Transition radiation and Bragg resonances

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Both spontaneous and stimulated radiation emitted by an electron beam traversing a periodic medium are studied in the approximation of neglecting the influence of the field on the motion of the electrons. The results are valid both in the vicinity of the Bragg domain as well as far from it. In the latter case, we recover previous results with some corrections. In the Bragg domains the result is of interest since in this case the emission of quanta increases considerably.

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I. INTRODUCTION

Ginzburg and Frank [1] first suggested the existence of "transition radiation" (TR) when charged particles move from one medium to another with different dielectric constant. TR has been studied both theoretically and experimentally for a long time [2–10]. The interest in the problem is due to the fact that TR is a bright source of x rays. As noted in Ref. [11], TR produced per electron is at least two orders of magnitude greater than synchrotron radiation. This effect grows considerably if TR is generated in a periodic medium where there is phase addition between the emitted photons at each interface [so-called resonant transition radiation (RTR)]. That is why the study of RTR is a subject of special interest both experimentally [11–18] and theoretically [18–22]. We refer to Refs. [11,12] for a discussion of the applications of RTR to medical imaging, spectroscopy, microscopy, and x-ray lasers. The effect of intensification of TR is even more pronounced in the vicinity of the Bragg domain which thus acquires special interest. Thus far, the problem has been discussed only in Ref. [22] where an exact treatment of the Maxwell equations in a periodic medium is presented. The results were, however, obtained in terms of the solutions of an infinite-rank system of coupled equations in a rather complicated manner. Only outside a Bragg domain can one decouple the system and derive an answer which has proved to be equivalent to the work in Refs. [18–20].

RTR in the Bragg domains is the main subject of this work; our results are based on the approximation of neglecting the influence of the radiated field on the motion of the radiating particle. RTR of wavelength λ which occurs in the direction given by the angle θ measured from the velocity $\mathbf{v}$ of the particle satisfies the following condition [23,24]:

$$\langle \epsilon \rangle^{1/2}\cos\theta = c/v - n\lambda/l \; , \tag{1}$$

where $\langle \epsilon \rangle$ is the average dielectric index, l is the period of the spatially varying dielectric index, c is the velocity of light, and n is an integer. At first, the theory of TR was studied within the Wentzel-Kramers-Brillouin (WKB) approximation [24], provided that the period l greatly exceeds the wavelength λ. However, as noted by Kaplan and Datta [18–20], progress in coating technology (for example, molecular-beam epitaxy [25]), has made it possible to grow periodic multilayered microstructures with layers less than 5 mm thick. Using these structures, one can obtain radiation in the range 1–30 nm with the aid of nonrelativistic electrons of energies 70–300 keV. To study such systems, it is necessary to go beyond the limits of the WKB approximation. For example, in Refs. [18–21], RTR generated by an electron traversing a periodic stratified medium was considered to be the result of interference of waves emitted at different interfaces ϵ_1/ϵ_2 and ϵ_2/ϵ_1. As pointed out by Pardo and Andre [22], this approach is justified if $|\Delta\epsilon/\epsilon| \ll 1$ and the system is not in the vicinity of a Bragg domain.

A number of people have also explored stimulated emission in cases of ordinary Cerenkov radiation [26–28] and TR [8–10,18–20]. It should be mentioned that the results for the stimulated emission deduced from quantum-mechanical and classical approaches are at variance, in contrast to the case of spontaneous emission [18,29] (for a detailed summary, see Ref. [18]). Concerning this point, we note that in the case of TR the current behaves classically in the sense that the radiation does not influence it (i.e., the recoil under emission is negligible) and consequently the quantum-mechanical and classical results should coincide [30]. In this paper, we adopt the quantum-mechanical approach so that the problems of spontaneous and stimulated emission can be treated within the same formalism.

The paper is organized as follows. In Sec. II we obtain a general expression for the spectral density of the radiated intensity in terms of the Bloch function of the electric field in a periodic medium $\epsilon(z)$ in the approximation in which we neglect the influence of the generated radiation on the motion of the particle. In Sec. III, the general result is specialized to the case of a periodic stratified structure. In the Appendix, we briefly outline a solution of the classical problem of TR and demonstrate the identity of the classical and quantum-mechanical results.

II. QUANTUM THEORY OF TRANSITION RADIATION IN A PERIODIC MEDIUM

We consider the case when the wavelength of the radiation λ is such that

$$\lambda \gg \lambda_e \ , \quad (2)$$

where λ_e is the de Broglie wavelength of the electron in the beam. Under this condition, the current behaves classically. The radiation of γ depends on the Lagrangian of the interaction between the electrons and the field [31],

$$\hat{L}^{\rm int} = \frac{e}{c}\hat{j}^{\mu}(\mathbf{r},t)\hat{A}_{\mu}(\mathbf{r},t) \ , \quad (3)$$

where $\hat{j}^{\mu}$ and $\hat{A}_{\mu}$ are the four-dimensional current density and potential, respectively. It is convenient to choose the Coulomb gauge of the potentials whereby

$$\hat{A}_0 = \hat{\phi} = 0, \ \ \mathrm{div}\epsilon\hat{\mathbf{A}} = 0, \ \ \hat{\mathbf{E}} = -c^{-1}\partial\hat{\mathbf{A}}/\partial t \ . \quad (4)$$

Thus we have from Eq. (3)

$$\hat{L}^{\rm int} = \frac{e}{c}\hat{\mathbf{j}}\cdot\hat{\mathbf{A}} \ . \quad (5)$$

Let us expand the operator $\hat{\mathbf{A}}$ in a series of Bloch modes in the periodic medium $\epsilon(z)$:

$$\hat{\mathbf{A}} = \sum_{\nu,\mathbf{k}} (\hbar/2\Omega\omega)^{1/2} c [\hat{a}_{\mathbf{k},\nu}\mathbf{e}_{\mathbf{k},\nu}(z)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + \hat{a}^{\dagger}_{\mathbf{k},\nu}\mathbf{e}^{*}_{\mathbf{k},\nu}(z)e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}] \ , \quad (6)$$

where Ω is a normalization volume and $\mathbf{k}$ is the Bloch wave vector. The vectorial function $\mathbf{e}_{\mathbf{k},\nu}(z)\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)]$ corresponds to the Bloch mode with a wave vector $\mathbf{k}$ and polarization ν, and it is normalized as follows:

$$\int_0^l |\mathbf{e}_{\mathbf{k},\nu}(z)|^2 \epsilon(z) dz = 4\pi l \ . \quad (7)$$

The frequency ω depends on both $\mathbf{k}$ and ν and is governed by the dispersion relation

$$\omega \equiv \omega_{\mathbf{k},\nu} = \omega_{\nu}(k_z, k_{\perp}) \ , \quad (8)$$

where the specific form of $\omega_{\nu}(k_z,k_{\perp})$ depends on the function $\epsilon(z)$. Under this condition, the Hamiltonian of the γ quanta is given by

$$\hat{H}_{\gamma} = \tfrac{1}{2}\sum_{\nu,\mathbf{k}} \hbar\omega_{\mathbf{k},\nu}(\hat{a}_{\mathbf{k},\nu}\hat{a}^{\dagger}_{\mathbf{k},\nu} + \hat{a}^{\dagger}_{\mathbf{k},\nu}\hat{a}_{\mathbf{k},\nu}) \ , \quad (9)$$

where $\hat{a}^{\dagger}$ and $\hat{a}$ are the creation and annihilation operators of the γ quantum in the state $\mathbf{k},\nu$. The expansion of the operator $\hat{\mathbf{j}}$ in plane waves is as follows:

$$\hat{\mathbf{j}}(\mathbf{r},t) = c^2(2\Omega)^{-1}\sum_{\mathbf{p},\mathbf{p}'}(\mathbf{p}/E + \mathbf{p}'/E')b^{\dagger}_{\mathbf{p}}b_{\mathbf{p}'} \times \exp[i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{r}/\hbar - i(E'-E)t/\hbar] \ , \quad (10)$$

where E and E' are the relativistic energies corresponding to momenta $\mathbf{p}$ and $\mathbf{p}'$ and $\hat{b}^{\dagger}$ and $\hat{b}$ are the creation and annihilation operators of the electron in the state $\mathbf{p}$. In the lowest order of the fine-structure constant, the radiation is dependent on the matrix element

$$\langle f | \int \hat{L}^{\rm int} dr | i \rangle = M_{if}(n_{\mathbf{k},\nu})^{1/2} \exp\left[i \left[\frac{E^f}{\hbar} + \omega - \frac{E^i}{\hbar} \right] t \right] \ , \quad (11)$$

where $n_{\mathbf{k},\nu}$ represents the number of photons in the final state in the mode $\mathbf{k},\nu$. To obtain the value M_{if}, we substitute Eqs. (6) and (9). The result is

$$M_{if} = \tfrac{1}{2}e(v^i + v^f)(\hbar/2\omega\Omega^3)^{1/2}(2\pi)^2\delta^2\left[\frac{\mathbf{p}^i_{\perp}}{\hbar} - \mathbf{k}_{\perp} - \frac{\mathbf{p}^f_{\perp}}{\hbar}\right] \times \int_{L/2}^{L/2} e^{*}_{\mathbf{k}}(z)\exp\left[i \left[\frac{p^i_z}{\hbar} - k_z - \frac{p^f_z}{\hbar} \right] z \right] dz \ , \quad (12)$$

where E^i, $\mathbf{p}^i$, and v^i (E^f, $\mathbf{p}^f$, and v^f) are the initial (final) energy, momentum, and z component of the velocity of the electron, and $e_{\mathbf{k}}(z)$ is the z projection of the Bloch mode vector $\mathbf{e}_{\mathbf{k},\nu}(z)$ with the polarization corresponding to the three vectors $\mathbf{k}$, $\mathbf{v}$, and $\mathbf{e}_{\mathbf{k},\nu}(z)$ being coplanar (the z direction is along the velocity $\mathbf{v}$). The normalization length L is $\Omega^{1/3}$. Next, we expand the function $e_{\mathbf{k}}(z)$ in a Fourier series

$$e_{\mathbf{k}}(z) = \sum_n e^n_{\mathbf{k}}\exp(in2\pi z/l) \quad (13)$$

and substitute it into Eq. (12). After integrating over z, we obtain the following result:

$$M_{if} = ev(\hbar/2\omega\Omega^3)^{1/2}(2\pi)^3 \times \sum_n e^{n*}_{\mathbf{k}}\delta\left[\frac{p^i_z}{\hbar} - k_z - \frac{2\pi n}{l} - \frac{p^f_z}{\hbar}\right] \times \delta^2\left[\frac{\mathbf{p}^i_{\perp}}{\hbar} - \mathbf{k}_{\perp} - \frac{\mathbf{p}^f_{\perp}}{\hbar}\right] . \quad (14)$$

Next, substitute this expression into the standard-quantum mechanical formula for the transition probability per unit time

$$\frac{dw}{dt} = \frac{2\pi}{\hbar^2}\int |M_{if}|^2 \delta\left[\frac{E^i}{\hbar} - \omega - \frac{E^f}{\hbar}\right]\Omega^2\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{d^3\mathbf{p}^f}{(2\pi\hbar)^3} \quad (15)$$

and perform the integration over $d^3\mathbf{p}^f$ recalling that

$$(2\pi)^3[\delta^3(\mathbf{k})]^2 = \Omega\delta^3(\mathbf{k}) \ . \quad (16)$$

As a result we obtain

$$\frac{dw}{dt} = \frac{e^2v^2}{4\pi\hbar}\sum_n \int |e^n_{\mathbf{k}}|^2 \delta\left[\frac{E^i}{\hbar} - \omega - \frac{E^f}{\hbar}\right]\frac{k_{\perp}dk_{\perp}dk_z}{\omega} \ . \quad (17)$$

In the subsequent calculations it is necessary to take into account the approximate equality

$$E^f \approx E^i - v\left[k_z + \frac{2\pi n}{l}\right]\hbar \quad (18)$$

which arises from both the momentum conservation condition and Eq. (2). Taking into account Eq. (18), we can rewrite the δ function in Eq. (17) in the following way:

$$\delta\left[v\left[k_z + \frac{2\pi n}{l}\right] - \omega\right] \quad (19)$$

and then integrate over $dk_\perp$. We obtain

$$\frac{dw}{dt}=\sum_n \int \frac{e^2 v^2}{4\pi\hbar}|e_{\mathbf{k}}^n|^2 \left[\frac{\partial\omega_{\nu_0}(k_z,k_\perp)}{\partial k_\perp^2}\right]^{-1} dk_z \;, \tag{20}$$

where v_0 is the polarization at which the three vectors **k**, **v**, and the electric field are coplanar. The last factor in the integrand of Eq. (20) stems from differentiating the argument of the δ function in Eq. (19) with Eq. (8) and suggests that the function $\omega_\nu(k_z,k_\perp)$ has only one branch, that is, there is no degeneracy on $k_\perp$.

To obtain the radiated power in the frequency interval $d\omega$, it is necessary in Eq. (20) to take into account the following relation:

$$\omega=vk_z+\frac{2\pi n}{l}v \;, \tag{21}$$

which arises from integrating the δ function in (19). The result is

$$dU=\frac{e^2 v}{4\pi}|e_{\mathbf{k}}^n|^2\omega \left[\frac{\partial\omega_{\nu_0}^2(k_z,k_\perp)}{\partial k_\perp^2}\right]^{-1} d\omega \;. \tag{22}$$

As a rule, authors are interested in the result for the power per unit area radiated by an electron beam with the current density J during a time t, i.e.,

$$\frac{dI}{d\omega}=\frac{1}{e}Jt\frac{dU}{dt}=\frac{Jev}{4\pi}t|e_{\mathbf{k}}^n|^2\omega\left[\frac{\partial\omega_{\nu_0}^2(k_z,k_\perp)}{\partial k_\perp^2}\right]^{-1} \;. \tag{23}$$

For further convenience we have multiplied and divided Eq. (20) by $\omega=\omega_{\nu_0}(k_z,k_\perp)$.

The result Eq. (23) should be compared with Eq. (29) of Ref. [18]. The two answers differ in two respects: (1) the factor $[\partial\omega_{\nu_0}^2/\partial k_\perp^2]^{-1}$ is replaced by c^{-2}, and (2) the factor $|e_{\mathbf{k}}^n|^2$ is replaced by $4\pi\sin^2\theta|c_{nz}|^2$ (our $e_{\mathbf{k}}^n/e_{\mathbf{k}}^0=c_{nz}$). The two formulas are equivalent to each other only far from the Bragg domains provided $|\epsilon(z)-1|\ll 1$. In this connection, it is worth recalling the approximation of Ref. [18]: the result for ordinary Cerenkov radiation is obtained first and then adapted to a spatially periodic medium $\epsilon(z)$ by a heuristic method which consists of multiplying the answer by $|c_{nz}|^2$. The stimulated emission is treated in Ref. [18] in an analogous manner, the result being Eq. (31) of Ref. [18]. Our exact result obtained below yields a correction to this approximation even when $|\epsilon(z)-1|\ll 1$ and far from the Bragg domains.

To treat stimulated emission, we assume that there are $n_{\mathbf{k}}\gg 1$ photons in a mode characterized by a wave vector **k** and a polarization ν_0. The standard quantum-mechanical formulas give the following rate of photons emitted (or absorbed) in this mode:

$$\Delta n_{\mathbf{k}}^{\pm}=\hbar^{-2}n_{\mathbf{k}}\int |M_{if}^{\pm}|^2 t^2\left[\frac{\sin^2\Delta^{\pm}}{(\Delta^{\pm})^2}\right]^2 \Omega\frac{d^3\mathbf{p}^f}{(2\pi\hbar)^3} \;, \tag{24}$$

where

$$\Delta^{\pm}=(t\,\delta E^{\pm}/2\hbar) \;, \tag{25}$$

$$\delta E^{\pm}=E^i-E^f\mp\hbar\omega_{\mathbf{k}} \;. \tag{26}$$

The upper (lower) sign corresponds to emission (absorption). M^+ is given by Eq. (14), and M^- differs from M^+ in the sign in front of both **k** and n in the argument of the δ) function in Eq. (14). Using momentum conservation $p^f=p^i\mp h\mathbf{k}$, we can obtain the following approximate relation:

$$\delta E^{\pm}=\mp\hbar\left\{\omega_{\mathbf{k}}-v\left[k_z+\frac{2\pi n}{l}\right]\mp\frac{h}{2m\gamma}\left[k_\perp^2+\gamma^{-2}\left[k_z+\frac{2\pi n}{l}\right]^2\right]\right\} \;,$$
$$\gamma=(1-v^2/c^2)^{-1/2} \;. \tag{27}$$

The last relation can be deduced by analogy with Eq. (18), if we consider the next term in the expansion in the smallness variable $(\hbar\omega_{\mathbf{k}}/E)$. Since $(|E^+|-|E^-|)$ is not zero, there is an effective number of emitted photons

$$\Delta n_{\mathbf{k}}=\Delta n_{\mathbf{k}}^{+}-\Delta n_{\mathbf{k}}^{-} \;. \tag{28}$$

Substituting Eqs. (15) and (24)–(27) into Eq. (28) and expanding the function $(\sin^2\eta/\eta^2)$ in powers of the small variable

$$\eta=\tfrac{1}{2}[\omega-v(k_z+2\pi n l^{-1})]t \;, \tag{29}$$

we obtain the following result for the gain Γ:

$$\Gamma\equiv\frac{\Delta n_{\mathbf{k}}}{n_{\mathbf{k}}}\frac{J\Omega}{ev} \;, \tag{30a}$$

$$\Gamma=\frac{Jev}{4\omega_{\mathbf{k}}m\gamma}|e_{\mathbf{k}}^n|^2\left[k_\perp^2+\gamma^{-2}\left[k_z+\frac{2\pi n}{l}\right]^2\right]Ft^3 \;, \tag{30b}$$

where, as in Ref. [18], we have called

$$F\equiv\frac{d}{d\eta}(\eta^{-2}\sin^2\eta) \;. \tag{31}$$

Finally, taking into account (23), we rewrite the result (30b) as

$$\Gamma=\frac{dI}{d\omega}\omega_{\mathbf{k}}^{-2}\left[\frac{\partial\omega_{\nu_0}^2(k_z,k_\perp)}{\partial k_\perp^2}\right]\frac{\pi F t^2}{m\gamma}\times\left[k_\perp^2+\gamma^{-2}\left[k_z+\frac{2\pi n}{l}\right]^2\right] \;. \tag{32}$$

Notice that far from the Bragg domains and when $|\epsilon(z)-1\ll 1$, Eq. (32) provides the result (31) of Ref. [18] only for $n=0$, whereas for values $n\neq 0$ it yields some corrections to that result.

III. RADIATION FROM PERIODIC STRATIFIED STRUCTURES

To use the results (23) and (32) for a given periodic function $\epsilon(z)$, it is necessary to have specific expressions for both the dispersion relation (8) to calculate the derivative $\partial\omega^2/\partial k_\perp^2$ and the coefficients $e_{\mathbf{k}}^n$ of the Fourier expansion of the Bloch function $e_{\mathbf{k}}(z)$, Eq.(13), in terms of $\epsilon(z)$. Let the function $\epsilon(z)$ have a periodic stratified structure and consist of layers with thicknesses l_1, l_2 with dielectric constants ϵ_1, ϵ_2, respectively (Fig. 1). Let us label these layers with two numbers (s,q), where q is the number of periods of the function $\epsilon(z)$ (q may be positive or negative integers, including zero) and s is the number of layers in a period (s may have values 1 or 2). As the function $\epsilon(z)$ is discontinuous, the electric field $\mathbf{E}(\mathbf{r})$ is circumscribed by the different analytical expressions for various layers. For a radiated photon mode, let us write the Bloch function of the z component of the electric field in the layer (s,q) as

$$E_{\mathbf{k}z}(\mathbf{r})=E_{\mathbf{k}sq}(z)\exp(i\mathbf{k}_\perp\cdot\mathbf{r}_\perp), \quad \mathbf{r}\in(s,q) \ . \tag{33}$$

As ϵ is a constant inside a layer, we may represent the function $E_{\mathbf{k}sq}(z)$ as

$$E_{\mathbf{k}sq}(z)=E^{+}_{\mathbf{k}sq}\exp(ik_{sz}z)+E^{-}_{\mathbf{k}sq}\exp(-ik_{sz}z) \ , \tag{34}$$

where $E^{\pm}_{\mathbf{k}sq}$ are some constants and

$$k_{sz}=(k_s^2-k_\perp^2)^{1/2}=(\epsilon_s c^{-2}\omega^2-k_\perp^2)^{1/2} \ , \quad s=1,2 \ . \tag{35}$$

All the values $E^{\pm}_{\mathbf{k}sq}$ are deduced from the appropriate boundary conditions if the constants $E^{\pm}_{\mathbf{k}s0}$ have been found. To find $E^{\pm}_{\mathbf{k}s0}$, recall that any radiated photon mode has the polarization at which the vectors $E(\mathbf{r})$, $\mathbf{v}$, and $\mathbf{k}$ are coplanar. For such a polarization the conditions of continuity of the normal component of the vector $\mathbf{D}$ and the tangent component of the vector $\mathbf{E}$ at $z=0$ and $z=l_1$ yield four equations:

$$\epsilon_1(E^{+}_{\mathbf{k}10}+E^{-}_{\mathbf{k}10})=\epsilon_2(E^{+}_{\mathbf{k}2-1}+E^{-}_{\mathbf{k}2-1}) \ , \tag{36}$$

$$\epsilon_1[E^{+}_{\mathbf{k}10}\exp(ik_{1z}l_1)+E^{-}_{\mathbf{k}10}\exp(-ik_{1z}l_1)] =\epsilon_2[E^{+}_{\mathbf{k}20}\exp(ik_{2z}l_1)+E^{-}_{\mathbf{k}20}\exp(-ik_{2z}l_1)] \ , \tag{37}$$

$$k_{1z}(E^{+}_{\mathbf{k}10}-E^{-}_{\mathbf{k}10})=k_{2z}(E^{+}_{\mathbf{k}2-1}-E^{-}_{\mathbf{k}2-1}) \ , \tag{38}$$

$$k_{1z}[E^{+}_{\mathbf{k}10}\exp(ik_{1z}l_1)-E^{-}_{\mathbf{k}10}\exp(-ik_{1z}l_1)] =k_{2z}[E^{+}_{\mathbf{k}20}\exp(ik_{2z}l_1)-E^{-}_{\mathbf{k}20}\exp(-ik_{2z}l_1)] \ . \tag{39}$$

Two additional equations can be deduced from the Bloch representation

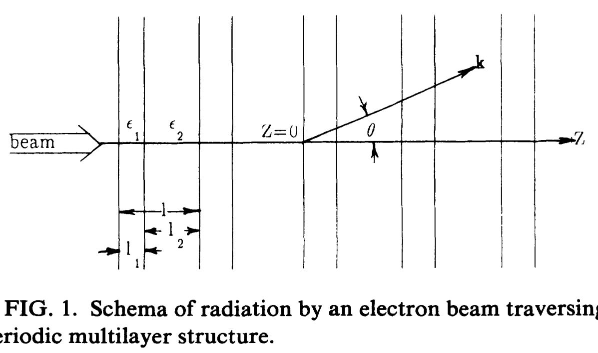

FIG. 1. Schema of radiation by an electron beam traversing a periodic multilayer structure.

$$E_{\mathbf{k}z}(\mathbf{r})=e_{\mathbf{k}}\exp(i\mathbf{k}\cdot\mathbf{r}) \tag{40}$$

where the function $e_{\mathbf{k}}(z)$ is periodic. From Eqs. (33), (34), and (40) it follows that

$$E^{\pm}_{\mathbf{k}sq}=E^{\pm}_{\mathbf{k}s0}\exp(\mp ik^{\mp}_{sz}ql) \ , \tag{41}$$

where

$$k^{\pm}_{sz}=k_{sz}\pm k_z, \quad s=1,2 \ . \tag{42}$$

At $q=1$, Eq. (41) yields

$$E^{\pm}_{\mathbf{k}2-1}=\exp(\pm ik^{\mp}_{2z}l)E^{\pm}_{\mathbf{k}20} \ . \tag{43}$$

The condition of compatibility of the system of six homogeneous equations (36)–(38) and (43) with the six unknowns yields the following relation:

$$\cos(k_zl)=\cos(k_{1z}l_1+k_{2z}l_2) -(2\epsilon_1\epsilon_2k_{1z}k_{2z})^{-1}(k_{1z}\epsilon_2-k_{2z}\epsilon_1)^2 \times\sin(k_{1z}l_1)\sin(k_{2z}l_2) \ . \tag{44}$$

This equality is a specific realization of the dispersion relation Eq. (8) for the periodic stratified structure and it allows us to compute the Bloch parameter k_z as a function of ω and $k_\perp$, taking into account Eqs. (35). Equation (44) can be compared with the dispersion relation obtained in Ref. [22] [see Eq. (39), which contains a determinant Δ of infinite rank]. The authors of Ref. [22] report that they have elaborated an algorithm for approximating Δ. As an example, they exhibit curves of the variation of both real and imaginary parts of the Bloch parameter versus the escape angle θ (Fig. 2 in Ref. [22]) provided $|\epsilon(z)-1|\ll 1$. But a detailed analysis shows that these curves contradict the exact equality (44) near the boundaries of the Bragg domains. Indeed, let us keep the frequency ω constant and expand Eq. (44) in powers of the small values $\Delta k_z^m\equiv k_z-k_z^m$ and $\Delta k_\perp^m\equiv k_\perp-k_\perp^m$, near the boundary values $k_z^m, k_\perp^m$ (where m is the number of the boundary). We derive

$$\pm[1-\tfrac{1}{2}(\Delta k_z^m l)^2]=\pm[1+A_m(\Delta k_\perp^m l)] \ , \tag{45}$$

where A_m are constants (it is worth mentioning that both the right- and left-hand sides of Eq. (44) are equal to ± 1 on the boundaries of Bragg domains). From Eq. (45) we conclude that the variation vs $|\Delta k_\perp^m|$ of $|\mathrm{Re}\Delta k_z^m|$ in the allowed zone near the boundary and of $|\mathrm{Im}\Delta k_z^m|$ vs $|\Delta k_\perp^m|$ in the forbidden zone near the same boundary, are identical. These conditions are not satisfied in Fig. 2 of Ref. [22] where the slope of $\mathrm{Im}\Delta k_z$ near the left-hand boundary of the Bragg domain is larger than near the right-hand boundary, while the opposite is true for $|\mathrm{Re}\Delta k_z|$.

In addition to the dispersion relation, we can also specify the general results (23) and (32) for the radiated intensity and the gain in the case of periodic stratified structures. To that end, we must deduce the Fourier coefficients $e^n_{\mathbf{k}}$ for the structures [as well as the dispersion relation which has been specified already in Eq. (44)]. The appropriate program can be realized on the basis of the system of Eqs. (36)–(39) and (43) under the conditions

Eqs. (44) and (6). It is convenient to write the results of rather lengthy calculations of $e_{\mathbf{k}}^n$ in the following forms:

$$e_{\mathbf{k}}^n = e_{1\mathbf{k}}^n + e_{2\mathbf{k}}^n , \tag{46}$$

$$e_{s\mathbf{k}}^n = 2(2\pi n - k_{sz}^- l)^{-1}\alpha_s^+ + 2(2\pi n + k_{sz}^+ l)^{-1}\alpha_s^- , \tag{47}$$

where $k^{\pm}$ are given by Eq. (42), and

$$\alpha_s^{\pm} = (-1)^{s+1}(4\pi)^{1/2}(R\epsilon_s)^{-1}a_s^{\pm} \times \sin\left[\pi n\frac{l_1}{l} \pm (-1)^s k_{sz}^{\mp} l_s\right] , \tag{48}$$

$$a_1^+ = 1, \quad a_1^- = \frac{(k_{1z}\epsilon_2 - k_{2z}\epsilon_1)\sin[(k_{2z}^+ l_2 - k_{1z}^- l_1)/2]}{(k_{1z}\epsilon_2 + k_{2z}\epsilon_1)\sin[(k_{1z}^+ l_1 + k_{2z}^+ l_2)/2]} , \tag{49}$$

$$a_2^{\pm} = \frac{(\epsilon_1 k_{2z} \pm \epsilon_2 k_{1z})\sin(k_{1z} l_1)}{2\epsilon_1 k_{2z}\sin[(k_{1z}^+ l_1 + k_{2z} l_2)/2]} , \tag{50}$$

$$\begin{aligned} R^2 = {} & (l_1/\epsilon_1 l)[1 + (k_{1z}^2/k_\perp^2)](1 + |a_1^-|^2) \\ & + (l_2/\epsilon_2 l)[1 + (k_{2z}^2/k_\perp^2)](|a_2^+|^2 + |a_2^-|^2) \\ & + (2a_1^-/\epsilon_1)[1 - (k_{1z}^2/k_\perp^2)][\sin(k_{1z}l_1)/k_{1z}l] \\ & + (2a_2^+ a_2^-/\epsilon_2)[1 - (k_{2z}^2/k_{1z}^2)] . \end{aligned} \tag{51}$$

If the inequality $|\Delta\epsilon/\epsilon| \ll 1$ holds, we may expand the results in powers of $\Delta\epsilon/\epsilon$. To the first order in $(\Delta\epsilon/\epsilon)$, we obtain a very compact expression. For example, the dispersion relation Eq. (44) has the form

$$k_z l = k_{1z} l_1 + k_{2z} l_2 . \tag{52}$$

This result means that the extensions of the Bragg domains (inside which the Bloch parameter k_z is imaginary) tend to zero to the first order in $(\Delta\epsilon/\epsilon)$. To the same order, we also have

$$\frac{dI}{d\omega} = (4\pi c^2)^{-1} eJv\omega t |e_{\mathbf{k}}^n|^2 , \tag{53}$$

where

$$e_{\mathbf{k}}^n = \pi^{-1/2}\frac{2(ng - \beta')}{n^2 g(2 - ng\beta')} R_n(\Delta\epsilon/\epsilon)\sin[(\pi n l_1/l)]\sin\theta , \tag{54}$$

$$\begin{aligned} R_n^{-2} = 1 + \frac{1}{16}\left[\frac{\Delta\epsilon}{\epsilon}\right]^2 \frac{\cos^2 2\theta}{\cos^4\theta}\left\{\sin\left[2\pi n\left[1 - \frac{1}{\beta' g n}\right]\right]\right\}^{-2} \\ \times\left\{\frac{l_1}{l}\sin^2\left[2\pi n\left[\frac{l_2}{l_1} - \frac{1}{\beta' g_2 n}\right]\right] + \frac{l_2}{l}\sin^2\left[2\pi n\left[\frac{l_1}{l_2} - \frac{1}{\beta' g_1 n}\right]\right]\right\} , \end{aligned} \tag{55}$$

where

$$g_1 = \lambda' l_1^{-1}, \quad g_2 = \lambda' l_2^{-1}, \quad g = \lambda' l^{-1} ,$$

$$\lambda' = \lambda\epsilon^{-1/2}, \quad \beta' = vc^{-1}\epsilon^{1/2}, \quad \tan\theta = k_z^{-1}k_\perp .$$

IV. CONCLUSIONS

In this paper we have considered both spontaneous and stimulated transition radiation which occurs when an electron beam traverses a periodic stratified medium. We have adopted an approximation that neglects the influence of the external field on the motion of the electron. It is interesting to find corrections to this approximation in the case of stimulated radiation in a strong external field (for example, our approximation yields zero gain for the mode with $k_\perp = 0$). But it is more important to investigate corrections arising from the imaginary part of the dielectric index. The latter can be done in the framework of the approach we have introduced. The result for the intensity of the spontaneous radiation is given by Eqs. (23) and (46)–(51) from which we have derived Eqs. (53)–(55) for the case $|\Delta\epsilon/\epsilon| \ll 1$. These results are valid both far from and near the Bragg domain. As can be observed from Eq. (55), the approximate equality $R_n \sim 1$ is realized far from the Bragg domain and from Eqs. (53) and (54) we can obtain a result that coincides with Eq. (5) of Ref. [18]; the latter, however, becomes invalid near the Bragg domain where $\beta' nq \sim 2$. On approaching the Bragg domain, the second term in the right-hand side of Eq. (55) grows sharply and prevents an unlimited growth of both Eqs. (54) and (53). This is the main prediction of our paper in the Bragg domain. Away from it, as noted in Ref. [11], the agreement between theory and experiment has been quite satisfactory. Regrettably, we cannot compare our results for the Bragg domains with experimental data since the Bragg resonances have not yet been investigated with ultrarelativistic electrons in periodic structures (Refs. [11–17]). However, due to the considerable increase of emitted γ's in those regions, it is hoped that experimental data will soon become available.

Finally, let us mention that, as $g \to 0$, Eqs. (53)–(55) become invalid in contrast with the exact results Eqs. (46)–(51) which are valid for any g. On the other hand, we recall that we are interested primarily in microstructures for which $g \geq 1$. As to the stimulated radiation, the result for the gain is given by Eq. (32) which corrects the approximate expression in Ref. [18] even far away from the Bragg domain.

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APPENDIX: CLASSICAL THEORY OF TRANSITION RADIATION

Starting from the Maxwell equations, we can deduce the equation for the z component of the induction vector $\mathbf{D}$ under the conditions which have been considered:

$$\left[\hat{O}+\frac{\hat{\omega}^2}{c^2}\right]D(\mathbf{r},t)=\frac{4\pi}{c^2}ev\frac{\partial}{\partial t}\delta(\mathbf{r}-\mathbf{v}t)+4\pi e\frac{\partial}{\partial z}\{\delta(\mathbf{r}-\mathbf{v}t)/\epsilon(z)\}\ , \quad (A1)$$

where we use the notation

$$D(\mathbf{r},t)\equiv D_z(z,t)\ , \quad (A2)$$

$$\hat{\omega}\equiv i\frac{\partial}{\partial t}\ , \quad (A3)$$

$$\hat{O}D\equiv\Delta[D/\epsilon(z)]+\frac{\partial}{\partial z}[D\epsilon'(z)/\epsilon(z)]\ . \quad (A4)$$

We can check that the operator $\hat{O}$ is Hermitian (it is worth noting that if we write the equation for $E_z(\mathbf{r},t)$, the analogous operator will not be Hermitian), so a system of its eigenfunctions is complete and orthogonal. Let us label eigenfunctions with the proper Bloch vector $\mathbf{k}$ and normalize the functions $D_{\mathbf{k}}(\mathbf{r})$ as follows:

$$\int D_{\mathbf{k}}^*(\mathbf{r})D_{\mathbf{k}'}(\mathbf{r})d\mathbf{r}=(2\pi)^3\delta(\mathbf{k}-\mathbf{k}')\ . \quad (A5)$$

The eigenfunctions $D_{\mathbf{k}}(\mathbf{r})$ satisfy the relation

$$\hat{O}D_{\mathbf{k}}(\mathbf{r})=-[\omega_{\nu_0}^2(k_z,k_\perp)/c^2]D_{\mathbf{k}}(\mathbf{r})\ , \quad (A6)$$

where $\omega_{\nu_0}(k_z,k_\perp)$ is the function defined by Eq. (8) with the polarization ν_0 specified below Eq. (20). Let us expand the function $D(\mathbf{r},t)$ in eigenfunctions $D_{\mathbf{k}}(\mathbf{r})$,

$$D(\mathbf{r},t)=\int S_{\mathbf{k}}(t)D_{\mathbf{k}}(\mathbf{r})d\mathbf{k}\ , \quad (A7)$$

and substitute Eq. (A7) into Eq. (A1). Multiplying the result by $D_{\mathbf{k}'}^*(\mathbf{r})$ and integrating over $d\mathbf{k}$ yields the following differential equation for the coefficient function $S_{\mathbf{k}}(t)$:

$$[\hat{\omega}^2-\omega_{\nu_0}^2(k_z,k_\perp)]S_{\mathbf{k}}(t)=e(2\pi)^{-2}\{v-[c^2/v\epsilon(vt)]\}\frac{d}{dt}D_{\mathbf{k}}^*(vt)\ , \quad (A8)$$

where

$$D_{\mathbf{k}}(z)\equiv D_{\mathbf{k}}(\mathbf{r}_\perp=0,z)\ . \quad (A9)$$

According to the Bloch theorem, we can represent the function $D_{\mathbf{k}}(\mathbf{r})$ as follows:

$$D_{\mathbf{k}}(\mathbf{r})=e^{i\mathbf{k}\cdot\mathbf{r}}d(z)=e^{i\mathbf{k}\cdot\mathbf{r}}\sum_m d_{\mathbf{k}}^m\exp\left[im\frac{2\pi z}{l}\right]\ . \quad (A10)$$

It is convenient to introduce the function$G_{\mathbf{k}}(z)$, where

$$\frac{1}{\epsilon(z)}\frac{d}{dz}D_{\mathbf{k}}(z)\equiv\frac{d}{dz}G_{\mathbf{k}}(z)\ . \quad (A11)$$

On the basis of Eqs. (A10) and (A11), it is possible to represent the function $G_{\mathbf{k}}(z)$ as

$$G_{\mathbf{k}}(z)=\exp(ik_zz)g_{\mathbf{k}}(z)=\exp(ik_zz)\sum_m g_{\mathbf{k}}^m\exp\left[im\frac{2\pi z}{l}\right]\ . \quad (A12)$$

Taking into account Eqs. (A10)–(A12), we can obtain the solution of Eq. (A8) in the following form:

$$S_{\mathbf{k}}(t)=\sum_n S_{\mathbf{k}}^n\exp(-i\omega_nt)\ , \quad (A13)$$

$$\omega_n=k_zv+n\frac{2\pi v}{l}\ , \quad (A14)$$

$$S_{\mathbf{k}}^n=-\frac{ie}{2\pi^2}\frac{\omega_n}{v}\frac{v^2d_{\mathbf{k}}^{n*}-c^2g_{\mathbf{k}}^{n*}}{\omega_n^2-\omega_{\nu_0}^2(k_z,k_\perp)}\ . \quad (A15)$$

To deduce the emission intensity we implement the method used by Landau and Lifshitz [32], according to which we must deduce the z component of the electric field acting on the particle itself (i.e., at the point $\mathbf{r}_\perp=0,z=vt$):

$$E(t)\equiv E_z(\mathbf{r}_\perp=0,z=vt,t)\ . \quad (A16)$$

Taking into account Eqs. (A16), (A13), and (A10) we obtain

$$E(t)=\int dk\ S_{\mathbf{k}}(t)E_{\mathbf{k}}(vt)\ , \quad (A17)$$

where the functions $E_{\mathbf{k}}(vt)$ may be represented as

$$E_{\mathbf{k}}(vt)=D_{\mathbf{k}}(vt)/\epsilon(vt)=\exp(ik_zvt)\tilde{e}_{\mathbf{k}}(vt)=\exp(ik_zvt)\sum_m\tilde{e}_{\mathbf{k}}^m\exp\left[im\frac{2\pi vt}{l}\right]\ , \quad (A18)$$

so that

$$E(t)=\int d\mathbf{k}\left[\sum_n S_{\mathbf{k}}^n\exp(-i\omega_nt)\right]\left[\sum_m\tilde{e}_{\mathbf{k}}^m\exp(i\omega_mt)\right]\ . \quad (A19)$$

It is worth noting that the functions $\tilde{e}_{\mathbf{k}}(z)$ and $e_{\mathbf{k}}(z)$ [see Eq. (13) above] differ only due to the normalization. The energy radiated by one electron is equal to the work of the electric field acting upon the electron. The work along the path of the electron per unit length is equal to the average force

$$F=-e\bar{E}(t)=-\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}E(t)dt\ .$$

Substituting Eq. (A19) for $E(t)$ and taking into account Eq. (15), we obtain the following representation for F:

$$F=\sum_n\int_{-\infty}^{\infty}F(\omega_n)d\omega_n\ , \quad (A20)$$

where

$$F(\omega_n)=\frac{ie^2}{2\pi^2}\int_0^\infty(\omega_n/v)\tilde{e}_{\mathbf{k}}^n\left[\frac{v^2d_{\mathbf{k}}^{n*}-c^2g_{\mathbf{k}}^{n*}}{\omega_n^2-\omega_{\nu_0}^2(k_z,k_\perp)}\right]\frac{\pi dk_\perp^2}{v}\ . \quad (A21)$$

The spectral density of the radiated intensity is then

$$\tilde{F}(\omega_n) \equiv F(\omega_n) + F(-\omega_n), \quad \omega_n > 0 \; . \tag{A22}$$

As Landau and Lifshitz [32] have shown, to calculate the integral (A21) it is necessary to consider the contribution only near the pole of the integrand (A21), provided that the pole must be rounded on the lower half plane of the complex plane of $k_\perp^2$ if $\omega_n < 0$. We obtain

$$\frac{dI}{d\omega} = e^{-1} J v t \tilde{F} = e J v t \omega_n [\partial \omega_{v_0}^2(k_z, k_\perp)/\partial k_\perp^2]^{-1} \times \mathrm{Re}\left[d_{\mathbf{k}}^n - \frac{c^2}{v^2} g_{\mathbf{k}}^n \right] \tilde{e}_{\mathbf{k}}^{n*} \; . \tag{A23}$$

To deduce this formula, we have taken into account the equalities

$$d_{-\mathbf{k}}^{-n} = d_{\mathbf{k}}^{n*}, \quad g_{-\mathbf{k}}^{-n} = g_{\mathbf{k}}^{n*}, \quad \tilde{e}_{-\mathbf{k}}^{-n} = \tilde{e}_{\mathbf{k}}^{n*} \; , \tag{A24}$$

which follow from the reality of the function $\epsilon(z)$.

To verify the equivalence of the results (A23) and (23), it is necessary to perform the proper algebraic transformations taking into account the following relations:

$$d_{\mathbf{k}}^n = \epsilon_1 \tilde{e}_{1\mathbf{k}}^n + \epsilon_2 \tilde{e}_{2\mathbf{k}}^n \; ,$$

$$g_{\mathbf{k}}^n = \tilde{e}_{\mathbf{k}}^n + \left| \frac{1}{\epsilon_2} - \frac{1}{\epsilon_1} \right|^2 \frac{\epsilon_1(\alpha_1^+ + \alpha_1^-)}{(2\pi n + k_z l)} \frac{R}{\tilde{R}} \; ,$$

$$\tilde{e}_{\mathbf{k}}^n = e_{\mathbf{k}}^n (R/\tilde{R}), \quad \tilde{e}_{sk}^n = e_{sk}^n (R/\tilde{R}), \quad s = 1,2 \; ,$$

$$(l/4\pi)\tilde{R}^2 = [1 + (a_1^-)^2] l_1 + [(a_2^+)^2 + (a_2^-)^2] l_2 + 2a_1^- l \frac{\sin k_{1z} l_1}{k_{1z} l} + 2a_2^+ a_2^- l \frac{\sin k_{2z} l_2}{k_{2z} l} \; ,$$

where $e_{\mathbf{k}}^n$, $a_s^{\pm}$, and R are given by Eqs. (46)–(51).

[1] V. L. Ginzburg and I. M. Frank, J. Phys. (Moscow) **9**, 3533 (1945).
[2] G. M. Garibyan, Zh. Eksp. Teor. Fiz. **33**, 1403 (1958) [Sov. Phys. JETP **6**, 1079 (1958)].
[3] M. L. Ter-Mikaelian, Nucl. Phys. **24**, 43 (1961).
[4] A. I. Alikhanyan, F. R. Arutyanyan, K. A. Ispiryan, and M. L. Ter-Mikaelian, Zh. Eksp. Teor. Fiz. **41**, 2002 (1961) [Sov. Phys. JETP **14**, 1421 (1962)].
[5] L. C. Yuan, C. L. Wang, H. Uto, and S. Prunseter, Phys. Lett. **31B**, 603 (1970).
[6] A. I. Alikhanyin, S. A. Kapkanian, A. G. Oganessian, and A. G. Tamanian, Phys. Rev. Lett. **30**, 109 (1973).
[7] L. L. DeRaad, Jr., Wu-yang Tsai, and T. Erber, Phys. Rev. D **18**, 2152 (1978).
[8] V. M. Arutyunyan and S. G. Oganesyan, Zh. Eksp. Teor. Fiz. **72**, 466 (1977) [Sov. Phys. JETP **45**, 244 (1977)].
[9] I. G. Ivanter, V. V. Lomonosov, and E. A. Nersesov, Zh. Eksp. Teor. Fiz. **85**, 830 (1983) [Sov. Phys. JETP, **58**, 479 (1983)].
[10] A. B. Khorev and V. A. Tsytovich, Zh. Eksp. Teor. Fiz. **89**, 1134 (1985) [Sov. Phys. JETP **62**, 653 (1985).
[11] M. A. Piestrup, M. J. Moran, D. G. Boyers, C. I. Pincus, J. O. Kephart, R. A. Gearhart, and X. K. Maruyama, Phys. Rev. A **43**, 2387 (1991).
[12] M. A. Piestrup, D. G. Boyers, C. I. Pincus. Qiang Li, G. D. Hallewell, M. J. Moran, D. M. Skopik, R. M. Silzer, X. K. Maruyama, D. D. Snyder, and G. B. Rothbart, Phys. Rev. A **45**, 1183 (1992).
[13] P. Goedtkindt, J. M. Salome, X. Artu, P. Dhez, M. Jablonka, N. Maene, F. Poortsmans, and L. Wartski, Nucl. Instrum. Methods Phys. Res. Sect. B **56/57**, 1060 (1991).
[14] M. L. Cherry, D. Muller, and T. A. Prince, Nucl. Instrum. Methods **115**, 141 (1974); M. J. Moran and P. J. Ebert, Nucl. Instrum. Methods Phys. Res. Sect. A **242**, 355 (1986).
[15] C. W. Fabjan and W. Struczinski, Phys. Lett. **57B**, 483 (1975).
[16] M. J. Moran, B. A. Dahling, P. J. Ebert, M. A. Piestrup, B. L. Berman, and J. O. Kephart, Phys. Rev. Lett. **57**, 1223 (1986).
[17] M. A. Piestrup, D. G. Boyers, Qiang Li, M. J. Moran, J. C. Bergstrom, H. S. Caplan, R. M. Silzer, D. M. Skopic, X. K. Maruyama, F. R. Buskirk, J. R. Neighbours, and G. B. Rothbart, Nucl. Instrum. Methods Phys. Res. Sect. B **40/41**, 965 (1989).
[18] S. Datta and A. E. Kaplan, Phys. Rev. A **31**, 790 (1985).
[19] A. E. Kaplan and S. Datta, Appl. Phys. Lett. **44**, 661 (1984).
[20] A. E. Kaplan and S. Datta, J. Lumin. **31**, 690 (1984).
[21] C. T. Law and A. E. Kaplan, Opt. Lett. **12**, 900 (1987).
[22] B. Pardo and J.-M. Andre, Phys. Rev. A **40**, 1918 (1989).
[23] G. M. Garibyan, Zh. Eksp. Teor. Fiz. **60**, 39 (1971) [Sov. Phys. JETP **33**, 23 (1971)].
[24] M. L. Ter-Mikaelian, *High Energy Electromagnetic Processes in Condensed Media* (Wiley-Interscience, New York, 1972).
[25] S. M. Zee, *Physics of Semiconductor Devices* (Interscience, New York, 1981).
[26] W. Becker and J. K. McIver, Phys. Rev. A **25**, 956 (1982).
[27] M. A. Piestrup, IEEE J. Quantum Electron. **QE-19**, 10 (1983).
[28] A. Gover and P. Sprangle, IEEE J. Quantum Electron. **QE-17**, 1196 (1981).
[29] D. Marcuse, *Principles of Quantum Electronics* (Academic, New York, 1980), p. 163.
[30] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965), p. 206.
[31] A. I. Akhiezer and V. B. Berestetskii, *Quantum Electrodynamics* (Interscience, New York, 1965).
[32] L. Landau and E. Lifshitz, *Electrodynamics of Continuous Media* (Mir, Moscow, 1969).