam is automatically taken care of. It does not appear explicitly in the calculation. In addition, the time evolution implies that the specifications of the experiment refer to the times of the interaction between atoms and light and not to spatial locations of the atoms.

This time evolution can be illustrated as in Fig. 1 as a sequence of splittings and recombinations of the atomic beam. For sufficiently localized wave packets the vertical axis can approximately be interpreted as the relative distance Δx^3 between two partial beams. Since atoms get simply a kick, i.e., some definite transfer of momentum by the lasers, this relative distance grows linearly in time even if a homogeneous gravitational field (acceleration) is present.

The resulting shifts $\Delta\varphi^{(-1)}$ and $\Delta\varphi^{(+1)}$ corresponding to the two interferometer geometries of Fig. 1 are given in Ref. [36] and Eqs. (28) and (31) of Ref. [35] (note the change in the notation). Specializing $H^{(1)}$ of Eq. (43) to our correction terms (30)–(32) we obtain, for the first order *phase shift*,

$$\Delta\varphi^{(\pm 1)} = \Delta\varphi_a^{(\pm 1)} + \Delta\varphi_\omega^{(\pm 1)} + \Delta\varphi_R \,. \tag{44}$$

The influence of the acceleration $\vec{a}$ is given by

$$\Delta\varphi_a^{(\pm 1)} \equiv \Delta\varphi_a = -ka_{\underline{3}}\, T(T+T')\,. \tag{45}$$

A rotation $\vec{\omega}$ leads to

$$\Delta\varphi_\omega^{(\pm 1)} \equiv \Delta\varphi_\omega = \frac{2}{m}\omega_{\underline{2}} p_1 k\, T(T+T')\,. \tag{46}$$

Both shifts agree for the two interferometer geometries (compare also Ref. [37]). The phase shift $\Delta\varphi_R$, which is caused by space-time curvature, is the sum $\Delta\varphi_{0103}^{(\pm 1)} + \Delta\varphi_{0303}^{(\pm 1)} + \Delta\varphi_{aR}$ of the terms

$$\Delta\varphi_{0103}^{(\pm 1)} \equiv \Delta\varphi_{0103} = -\frac{c^2}{2m} R_{\underline{0103}}\, kp_1 T[2T^2 + 3TT' + (T')^2]\,, \tag{47}$$

$$\Delta\varphi_{0303}^{(-1)} = -\frac{\hbar c^2}{2m} R_{\underline{0303}}\, k^2(\tfrac{2}{3}T^3 + T^2T')\,, \tag{48}$$

$$\Delta\varphi_{0303}^{(+1)} = -\frac{\hbar c^2}{2m} R_{\underline{0303}}\, k^2 T[4T^2 + 6TT' + 3(T')^2]\,, \tag{49}$$

$$\Delta\varphi_{aR} = \frac{c^2}{12} R_{\underline{030m}} k a_{\underline{m}} T(T+T')(7T^2+7TT'+2T'^2)\,. \tag{50}$$

For convenience we have reintroduced $\hbar$ and c and have set $k = 2\pi/\lambda$, where λ is the wavelength of the lasers. v_1 is the initial mean velocity of the atoms incident in the x^1 direction and p_1 the corresponding initial momentum.

The calculation of the phase shift done in Ref. [35] includes only the derivation of Eqs. (47)–(49). The mixed term (50), which includes also the acceleration, must be calculated in a different manner. Details will be given elsewhere [36]. Because this expression cannot be found in an already published paper we give some heuristic arguments which may clarify its structure. Note that the phase shift is a pure number. If curvature is (to first order) included $\Delta\varphi_{aR}$ must be proportional to $R_{\underline{0l0m}} v_{\underline{l}} w_{\underline{m}}$, where v and w are certain vectors. Furthermore, $\Delta\varphi_{aR}$ is an interference effect and must therefore include something which indicates that the atomic beam was split and recombined. The only quantity that can do this is the wave vector $k_{\underline{m}} = k\delta^3_{\underline{m}}$ of the lasers. Hence $\Delta\varphi \propto R_{\underline{0l0m}} k_{\underline{l}}$. The rest of the argument is based on the comparison with the curvature shift (47), which is in general proportional to $R_{\underline{0m03}} k p_{\underline{m}}$. If the atoms were classical balls their momentum in an accelerated frame of reference would change according to $\vec{p} \to \vec{p} + m\vec{a}T$. If we do this replacement in Eq. (47) we get an additional expression of the form (50). Only the exact form of the fourth order polynomial in the flight times T, T' differs.

In Ref. [36] a nonperturbative approach is performed. It leads to the interesting result that all acceleration dependent phase shifts are linear in $\vec{a}$. No powers of $\vec{a}$ can appear if only acceleration, rotation, and curvature are considered. This is one point why it may be of advantage to use atoms instead of neutrons. The work of Anandan [14] demonstrates clearly that for neutrons there are several contributions to the phase shift from different orders of the acceleration alone so that it may become more difficult to distinguish between acceleration and curvature effects. Note also that the approach of Anandan uses the WKB approximation and is therefore completely different from our calculations which are based on a unitary time evolution and make no reference to classical paths.

VIII. ESTIMATION OF THE MAGNITUDE OF THE EXPERIMENTAL EFFECTS

In the following we want to discuss in detail the measurability of the influence of space-time curvature on the phase shift. We thereby take as a basis the specifications of two already existing experimental setups described by Riehle *et al.* [6] and Sterr *et al.* [38]. In Ref. [6] the intercombination transition $^3P_1 \to {}^1S_0$ of ^{40}Ca has been used. In Ref. [38] the laser waves are resonant with the intercombination transition $^3P_1 \to {}^1S_0$ of ^{24}Mg. In the first and third row of Table I the lifetime τ_l of the excited metastable $^3P_1$, the wavelength λ of the transition, the mass m, and the initial velocity v_1 of the atoms in the x^1 direction as well as the times of flight T, T' between the laser beams in the respective apparatus are given.

A. Space-time curvature of the Earth

In a first step we want to examine whether the space-time curvature caused by the Earth may be measurable. Acceleration, rotation, and Riemannian curvature tensor components are then given by Eqs. (C13), (C14), and (C15), respectively, in Appendix C. The three angles α, β, and γ describe the orientation of the interferometer (which is fixed to the coordinate system as described in Sec. VII) relative to the Earth. They are defined as follows. In the initial orientation for which all three angles vanish, x^3 points towards the ceiling of the laboratory, x^1 to the south, and x^2 to the east. We now perform three rotations. The first is around the x^3 axis with the angle α and turns the apparatus so that the x^1 axis does not point further in north-south direction. The second is around the new x^2 axis (this is the axis perpendicular to the atom beam and the laser waves) with angle β. The third rotation is around the resulting x^1 axis (turning around the incident atomic beam) with angle γ (compare Fig. 2 for the case $\alpha = 0$).

To get an impression of the order of magnitude of the phase shifts caused by gravitational acceleration, rotation, and space-time curvature on Earth, we have worked out on the basis of Table I and Eqs. (C13)–(C15) the respective maximal values of $\Delta\varphi_a$, $\Delta\varphi_\omega$, and $\Delta\varphi_R$. The results are listed in Table II. They are obtained in each case for an optimally adjusted orientation of the interferometer. Note that for a maximal influence of $\vec{a}$ an orientation is needed other than that for $\vec{\omega}$ or the curvature terms.

Table II clearly demonstrates that for the two existing interferometers No. 1 and No. 3, only $\Delta\varphi_a$ is measurable. The influence of the Earth's rotation is too small and the influence of the space-time curvature is many orders of magnitudes too small.

Because of the cubic time dependence of the curvature induced phase shift this situation changes if the flight times are in the order of 1 s. To enlarge the times of flight T, T' between the lasers it is necessary to slow down the atoms and to build a device with larger distances between the laser beams. Modern laser cooling techniques allow us to build spectrometers in which the mean velocity of the atoms is as low as 2 m s^{-1} (see, e.g., Ref. [38]). There are some limitations on the magnitude of T and T'. Since in the interferometer geometry of Fig. 1 leading to $b^{(-1)}$ one part of the atomic beam is excited between the laser pairs 1-2 and 3-4 the corresponding time of flight T is limited to be at best of the order of the lifetime τ_l of the excited state. If it is substantially larger the coherence of the atoms will be destroyed by spontaneous emission happening between the copropagating laser beams. However, this argument does not hold for the time of flight T' between the second and third lasers since all atoms moving to the $b^{(-1)}$ output are unexcited. It is clear that even T' cannot be made very large since it is very difficult to collimate the atomic beam over large distances. The loss of atomic flux for a larger T' leads to a growing integration time for a given accuracy of the statistics. Referring to the experiment of Kasevich and Chu [3] where the time of flight is about 0.5 s and the atomic flux decreases with a factor of about 30 we think that a total time of flight of 1 or 2 s should be possible within the near future. This leads to the modified specifications No. 2 and No. 4 of the Ramsey device given in Table I.

TABLE I. Lifetime, wavelength, atomic mass, atomic mean velocity, and flight times of the atoms between the laser beams of two existing and two hypothetical Ramsey devices. The question mark means that we have not succeeded in finding the corresponding value in the literature.

No.	Reference	τ_l (ms)	λ (nm)	m (kg)	v_1 (m/s)	T (ms)	T' (ms)
1	[6]	0.4	657	6.7×10^{-26}	700	1.86×10^{-2}	4.7×10^{-2}
2	[6] modified	0.4	657	6.7×10^{-26}	2	0.2	1000
3	[38]	4.6	457	4×10^{-26}	700	1.7×10^{-2}	6×10^{-2} (?)
4	[38] modified	4.6	457	4×10^{-26}	5	3	1020

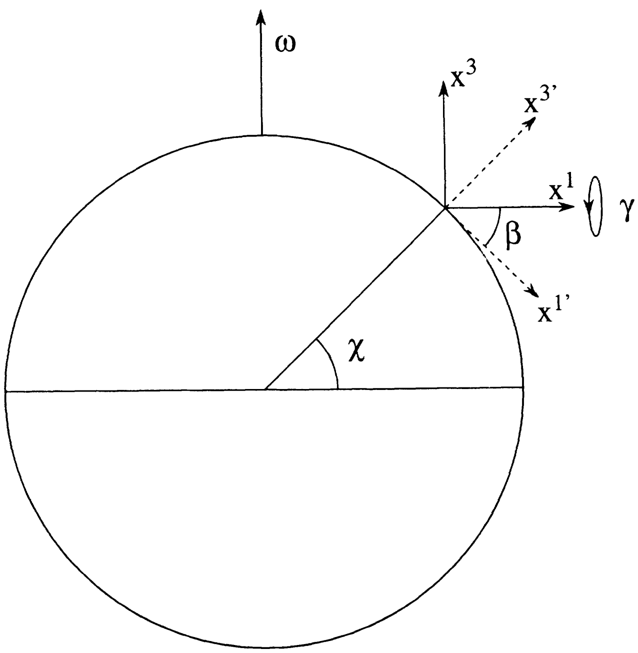

FIG. 2. Orientations of the interferometer for the measurement of the space-time curvature of the Earth. The parallel latitude of the laboratory is $\chi = \pi/2 - \vartheta$. The dashed axes show the orientation of the interferometer for $\alpha = \beta = \gamma = 0$. For $\alpha = 0$ a rotation of the interferometer around the $x^{2'}$ axis by the angle β leads to the solid line axes. A subsequent γ rotation is indicated. The case $-\beta = \vartheta = \chi = \pi/4$ is shown. The atom beam entering the interferometer is directed along the x^1 axis and the laser beams are parallel to the x^3 axis.

Based on these modifications we obtain, for the phase shifts, the results given in the rows $2_\oplus$ and $4_\oplus$ of Table II. The curvature induced phase shift $\Delta\varphi_R$ has now become large enough to be measurable if one manages to separate it from the shifts $\Delta\varphi_a$ and $\Delta\varphi_\omega$, which have become very large and will contribute to the resulting shift $\Delta\varphi$ of Eq. (44). This separation could formally be achieved by laser and atomic beam reversal [35]. This employs the fact that the shifts (45)–(50) have different powers of k and p_1 and therefore behave differently under beam reversal. For instance, to isolate $\Delta\varphi_{0103}$ one can invert the atomic beam ($p_1 \to -p_1$) and set the device on a turntable so that $\Delta\varphi_\omega \approx 0$ or perform the experiment at a parallel latitude $\chi = \pi/2 - \vartheta$ where ω_2 of Eq. (C14) is zero for certain orientations. In addition, it is necessary to perform the experiment at least in two different orientations (by changing the orientation of the lasers) in order to vary the magnitude of $R_{\underline{0}1\underline{0}3}$ and therefore the isolated phase shift $\Delta\varphi_{0103}$. Similarly, one could reverse the laser beams ($k \to -k$) in order to isolate $\Delta\varphi_{0303}$. In this case both T and T' have to be in the order of 1 s, otherwise $\Delta\varphi_{0303}$ would be too small.

Unfortunately, for all practical realizations these changes in the orientation are sources of big errors since the acceleration induced phase shift is much larger than the curvature shift. According to row $4_\oplus$ of Table II we have $\Delta\varphi_a \approx 10^6 \Delta\varphi_R$. In order to get proper results the relative error in the contribution of the acceleration has to be of the order of 10^{-6}. Neglecting for simplicity the centrifugal force Eq. (C13) states that the relevant component of the acceleration is given by $a_{\underline{3}} \approx \cos\beta \cos\gamma \, 9.81$ m s^{-2}. To estimate the corresponding allowed error $\delta\beta$ in the fixation of the angle β we expand $a_{\underline{3}}$ around $\beta = 0$. Then the error $\delta a_{\underline{3}}$ is proportional to $\delta\beta^2$, which shows that $\delta\beta$ has to be smaller than about 10^{-3} rad, which is too difficult to manage. Furthermore, for the orientation $\beta = 0$ the component $R_{\underline{0}1\underline{0}3}$ of the curvature tensor vanishes so that we would lose a part of the effect. On the other hand, the expansion around $\beta = \pi/4$ for which $R_{\underline{0}1\underline{0}3}$ is maximal leads to $\delta\beta < 10^{-6}$ rad, which is clearly out of reach.

It should be stressed that the experimental situation is even worsened by the fact that the atomic beam is bent through the acceleration of the Earth. Thus, to match the atoms, the laser beams have to be adjusted after each change of an orientation, although the times T and T' can be held constant by using laser pulses as in Ref. [3].

Anandan [39] and Clauser [8] have made a proposal to eliminate the rotational and accelerational phase shift, respectively. An interferometer with crossing beams ("figure eight") is only sensitive to relative acceleration (curvature). To obtain this configuration in using four running laser waves the waves must be traveling in the same direction. In this case always one of the atomic partial beams must be excited. We therefore lose the possibility of enlarging the times of flight T and T'.

Accordingly, our final conclusion is that it is not possi-

TABLE II. Calculated phase shifts induced by acceleration, rotation, and space-time curvature for the Ramsey devices specified in Table I. $\oplus$ and Pb indicate the measurement of the influence of the Earth or of two lead blocks, respectively. The orientation of the interferometer with respect to the Earth or to the lead blocks is given by the angles α, β, γ, and ϑ. See text and Figs. 2 and 3 for details.

No.	↓ Reference orientation →	$\Delta\varphi_a$ $\beta,\gamma=0$	$\Delta\varphi_\omega$ $\gamma=0,\alpha=\vartheta=\pi/2$	$\Delta\varphi_R$ $\beta=\pi/4,\gamma=0$
$1_\oplus$	Earth, [6]	-0.11	1.9×10^{-4}	-8×10^{-10}
$2_\oplus$	Earth, [6] modified	-1.9×10^{4}	0.55	-0.01
$3_\oplus$	Earth, [38]	-0.17	2.8×10^{-4}	-1.4×10^{-9}
$4_\oplus$	Earth, [38] modified	-4×10^{5}	11.8	-0.4
4_{Pb}	lead, [38] modified			-0.08

ble to demonstrate the influence of space-time curvature on Ramsey interference if the gravitational field of the Earth is used.

To close this section we remark that even if both flight times T and T' are of the order of 1 s one would neither be able to detect the reaction of quantum systems on gravitational radiation (because of the corresponding $R_{0i0j} \approx 5 \times 10^{-28}$ m^{-2}) nor the Lense-Thirring effect, i.e., the dragging of the inertial reference frame in free motion caused by the rotation of the Earth (because of the relevant $\omega_{\rm LT} \approx 5 \times 10^{-14}$ s^{-1}).

B. The space-time curvature of two lead blocks

The origin of the difficulties with the Earth's gravitational field is the fact that the influence of curvature, although measurable as far as its magnitude is concerned, cannot practically be separated from the influence of the acceleration, which is larger by many orders of magnitude. We therefore have to look for other sources of gravitational fields where this separation can be performed easier.

That bodies of laboratory size can produce in their vicinity a curvature comparable to that of the Earth can be seen by the following heuristic argument. Consider for this purpose a spherical body with homogeneous mass density ρ, radius R, and mass $M = 4\pi\rho R^3/3$. Its Newtonian potential at the distance $r \geq R$ is proportional to M/r. As discussed in Sec. V, the acceleration and curvature registered by an observer at rest relative to the body are given by the first and second derivatives of the potential, respectively. Thus $R_{0l0m} \propto M/r^3$ (here we neglect any angular dependence). Accordingly, we obtain near the surface of the body $R_{0l0m} \propto \rho$. The curvature is therefore independent of the radius of the body. It is the same for the Earth and a laboratory sized body if the respective mass densities ρ agree.

The question remains whether for laboratory sized bodies the influence of the curvature can at all be separated from the influence of (a) the acceleration produced by the body itself and (b) the Earth's rotation and gravitational field. To eliminate influence (a) we simply work with two appropriately shaped bodies and place the center of the interferometer at a point $\vec{x} = \vec{0}$, where the first derivative of the generated potential vanishes: $\vec{a} = \vec{\nabla}\phi(0) = \vec{0}$. Note that we are dealing with a Taylor expansion of $\phi(\vec{x})$ [compare Eq. (42)]. In the Newtonian picture the space dependence of $\vec{a}$ is represented by $R_{0l0m}(0)$. A separation from influence (b) is easy. For a fixed interferometer we remove the laboratory sized bodies, thereby changing the interference result by $\Delta\varphi_R$. This is the measured effect.

In the following we discuss a setup with two identical squared blocks of lead which are parallely oriented. They build a gap. The point O in the middle of the gap is taken as the point $\vec{x} = \vec{0}$ discussed above. Here we put the interferometer and orient it relative to the Earth's gravitational field in the following way (compare Fig. 3): The laser beams are traveling horizontally (in the x^3 direction). The atomic beam moves initially vertically upwards (in the x^1 direction) with initial velocity

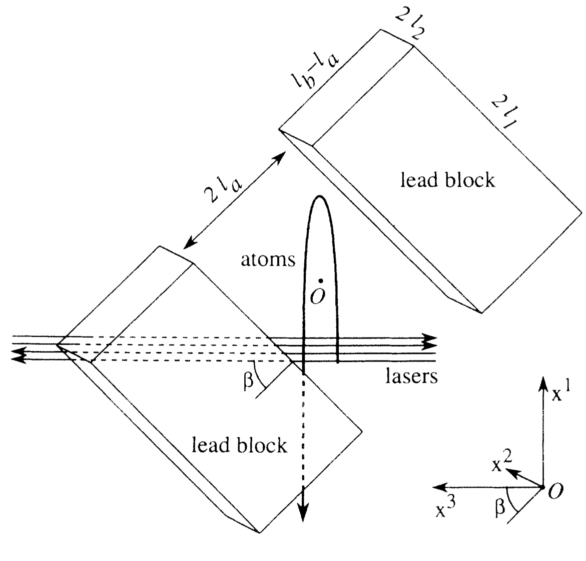

FIG. 3. Orientation of interferometer, lead blocks, and Earth gravitational field for the measurement of the space-time curvature of the lead blocks: The origin of the coordinate system is in the center O of the interferometer in the middle of the gap. The x^1 direction is orthogonal to the surface of the Earth. The interferometer lies in the plane $x^2 = 0$. It forms a kind of atomic fountain. The envelope of the atomic partial beams is shown with a strongly overdone separation of the rising and falling part. The lead blocks are rotated by an angle $-\beta$ around the x^2 axis from the position in which their surfaces are parallel to the coordinate planes. β is chosen to be $\pi/4$ so that the curvature effect becomes maximal.

v_1. This setup could be called a Ramsey fountain. The plane surfaces of the two squared blocks are not chosen to be parallel to the coordinate planes. Instead the blocks are tilted from this position by a rotation with angle $-\pi/4$ around the x^2 axis orthogonal to the interferometer plane. This amounts to the choice $\beta = \pi/4$ in Eq. (D6).

To estimate according to Eqs. (47), (48), and (50) the magnitude of the phase shift $\Delta\varphi_R$ caused by space-time curvature we use two lead blocks specified by $l_1 = l_2 = 0.7$ m, $l_a = 0.5$ m, and $l_b = 1.3$ m as in Fig. 3. The corresponding volume is 1.57 m^3, which amounts to a weight of 17.2×10^3 kg for one block ($\rho = 1.1 \times 10^4$ kg m^{-3}). The acceleration entering $\Delta\varphi_{aR}$ is the negative Earth's acceleration ($a_{\underline{1}} = 9.81$ m s^{-2} and $a_{\underline{2}} = a_3 = 0$). The components of the curvature tensor can be obtained from the result (D6) of Appendix D by inserting $\beta = \pi/4$ and replacing $R_{\underline{0'1'0'1'}}$ by R_{0101} of Eq. (D5), etc. The function $g(u, v, w)$ is defined in Eq. (D2). We omit the details of the calculation. The result is

$$R_{\underline{0103}} \approx 2.98\frac{G\rho}{c^2} \approx 2.43 \times 10^{-23}\ \mathrm{m}^{-2}\,,$$

$$R_{\underline{0303}} \approx -0.99\frac{G\rho}{c^2} \approx 8.5 \times 10^{-24}\ \mathrm{m}^{-2}\,, \qquad (51)$$

where G is Newton's constant.

As the interferometer we choose the modified Ramsey device specified in row 4 of Table I. The traveling time

T' between lasers 2 and 3 is then given by $T' = 2v_1/a_{\underline{1}} \approx 1.02$ s. The height of the fountain which has the point O in its middle is $v_1^2/(2a_{\underline{1}}) \approx 1.27$ m, so that the atoms do not enter the blocks. For the contributing phase shifts we obtain $\Delta\varphi_{0103} \approx -0.236$, $\Delta\varphi_{aR} \approx 0.158$, and $\Delta\varphi_{0303}^{(-1)} \approx 10^{-6}$. The resulting curvature induced phase shift is then

$$\Delta\varphi_R \approx -0.08\,. \tag{52}$$

This should be measurable. The essential limitation of the Ramsey setup is the fact that the time T must be smaller than the lifetime of the excited state of the atoms (compare Fig. 1). There is no such limitation in the fountain of Kasevich and Chu [3] so that for it we can expect the phase shift to be larger by one or two powers of 10.

To sum up, for the gravitational field of two lead blocks we have shown that the curvature induced shift $\Delta\varphi_R$ of the Ramsey fringes is within the range of measurability for experimental setups which may be realized in the near future. They should make it possible to demonstrate experimentally for the first time the influence of space-time curvature on a quantum system under laboratory conditions.

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APPENDIX A

We use the conventions of [23], i.e., $\eta_{\alpha\beta} = \mathrm{diag}(-1,1,1,1)$ and $R^{\mu}{}_{\nu\alpha\beta} = \Gamma^{\mu}{}_{\nu\beta,\alpha} - \cdots$. Latin indices run from 1 to 3 and tetrad indices are underlined. We sum over equal indices regardless of their position (upper or lower). Nonsummation over equal indices is denoted by setting one of them in parentheses. The tetrads fulfill $e^{\underline{\alpha}}{}_{\mu} e^{\underline{\beta}}{}_{\nu} g^{\mu\nu} = \eta^{\underline{\alpha}\underline{\beta}}$. The Dirac matrices obey the anticommutator relation $\{\gamma_{\underline{\alpha}}, \gamma_{\underline{\beta}}\} = 2\eta_{\underline{\alpha}\underline{\beta}} \mathbf{1}_4$. Note that the sign convention for the metric implies that the $\gamma_{\underline{\mu}}$ matrices have an additional factor of i compared to the matrices used in particle physics. We define $\alpha_{\underline{i}} \equiv \gamma_{\underline{0}}\gamma_{\underline{i}}$, $\gamma_{\underline{5}} \equiv i\gamma^{\underline{0}}\gamma^{\underline{1}}\gamma^{\underline{2}}\gamma^{\underline{3}}$, and $\Sigma_{\underline{k}} \equiv -i\varepsilon_{ijk}\gamma_{\underline{i}}\gamma_{\underline{j}}/2$. The spinor connection is given by

$$\Gamma_\mu = -\tfrac{1}{4}\gamma_{\underline{\alpha}}\gamma_{\underline{\beta}}\, e^{\underline{\alpha}\nu}\nabla_\mu e^{\underline{\beta}}{}_{\nu}\,, \tag{A1}$$

where ∇_μ is the covariant derivative acting on vectors. The hypersurface independent scalar product between spinors is [18]

$$(\phi,\psi) = -\int d^3x\sqrt{-g}\;\phi^+\gamma^{\underline{0}}\, e^0{}_{\underline{\mu}}\,\gamma^{\underline{\mu}}\,\psi. \tag{A2}$$

The contravariant components of the metric in Fermi coordinates are up to order $O((x^l)^2)$ [compare (8)]

$$\begin{aligned} g^{00} &= -1 + 2(\vec{a}\cdot\vec{x}) - 3(\vec{a}\cdot\vec{x})^2 + R_{\underline{0}k\underline{0}l}x^k x^l\,,\\ g^{0i} &= (1-2\vec{a}\cdot\vec{x})(\vec{\omega}\times\vec{x})^i - \tfrac{2}{3}R_{\underline{0}\underline{l}\underline{i}\underline{m}}x^l x^m\,,\\ g^{ij} &= \delta^{ij} + \tfrac{1}{3}R_{\underline{i}\underline{l}\underline{j}\underline{m}}x^l x^m - (\vec{\omega}\times\vec{x})^i(\vec{\omega}\times\vec{x})^j\,. \end{aligned} \tag{A3}$$

After some lengthy algebra we find for the Christoffel symbols [to $O(x^l)$]

$$\begin{aligned} \Gamma^0{}_{00} &= \dot{\vec{a}}\cdot\vec{x} + \vec{a}\cdot(\vec{\omega}\times\vec{x})\,,\\ \Gamma^i{}_{00} &= R_{\underline{0i0l}}x^l + (1+\vec{a}\cdot\vec{x})a_{\underline{i}} + (\dot{\vec{\omega}}\times\vec{x})_i\\ &\quad -[(\vec{\omega}\times\vec{x})\times\vec{\omega}]_i\,,\\ \Gamma^0{}_{0i} &= R_{\underline{0i0l}}x^l + (1-\vec{a}\cdot\vec{x})a_{\underline{i}}\,,\\ \Gamma^j{}_{0i} &= R_{\underline{0mij}}x^m - \varepsilon_{\underline{j}il}\omega^{\underline{l}} - a_{\underline{i}}(\vec{\omega}\times\vec{x})_j\,,\\ \Gamma^0{}_{ij} &= -\tfrac{1}{3}(R_{\underline{0imj}} + R_{\underline{0jmi}})x^m\,,\\ \Gamma^k{}_{ij} &= -\tfrac{1}{3}(R_{\underline{kijm}} + R_{\underline{kjim}})x^m \end{aligned} \tag{A4}$$

and for the tetrads

$$\begin{aligned} e^{\underline{0}}{}_0 &= 1 + (\vec{a}\cdot\vec{x}) - \tfrac{1}{2}R^{\underline{0}}{}_{\underline{l0m}}x^l x^m\,,\\ e^{\underline{j}}{}_0 &= -\tfrac{1}{2}R^{\underline{j}}{}_{\underline{l0m}}x^l x^m + \varepsilon_{\underline{jlm}}\omega^{\underline{l}}x^m\,,\\ e^{\underline{0}}{}_i &= -\tfrac{1}{6}R^{\underline{0}}{}_{\underline{lim}}x^l x^m\,,\\ e^{\underline{j}}{}_i &= \delta^j{}_i - \tfrac{1}{6}R^{\underline{j}}{}_{\underline{lim}}x^l x^m\,. \end{aligned} \tag{A5}$$

The spinor connections in Fermi coordinates are

$$\begin{aligned} \Gamma_0 &= \tfrac{1}{2}\gamma_{\underline{0}}\gamma_{\underline{i}}(R_{\underline{i00m}}x^m - a_{\underline{i}})\\ &\quad + \tfrac{1}{4}\gamma_{\underline{i}}\gamma_{\underline{j}}(R_{\underline{ij0m}}x^m + \varepsilon_{\underline{ijl}}\omega_{\underline{l}})\,,\\ \Gamma_i &= \tfrac{1}{4}\gamma_{\underline{0}}\gamma_{\underline{j}}R_{\underline{0jmi}}x^m + \tfrac{1}{8}\gamma_{\underline{j}}\gamma_{\underline{k}}R_{\underline{jkim}}x^m \end{aligned} \tag{A6}$$

and the scalar product is given by

$$(\phi,\psi) = \int d^3x\phi^+[1 - \tfrac{1}{6}R_{\underline{ilim}}x^l x^m - \tfrac{1}{6}R_{\underline{0lim}}x^l x^m \alpha_{\underline{i}}]\psi\,. \tag{A7}$$

APPENDIX B

Here we generalize the calculation of the corrections to the Coulomb potential in Fermi coordinates given by Parker [18] to the case of an accelerating and rotating observer.

With reference to the vector potential A_μ the covariant Maxwell equations in curved spacetime are given by

$$\nabla^\lambda\nabla_\lambda A_\mu - R_\mu{}^\nu A_\nu = -4\pi j_\mu\,, \tag{B1}$$

where we have used the Lorentz gauge $\nabla_\mu A^\mu = 0$. In order to determine the influence of noninertial motion and space-time curvature on the electromagnetic field of a point charge we write the Maxwell equations in adapted Fermi coordinates. Since the charge is now resting at the origin of the coordinate system the current is given by

$$j_\mu = -Ze\delta(\vec{x})\;(1,0,0,0)\,. \tag{B2}$$

Inserting the metric (8) and the current (B2) into Eq. (B1) we arrive at

$$[\delta^{ij} + \tfrac{1}{3}R_{\underline{iljm}}x^l x^m - (\vec{\omega}\times\vec{x})_i(\vec{\omega}\times\vec{x})_j]\partial_i\partial_j A_0 + A_{0,i}\{-\tfrac{2}{3}R_{\underline{im}}x^m - \tfrac{5}{3}R_{\underline{0i0m}}x^m + (\vec{a}\cdot\vec{x}-1)a_{\underline{i}}$$
$$+[(\vec{\omega}\times\vec{x})\times\vec{\omega}]_i\} + A_{i,j}\{2R_{\underline{0mij}}x^m + 2[(\vec{\omega}\times\vec{x})_i a_{\underline{j}} - (\vec{\omega}\times\vec{x})_j a_{\underline{i}}] + 2\varepsilon_{\underline{ijl}}\omega_{\underline{l}}\} = 4\pi Ze\delta(\vec{x}) \quad \text{(B3)}$$

and

$$[\delta^{ij} + \tfrac{1}{3}R_{\underline{iljm}}x^l x^m - (\vec{\omega}\times\vec{x})_i(\vec{\omega}\times\vec{x})_j]\partial_i\partial_j A_k + A_{0,j}[\tfrac{2}{3}(R_{\underline{0kmj}} + R_{\underline{0jmk}})x^m - 2(\vec{\omega}\times\vec{x})_j a_{\underline{k}}]$$
$$+A_{j,i}[\tfrac{2}{3}(R_{\underline{jikm}} + R_{\underline{jkim}})x^m + 2\varepsilon_{\underline{jkl}}\omega_{\underline{l}}(\vec{\omega}\times\vec{x})_i] + A_{k,j}\{\tfrac{1}{3}(R_{\underline{0j0m}} - 2R_{\underline{jm}})x^m + (1-\vec{a}\cdot\vec{x})a_{\underline{j}}$$
$$+[(\vec{\omega}\times\vec{x})\times\vec{\omega}]_j\} - \tfrac{2}{3}R_{\underline{k}}{}^{\lambda}A_\lambda + \tfrac{1}{3}R_{\underline{0k0j}}A_j + (\vec{\omega}\times\vec{a})_{\underline{k}}A_0 - a_{\underline{k}}(\vec{a}\cdot\vec{A}) + [\vec{\omega}\times(\vec{A}\times\vec{\omega})]_k = 0\,. \quad \text{(B4)}$$

All equations are understood to be correct only up to second order in the spatial coordinates x^l, derivatives of second order equations are of first order, and so on.

We now divide the electromagnetic potential into two parts, one unperturbed, which is simply the Coulomb potential in flat space-time, and one $A_\mu^{(1)}$, which contains the corrections up to order $O(x^l)$:

$$A_0 = -\frac{Ze}{r} + A_0^{(1)}\,, \quad A_i = A_i^{(1)}\,. \quad \text{(B5)}$$

The second derivatives of $A_\mu^{(1)}$ are then of the order $O((x^l)^{-1})$, so that we can drop all terms which are of higher order in the Maxwell equations. This leads to

$$\partial_i\partial_i A_0^{(1)} - \vec{a}\cdot\vec{\nabla}A_0^{(1)} + 2\varepsilon_{\underline{ijl}}\omega_{\underline{l}}A_{i,j}^{(1)}$$
$$= \frac{Ze}{r^3}\{\vec{a}\cdot\vec{x}(1-\vec{a}\cdot\vec{x}) + (\vec{\omega}\times\vec{x})^2$$
$$+\tfrac{1}{3}(R_{\underline{lm}} + 4R_{\underline{0l0m}})x^l x^m\}\,,$$
$$\text{(B6)}$$
$$\partial_i\partial_i A_k^{(1)} + a_{\underline{j}}A_{k,j}^{(1)}$$
$$= \frac{Ze}{r^3}\{-\tfrac{2}{3}R_{\underline{0jmk}}x^m x^j + \tfrac{2}{3}R_{\underline{k0}}r^2 + (\vec{\omega}\times\vec{a})_{\underline{k}}r^2\}\,.$$

The curvature terms of Eq. (B6) are in disagreement with Eq. (7.9) of Ref. [18]. The solution of these equations is found to be

$$A_0^{(1)} = -\frac{Ze}{2}\frac{\vec{a}\cdot\vec{x}}{r} + Ze\{\tfrac{1}{4}\vec{\omega}^2 - \tfrac{3}{8}\vec{a}^2 + \tfrac{1}{12}(R + 5R_{\underline{00}})\}r$$
$$+Ze\{\tfrac{1}{4}\omega_{\underline{l}}\omega_{\underline{m}} + \tfrac{1}{8}a_{\underline{l}}a_{\underline{m}} - \tfrac{1}{12}(R_{\underline{lm}} + 4R_{\underline{0l0m}})\}\frac{x^l x^m}{r}\,,$$
$$\text{(B7)}$$
$$A_k^{(1)} = \frac{Ze}{2}(R_{\underline{0k}} + (\vec{\omega}\times\vec{a})_{\underline{k}})r + \frac{Ze}{6}R_{\underline{0lmk}}\frac{x^l x^m}{r}\,,$$

which satisfies the Lorentz condition. Again, the curvature terms are slightly different from those given in Ref. [18].

APPENDIX C

In this appendix we sketch the derivation of the components of the Riemannian curvature tensor in Fermi coordinates for the rotating Earth, modeled by the Schwarzschild metric, as seen by an observer resting on it and for two lead blocks in the limit of weak gravity.

The first case was already studied by Parker and Pimentel [40] for an observer in radial and circular geodesic motion. We will modify their results for the world line $P_0(\tau)$ given by $\dot{r} = \dot{\vartheta} = 0$ and $\dot{\varphi} := \tilde{\omega}$ with $\tilde{\omega} = 2\pi$ day^{-1} given by the rotation of the Earth. The overdot denotes the derivative with respect to the observers proper time τ. Let $R_s = 2GM/c^2$ be the Schwarzschild radius of the central body where M is its mass and c is reinserted ($R_s \approx 8.9$ mm for the Earth). G is Newton's constant. Defining $X \equiv 1 - R_s/r$, the components of the curvature tensor in standard Schwarzschild coordinates (see, e.g., Ref. [40]) are given by

$$R_{rtrt} = -\frac{R_s}{r^3}\,, \quad R_{\vartheta t\vartheta t} = \frac{R_s X}{2r}\,,$$
$$R_{\varphi t\varphi t} = \frac{R_s X}{2r}\sin^2\vartheta\,, \quad R_{r\vartheta r\vartheta} = -\frac{R_s}{2rX}\,, \quad \text{(C1)}$$
$$R_{\vartheta\varphi\vartheta\varphi} = R_s r\sin^2\vartheta, \quad R_{r\varphi r\varphi} = -\frac{R_s}{2rX}\sin^2\vartheta\,.$$

The acceleration a^μ of the observer can be read off from its equation of motion

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda = a^\mu\,, \quad \text{(C2)}$$

which describes the world line $P_0(\tau)$ with tangent vector $u^\mu = \dot{x}^\mu$. Taking into account the conditions $u^\mu u_\mu = -1$ and $u^\mu a_\mu = 0$, we find

$$a^r = \frac{R_s}{2r^2} - \tilde{\omega}^2 r\left(1 - \frac{3R_s}{2r}\right)\sin^2\vartheta\,, \quad a^\vartheta = -\tilde{\omega}^2\sin\vartheta\cos\vartheta\,. \quad \text{(C3)}$$

We observe that the first part of a^r is the negative of Newton's acceleration. This is reasonable with regard to the physical meaning of a^μ since the surface of the Earth prevents the observer from falling freely. The second part and a^ϑ represent the centrifugal force. Note that in a^r a general relativistic correction $-3R_s/2$ is present. This is negligible for the Earth, but is of interest in the case of a black hole where for $r < 1.5R_s$ the radial component of the centrifugal force changes direction (see also Ref. [41]).

We assume the comoving tetrad ($e^{\alpha}_{\underline{0}'} = u^{\alpha}$) to be fixed to the rotating Earth according to $e_{\underline{1}'} \sim \partial_{\vartheta}$ and $e_{\underline{3}'} \sim \partial_r$ (dashed lines in Fig. 2). With $e_{\underline{0}'} \propto \partial_{\tau}$ and $e_{\underline{\alpha}'} \cdot e_{\underline{\beta}'} = \eta_{\underline{\alpha}'\underline{\beta}'}$ it is found to be

$$e_{\underline{0}'} = \frac{W}{\sqrt{X}}\partial_t + \tilde{\omega}\partial_{\varphi}\,, e_{\underline{1}'} = r^{-1}\partial_{\vartheta}\,,$$
$$e_{\underline{2}'} = \frac{\tilde{\omega} r \sin\vartheta}{\sqrt{X}}\partial_t + \frac{W}{r\sin\vartheta}\partial_{\varphi}\,, e_{\underline{3}'} = \sqrt{X}\partial_r\,, \tag{C4}$$

where $W \equiv \sqrt{1 + r^2\tilde{\omega}^2 \sin^2\vartheta}$. The rotation ω^{α} of the tetrad represents physically the rotation relative to a Fermi transported tetrad, which may be fixed by gyroscopes. It is this rotation that enters the metric in Fermi coordinates as in (8). Mathematically ω^{α} is given by (compare Ref. [23])

$$\frac{De^{\mu}_{\underline{\alpha}'}}{D\tau} = -[a^{\mu}u^{\nu} - a^{\nu}u^{\mu} + u_{\alpha}\omega_{\beta}\varepsilon^{\alpha\beta\mu\nu}]e_{\underline{\alpha}'\nu}\,. \tag{C5}$$

Inserting (C5) we find

$$\omega_r = \frac{W\tilde{\omega}\cos\vartheta}{\sqrt{X}}\,, \quad \omega_{\vartheta} = -\frac{W\tilde{\omega} r\sin\vartheta}{\sqrt{X}}\left(1 - \frac{3R_s}{2r}\right). \tag{C6}$$

Note again the general relativistic corrections proportional to R_s/r.

Still referring to the tetrad (C4) we work out the components of a_{μ}, ω_{μ}, and $R_{\mu\nu\rho\sigma}$ on the world line $P_0(\tau)$ in the respective Fermi coordinates adjusted to the tetrad. This amounts to a projection with the tetrad. Using

$$R_{\underline{\alpha}'\underline{\beta}'\underline{\gamma}'\underline{\delta}'} = R_{\mu\nu\rho\sigma}e^{\mu}_{\underline{\alpha}'}e^{\nu}_{\underline{\beta}'}e^{\rho}_{\underline{\gamma}'}e^{\sigma}_{\underline{\delta}'} \tag{C7}$$

we find, for the curvature in the Fermi coordinates of the observer,

$$R_{\underline{0}'\underline{1}'\underline{0}'\underline{1}'} = -R_{\underline{2}'\underline{3}'\underline{2}'\underline{3}'} = \frac{R_s}{2r^3}\{1 + 3r^2\tilde{\omega}^2\sin^2\vartheta\}\,,$$
$$R_{\underline{0}'\underline{2}'\underline{0}'\underline{2}'} = -R_{\underline{1}'\underline{3}'\underline{1}'\underline{3}'} = \frac{R_s}{2r^3}\,,$$
$$R_{\underline{0}'\underline{3}'\underline{0}'\underline{3}'} = -R_{\underline{1}'\underline{2}'\underline{1}'\underline{2}'} = -\frac{R_s}{r^3}\{1 + \frac{3}{2}r^2\tilde{\omega}^2\sin^2\vartheta\}\,,$$
$$R_{\underline{0}'\underline{1}'\underline{2}'\underline{1}'} = -R_{\underline{0}'\underline{3}'\underline{2}'\underline{3}'} = \frac{3R_s}{2r^2}W\tilde{\omega}\sin\vartheta\,. \tag{C8}$$

For the freely falling observer see Ref. [40]. The case of gravitational waves is discussed in Ref. [42]. Correspondingly we obtain, for the acceleration,

$$a_{\underline{1}'} = -r\tilde{\omega}^2\sin\vartheta\cos\vartheta\,,$$
$$a_{\underline{3}'} = \frac{R_s}{2r^2\sqrt{X}} - \frac{\tilde{\omega}^2}{\sqrt{X}}\left(r - \frac{3}{2}R_s\right)\sin^2\vartheta\,, \tag{C9}$$

and for the rotation,

$$\omega_{\underline{1}'} = -\frac{W\tilde{\omega}\sin\vartheta}{\sqrt{X}}\left(1 - \frac{3R_s}{2r}\right)\,, \quad \omega_{\underline{3}'} = W\tilde{\omega}\cos\vartheta\,. \tag{C10}$$

To describe arbitrary orientations of the experimental setup "fixed" to the tetrad, we go over to a different orientation of the three vectors $e_{\underline{i}'} \to e_{\underline{i}}$ by means of a Lorentz transformation. Vector components change according to

$$e_{\underline{\alpha}} = \Lambda_{\underline{\alpha}}{}^{\underline{\beta}'}\, e_{\underline{\beta}'}\,,$$
$$R_{\underline{\alpha\beta\gamma\delta}} = \Lambda_{\underline{\alpha}}{}^{\underline{\mu}'}\,\Lambda_{\underline{\beta}}{}^{\underline{\nu}'}\,\Lambda_{\underline{\gamma}}{}^{\underline{\rho}'}\,\Lambda_{\underline{\delta}}{}^{\underline{\sigma}'}\,R_{\underline{\mu}'\underline{\nu}'\underline{\rho}'\underline{\sigma}'}\,. \tag{C11}$$

In detail we perform three rotations: the first around the x^3 axis with angle α (turning the apparatus on the earth's surface), the second around the *new* x^2 axis with angle β (the axis perpendicular to the atom beam and the laser waves), and the third around the *new* x^1 axis with angle γ (turning around the atomic beam). This results in a matrix

$$\Lambda_{\underline{\alpha}}{}^{\underline{\beta}'} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha\cos\beta & \sin\alpha\cos\beta & -\sin\beta \\ 0 & -\sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma & \cos\alpha\cos\gamma + \sin\alpha\sin\beta\sin\gamma & \cos\beta\sin\gamma \\ 0 & \sin\alpha\sin\gamma + \cos\alpha\sin\beta\cos\gamma & -\cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma & \cos\beta\cos\gamma \end{pmatrix}. \tag{C12}$$

The relevant components of the various physical quantities which enter into the phase shift (44) are now given by

$$a_{\underline{3}} = \frac{R_s}{2r^2\sqrt{X}}\cos\beta\cos\gamma - \tilde{\omega}^2\sin\vartheta\left\{r\cos\vartheta[\sin\alpha\sin\gamma + \cos\alpha\sin\beta\cos\gamma] + \frac{r - 3R_s/2}{\sqrt{X}}\sin\vartheta\cos\beta\cos\gamma\right\}, \tag{C13}$$

$$\omega_{\underline{2}} = W\tilde{\omega}\left\{\cos\vartheta\cos\beta\sin\gamma + \frac{2r - 3R_s}{2r\sqrt{X}}\sin\vartheta[\sin\alpha\cos\gamma - \cos\alpha\sin\beta\sin\gamma]\right\}, \tag{C14}$$

$$R_{0103} = \frac{3R_s}{2r^3}\cos\beta\,\{\sin\beta\cos\gamma + r^2\tilde{\omega}^2\sin^2\vartheta\,[\sin\alpha\cos\alpha\sin\gamma + (1+\cos^2\alpha)\sin\beta\cos\gamma]\}\,, \tag{C15}$$

$$R_{0303} = \frac{R_s}{2r^3}\{1 - 3W^2\cos^2\beta\cos^2\gamma + 3r^2\tilde{\omega}^2\sin^2\vartheta\,(\sin\alpha\sin\gamma + \cos\alpha\sin\beta\cos\gamma)^2\}\,.$$

In Sec. VII r becomes the radius $R_\oplus \approx 6378$ km of the Earth.

APPENDIX D

We derive the components R_{0l0m} of the Riemannian curvature tensor for two squared lead blocks of laboratory size. The limit of weak gravity discussed in Sec. V will be appropriate and we may therefore base our calculation on Eq. (40) using the Newtonian potential $\phi(\vec{x})$ given by Eq. (36). We make again the convention that the frame of reference remains always fixed to the interferometer as specified in Sec. VII.

In a first step of the calculation we orientate the two identical blocks such that their surfaces are parallel to the coordinate planes and that they are separated in the x^3 direction (direction of the laser beams) by $2l_a$. The first block has constant mass density ρ in the volume $-l_1 < x^1 < l_1$, $-l_2 < x^2 < l_2$, and $0 < l_a < x^3 < l_b$. The second block of the same mass density ρ lies symmetrically with respect to the plane $x^3 = 0$. The center of the interferometer is in the middle of the gap in the origin $\vec{x} = \vec{0}$ of the coordinate system with an atomic beam moving initially in the x^1 direction. Because of the symmetry of the setup the acceleration $\vec{a}_{Pb}$ caused by the blocks is vanishing at this position,

$$\vec{a}_{\rm Pb}(0) = 0\,. \tag{D1}$$

Inserting the specification in Eq. (40) and introducing the functions

$$g(u,v,w) := \operatorname{sgn}(uvw)\arcsin\left[\frac{(w^2-u^2)(v^2+u^2)-2u^2w^2}{(v^2+u^2)(w^2+u^2)}\right], \tag{D2}$$

$$h(u,v) := \frac{1}{v}\operatorname{arcsinh}\left(\frac{u}{v}\right)$$

and the abbreviations $l_1^\pm := \pm l_1 - x^1$, $l_2^\pm := \pm l_2 - x^2$, and $w_q^\pm := \sqrt{(l_1^\pm)^2 + (x^3+q)^2}$ as well as $l_{a\pm} := l_a \pm x^3$ and $l_{b\pm} := l_b \pm x^3$, we find, for the relevant components,

$$R_{0103}(\vec{x}) = \frac{-\kappa\rho}{8\pi}\{h(l_2^+,w_a^+) - h(l_2^-,w_a^+) - h(l_2^+,w_a^-) + h(l_2^-,w_a^-) - h(l_2^+,w_b^+) + h(l_2^-,w_b^+) + h(l_2^+,w_b^-) - h(l_2^-,w_b^-) + h(l_2^+,w_{-b}^+) - h(l_2^-,w_{-b}^+) - h(l_2^+,w_{-b}^-) + h(l_2^-,w_{-b}^-) - h(l_2^+,w_{-a}^+) + h(l_2^-,w_{-a}^+) + h(l_2^+,w_{-a}^-) - h(l_2^-,w_{-a}^-)\} \tag{D3}$$

and

$$R_{0303}(\vec{x}) = \frac{\kappa\rho}{16\pi}\{-g(l_{a+},l_1^+,l_2^+) + g(l_{a+},l_1^-,l_2^+) + g(l_{a+},l_1^+,l_2^-) - g(l_{a+},l_1^-,l_2^-) + g(l_{b+},l_1^+,l_2^+) - g(l_{b+},l_1^-,l_2^+) - g(l_{b+},l_1^+,l_2^-) + g(l_{b+},l_1^-,l_2^-) + g(l_{b-},l_1^+,l_2^+) - g(l_{b-},l_1^-,l_2^+) - g(l_{b-},l_1^+,l_2^-) + g(l_{b--},l_1^-,l_2^-) - g(l_{a-},l_1^+,l_2^+) + g(l_{a-},l_1^-,l_2^+) + g(l_{a-},l_1^+,l_2^-) - g(l_{a-},l_1^-,l_2^-)\}\,. \tag{D4}$$

The other terms $R_{0l0m}(\vec{x})$ are of a similar structure. We have worked out the space dependence of R_{0l0m} to be able to test whether the curvature components are sufficiently constant throughout the interferometer. For the specification of the experimental setup given in Sec. VIII this turns roughly out to be the case. Our approximation of Sec. IV is therefore justified. Hence it is sufficient to restrict to the curvature at the point O in the middle between the blocks ($\vec{x} = \vec{0}$):

$$\begin{aligned} R_{0101} &= \frac{\kappa\rho}{2\pi}[g(l_1,l_2,l_b) - g(l_1,l_2,l_a)]\,,\\ R_{0202} &= \frac{\kappa\rho}{2\pi}[g(l_2,l_1,l_b) - g(l_2,l_1,l_a)]\,,\\ R_{0303} &= \frac{\kappa\rho}{2\pi}[g(l_b,l_1,l_2) - g(l_a,l_1,l_2)]\,. \end{aligned} \tag{D5}$$

All other components R_{0l0m} are zero.

In Eq. (D5) we have worked out the coordinate components of the Riemann tensor which correspond to Eq. (C1) in the calculation for the Earth's gravitational field. In the approximation of a weak gravitational field the coordinate components (D5) agree according to Eq. (41) with the projections on the comoving tetrad representing the measured quantities so that Eq. (D5) is in fact already the analog of Eq. (C8): $R_{\underline{0'l'0'm'}} = R_{0l0m}$.

Finally, for the experimental application, we change the orientation of the interferometer (to which the coordinate system remains attached) relative to the blocks. We rotate the interferometer around the x^2 axis orthogonal to its plane by an angle β (or equivalently the blocks by $-\beta$). The resulting setup is shown for $\beta = \pi/4$ in Fig. 3 where in addition the x^1 axis is oriented vertically on the Earth. The components of the curvature generated by the rotated blocks are worked out according to Eq. (C11) in using Eq. (C12) with $\alpha = \gamma = 0$. We obtain

$$R_{\underline{0103}} = (R_{\underline{0'1'0'1'}} - R_{\underline{0'3'0'3'}}) \sin\beta\cos\beta\,,$$
$$R_{\underline{0303}} = R_{\underline{0'1'0'1'}} \sin^2\beta + R_{\underline{0'3'0'3'}} \cos^2\beta\,, \tag{D6}$$

where the primed components agree with those of Eq. (D5). This is the final result, which may directly be inserted into Eqs. (47)–(50).

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