Connection of a type of q-deformed binomial state with q-spin coherent states

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Using the q analog of the Holstein-Primakoff boson realization of the su(2) generators, we show that a type of q-deformed binomial state that corresponds to the Heine distribution can be identified as an $\mathrm{su}(2)_q$ coherent state. This fact is a q extension of the fact that the ordinary binomial state is a particular su(2) coherent state when the Holstein-Primakoff transformation is employed.

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As is known, the study of photon statistics distribution is an important topic in quantum optics. In Ref. [1] the concept of binomial states was proposed and investigated experimentally. Theoretically, the binomial state is interplated between the number state and the coherent state and is a linear combination of the $M+1$ number states with coefficients chosen such that the counting probability distribution is binomial, i.e.,

$$|\eta,M\rangle=\sum_{n=0}^{M}\beta_n^M|n\rangle,\quad \beta_n^M\equiv\left[\begin{bmatrix}M\\ n\end{bmatrix}\eta^n(1-\eta)^{M-n}\right]^{1/2}, \tag{1}$$

where $0<\eta<1$. As shown by Stoler, Saleh, and Teich, the binomial state $|\eta,M\rangle$ is antibunched, sub-Poissonian, and squeezed for a certain parameter range. Moreover, $|\eta,M\rangle$ possesses the properties [1]

$$\langle\eta,M|\eta,M\rangle=1,\quad \langle\eta,M|a^\dagger a|\eta,M\rangle=\eta M\ . \tag{2}$$

On the other hand, the spin coherent state (similar to what are sometimes referred to in the literature as the atomic coherent state or the Bloch coherent state), which has been applied in many branches of physics, is defined as [2–4]

$$|\theta,\varphi\rangle=e^{\xi\hat S_+-\xi^*\hat S_-}|S\rangle,\quad \xi\equiv(\theta/2)e^{-i\varphi}\ , \tag{3}$$

where the non-self-adjoint spin operators satisfy the su(2) algebra $[S_+,S_-]=2S_3$, and the state $|S\rangle$ is the highest-weight state satisfying $\hat S_+|S\rangle=0$, $\hat S_3|S\rangle=S|S\rangle$. Using the disentangling [3–5]

$$e^{\xi\hat S_+-\xi^*\hat S_-}=e^{-\tau^*\hat S_-}e^{-\ln(1+|\tau|^2)\hat S_3}e^{\tau\hat S_+},\quad \tau=e^{-i\varphi}\tan\frac{\theta}{2}\ , \tag{4}$$

the expression (3) becomes

$$|\theta,\varphi\rangle=(1+|\tau|^2)^{-S}e^{-\tau^*\hat S_-}|S\rangle\equiv|\tau\rangle\ . \tag{5}$$

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As one can see from Refs. [2–5], using the group contraction method the spin coherent state can be contracted to the coherent state of harmonic oscillator. Using $\hat S_-|S_3\rangle=[S(S+1)-S_3(S_3-1)]^{1/2}|S_3-1\rangle$, we easily know that the expression $|\tau\rangle=(1+|\tau|^2)^{-S}\times\sum_{p=0}^{2S}\binom{2S}{p}^{1/2}(-\tau^*)^m|S-p\rangle$, so $|\langle S-p|\tau\rangle|^2=\binom{2S}{p}(|\tau|^2)^p(1+|\tau|^2)^{-2S}$ gives the probability that a system described by $|\tau\rangle$ is in the projected state $|S-p\rangle$, which is a binomial distribution [5].

We point out that, using the Holstein-Primakoff boson realization of spin operators, the binomial state is actually a particular spin coherent state. Then we extend the discussion to the q-deformed case, e.g., we try to reveal some connection between a type of q-deformed binomial state, which corresponds to the Heine distribution, and the q-deformed su(2) coherent state. For this purpose we introduce a q analog of the Holstein-Primakoff boson realization of q-spin operators. Our conclusion is that this kind of q-deformed binomial state can be identified as a particular set of the q-deformed su(2) coherent state. Note that the q-binomial state introduced here is different from the other type of q-binomial state defined earlier in Ref. [10] in that the former corresponds to the Heine distribution whereas the latter corresponds to the Euler distribution. They are different limitations of the q-Poisson distribution. Later, we discuss some properties of our new q-binomial state.

Let us recall the Holstein-Primakoff transformation (HPT),

$$\hat S_+=\sqrt{2S-a^\dagger a}\,a,\quad \hat S_-=a^\dagger\sqrt{2S-a^\dagger a}\ ,\qquad \hat S_3=S-a^\dagger a\ , \tag{6}$$

where $a^\dagger(a)$ are boson creation (annihilation) operators satisfying $[a,a^\dagger]=1$. $a^\dagger a=S-\hat S_3$ is called the spin deviation occupation number operator possessing the eigenstates $|n\rangle$, $n\le 2S$. Using HPT we are able to reexpress the equation

$$\hat S_+|S_3\rangle=[S(S+1)-S_3(S_3+1)]^{1/2}|S_3+1\rangle \tag{7}$$

as

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$$\hat{S}_+|n\rangle=\sqrt{2S-(n-1)}\sqrt{n}\,|n-1\rangle \ . \quad (8)$$

Equations (7) and (8) tell us that $\hat{S}_+$ operating on $|S_3\rangle$ gives rise to a state having S_3+1; this means that the occupation number n decreases by one unit, becoming $n-1$. Similarly, we have

$$\hat{S}_-|n\rangle=\sqrt{2S-n}\,\sqrt{n+1}|n+1\rangle \ . \quad (9)$$

By noticing that the highest-weight state $|S\rangle$ is now expressed as $|0\rangle$, using $a^\dagger|n\rangle=\sqrt{n+1}|n+1\rangle$ we can calculate

$$\hat{S}_-^m|S\rangle=(a^\dagger\sqrt{2S-a^\dagger a}\,)^m|0\rangle = \begin{bmatrix}2S\\ m\end{bmatrix}^{1/2} m!|m\rangle, \quad |m\rangle=\frac{a^{\dagger m}}{\sqrt{m!}}|0\rangle \ . \quad (10)$$

It then follows from (5) and (10) that

$$|\tau\rangle=(1+|\tau|^2)^{-S}\sum_{m=0}^{\infty}\frac{\hat{S}_-^m}{m!}(-\tau^*)^m|S\rangle = (1+|\tau|^2)^{-S}\sum_{m=0}^{2S}\begin{bmatrix}2S\\ m\end{bmatrix}^{1/2}(-\tau^*)^m|m\rangle \ . \quad (11)$$

On the other hand, let the number M in Eq. (1) be $2S$; we can express $|\eta,M\rangle$ as

$$|\eta,2S\rangle=\sum_{m=0}^{2S}\left[\begin{bmatrix}2S\\ m\end{bmatrix}\eta^m(1-\eta)^{2S-m}\right]^{1/2}|m\rangle$$
$$=(1-\eta)^S\sum_{m=0}^{2S}\frac{1}{m!}(a^\dagger\sqrt{2S-a^\dagger a}\,)^m|0\rangle \times[\eta/(1-\eta)]^{m/2}$$
$$=\left(\frac{1}{1-\eta}\right)^{-S}\sum_{m=0}^{\infty}\frac{\hat{S}_-^m}{m!}|S\rangle[\eta/(1-\eta)]^{m/2}$$
$$=|\tau\rangle\Big|_{\tau^*=-(\eta/1-\eta)^{1/2}} , \quad (12)$$

which turns out to be a particular spin coherent state when one compares Eqs. (12) and (11).

As a q extension we define the q-binomial state as

$$|\eta,M\rangle_q=\sum_{n=0}^{M}(p_B(n;M,\eta,q))^{1/2}|n\rangle_q \ , \quad (13)$$

where $0<q<1$ and $\eta>0$,

$$p_B(n;M,\eta,q)=\frac{1}{(1+\eta)_q^M}\begin{bmatrix}M\\ n\end{bmatrix}_q q^{n(n-1)/2}\eta^n \ . \quad (14)$$

Here the q combinatorial and q factorial are defined as

$$\begin{bmatrix}M\\ n\end{bmatrix}_q=\frac{[M]_q!}{[n]_q![M-n]_q!} \ ,$$
$$[n]_q!=[n]_q[n-1]_q\cdots[1]_q, \quad [n]_q=\frac{1-q^n}{1-q} \ , \quad (15)$$

whereas the q-binomial expansion is given by

$$(1+\eta)_q^M=(1+\eta)(1+q\eta)\ldots(1+q^{M-1}\eta) = (1+\eta)(1+q\eta)_q^{M-1} \ . \quad (16)$$

The state $|n\rangle_q$ is the q-deformed number state, the eigenstates of N_q,

$$|n\rangle_q=(a_q^{\dagger n}/\sqrt{[n]_q!})|0\rangle_q \ , \quad (17)$$

where the q operators $a_q^\dagger(a_q)$ and N_q satisfy the q-Heisenberg algebra [6,7]

$$a_q a_q^\dagger-q a_q^\dagger a_q=1, \quad [N_q,a_q^\dagger]=a_q^\dagger, \quad [N_q,a_q]=-a_q \ . \quad (18)$$

$P_B(n;M,\eta,q)$ in Eq. (14) may be called the probability mass function (PMF) of the q-deformed binomial distribution because when $q=1$ and $\eta=\tan^2\alpha$, $P_B(n,M,\eta,q)$ reduces to the ordinary binomial distribution, i.e.,

$$P_B(n;M,\tan^2\alpha,1)=\begin{bmatrix}M\\ n\end{bmatrix}(\cos^2\alpha)^M(\tan^2\alpha)^n = \begin{bmatrix}M\\ n\end{bmatrix}(\sin^2\alpha)^n(1-\sin^2\alpha)^{M-n} \ . \quad (19)$$

In terms of the q-binomial theorem [8–10]

$$(x+y)_q^M=\sum_{n=0}^{M}\begin{bmatrix}M\\ n\end{bmatrix}_q q^{n(n-1)/2}x^{M-n}y^n \quad (20)$$

we see the normalization of $P_B(n;M,\eta,q)$

$$\sum_{n=0}^{M}P_B(n;M,\eta,q)=\frac{1}{(1+\eta)_q^M}\sum_{n=0}^{M}\begin{bmatrix}M\\ n\end{bmatrix}_q q^{n(n-1)/2}\eta^n=1 \ , \quad (21)$$

which leads us to know the mean

$$_q\langle\eta,M|\eta,M\rangle_q=1 \ , \quad (22)$$

$$_q\langle\eta,M|a_q^\dagger a_q|\eta,M\rangle_q=\sum_{n=0}^{M}\frac{1}{(1+\eta)_q^M}\begin{bmatrix}M\\ n\end{bmatrix}_q q^{n(n-1)/2}\eta^n[n]_q = \eta[M]_q/1+\eta \ . \quad (23)$$

When $M\to\infty$, $P_B(n;M,\eta,q)$ approaches the Heine distribution P_H [11], which is one of the q analogs of the ordinary Poisson distribution, i.e.,

$$P_H(n;\eta,q)=\frac{(\eta/1-q)^n}{[n]_{q^{-1}}!}(e_{q^{-1}}^{\eta/1-q})^{-1} \quad \left[\lim_{M\to\infty}(1+\eta)_q^M=e_{q^{-1}}^{\eta/1-q}\right] , \quad (24)$$

where

$$[n]_{q^{-1}}=\frac{1-q^{-n}}{1-q^{-1}}=q^{1-n}[n]_q, \quad e_{q^{-1}}^x=\sum_{n=0}^{\infty}\frac{x^n}{[n]_{q^{-1}}!} \ . \quad (25)$$

It must be pointed out that the q-binomial state $|\eta,M\rangle_q$ is quite different from the one introduced in our earlier paper [10], where the q deformation of the binomial distribution is given by

$$\begin{bmatrix}M\\ n\end{bmatrix}_q\eta^n(1-\eta)_q^{M-n} \ , \quad (26)$$

whose limiting form as $M \to \infty$ is the so-called Euler distribution [11], i.e.,

$$\{(\eta/1-q)^n/[n]_q!\}(e_q^{\eta/1-q})^{-1} . \tag{27}$$

The reason we introduced the new q analog of binomial state (13) lies in the fact that it is a good candidate for comparison with the q-spin coherent state [su(2)$_q$ coherent state]. The su(2)$_q$ algebra, as studied in Ref. [12], is defined as

$$[\hat{J}_3,\hat{J}_\pm]=\pm\hat{J}_\pm, \quad [\hat{J}_+,\hat{J}_-]=\frac{q^{\hat{J}_3}-q^{-\hat{J}_3}}{q^{1/2}-q^{-1/2}} . \tag{28}$$

For the sake of later convenience we introduce the transformation

$$\hat{S}'_+=q^{(1/2)\hat{J}_3-(1/4)}\hat{J}_+, \quad \hat{S}'_-=\hat{J}_-q^{(1/2)\hat{J}_3-(1/4)}, \quad \hat{S}'_3=\hat{J}_3 \tag{29}$$

to change Eq. (28) to

$$[\hat{S}'_3,S'_\pm]=\pm\hat{S}'_\pm ,$$
$$[\hat{S}'_+,\hat{S}'_-]_{q^{-1}}\equiv\hat{S}'_+\hat{S}'_- - q^{-1}\hat{S}'_-\hat{S}'_+=(1-q^{2\hat{S}'_3})/1-q . \tag{30}$$

According to $[n]_q=(1-q^n)/(1-q)$ we can write $[\hat{S}'_+,\hat{S}'_-]_{q^{-1}}=[2\hat{S}'_3]_q$. Then we introduce the q analog of the Holstein-Primakoff realization for su(2)$_q$ generators

$$\hat{S}'_+=q^{-(1/2)N_q}\sqrt{[2S-N_q]_q}a_q,$$
$$\hat{S}'_-=a_q^\dagger\sqrt{[2S-N_q]_q}q^{-(1/2)N_q} , \tag{31}$$
$$\hat{S}'_3=S-N_q ,$$

where $a_q^\dagger$, a_q, and N_q are given by Eq. (18). For any non-negative integer or half-integers the su(2)$_q$ algebra has the irreducible representation $|S,S_3\rangle_q$ with $S_3=-S,-S+1,\ldots,S-1,S$. Because $N_q=S-\hat{S}'_3$, the eigenstates of $\hat{S}'_3$ can be expressed as the q-number state $|n=S-S_3\rangle_q\equiv|n\rangle_q$; thus using

$$a_q|n\rangle_q=\sqrt{[n]_q}|n-1\rangle_q, \quad a_q^\dagger|n\rangle_q=\sqrt{[n+1]_q}|n+1\rangle_q, \tag{32}$$

we can derive

$$\hat{S}'_+|n\rangle_q=q^{-N_q/2}\sqrt{[2S-N_q]_q}a_q|n\rangle_q$$
$$=q^{-(1/2)(n-1)}\sqrt{[n]_q[2S-n+1]_q}|n-1\rangle_q$$
$$=\sqrt{[S-S_3]_{q^{-1}}[S+S_3+1]_q}|S,S_3+1\rangle_q , \tag{33}$$

$$\hat{S}'_-|n\rangle_q=q^{-n/2}\sqrt{[n+1]_q[2S-n]_q}|n+1\rangle_q$$
$$=\sqrt{[S+S_3]_q[S-S_3+1]_{q^{-1}}}|S,S_3-1\rangle_q . \tag{34}$$

Noticing that the highest-weight state $|S,S\rangle_q$ is now expressed as $|n=0\rangle_q$, we have

$$\hat{S}'^m_-|S,S\rangle_q=(a_q^\dagger\sqrt{[2S-N_q]_q}q^{-N_q/2})^m|0\rangle_q$$
$$=q^{-m(m-1)/4}\left[\begin{bmatrix}2S\\ m\end{bmatrix}_q\right]^{1/2}[m]_q!|m\rangle_q . \tag{35}$$

On the other hand, following Ref. [3] and using (35) we can construct an su(2)$_q$ coherent state

$$|Z\rangle_q^S=\{(1+|Z|^2)_q^{2S}\}^{-1/2}e_{q^{-1}}^{-Z^*\hat{S}'_-}|S,S\rangle_q$$
$$=\{(1+|Z|^2)_q^{2S}\}^{-1/2}$$
$$\times\sum_{m=0}^{\infty}\{(-Z^*)^m/[m]_q!\}\hat{S}'^m_-|0\rangle_q q^{m(m-1)/2} . \tag{36}$$

This state is normalized because

$${}_q^S\langle Z|Z\rangle_q^S=\frac{1}{(1+|Z|^2)_q^{2S}}\sum_{m=0}^{2S}\begin{bmatrix}2S\\ m\end{bmatrix}_q q^{m(m-1)/2}|Z|^{2m}=1 . \tag{37}$$

By comparing Eq. (13) with (36) we can identify the q-deformed binomial state $|\eta,M=2S\rangle_q$ as the su(2)$_q$ coherent state, i.e.,

$$|\eta,2S\rangle_q=\sum_{m=0}^{2S}\left[\frac{1}{(1+\eta)_q^{2S}}\begin{bmatrix}2S\\ m\end{bmatrix}_q q^{m(m-1)/2}\eta^m\right]^{1/2}|m\rangle_q$$
$$=|Z\rangle_q^S\Big|_{Z^*=-\sqrt{\eta}} . \tag{38}$$

Hence Eq. (38) is a q extension of Eq. (12).

We emphasize that although there are two ways to define the q-binomial state [see Eqs. (13) and [10]], only the q-binomial state $|\eta,M\rangle_q$, whose limit distribution as $M\to\infty$ goes to the Heine distribution (not the Euler distribution), can be identified as a $|Z\rangle_q^S$.

Now we examine whether the new q-binomial state $|\eta,M\rangle_q$ is antibunched. Using (16) we have

$$a_q|\eta,M\rangle_q=\sum_{n=1}^{M}\sqrt{P_B(n;M,\eta,q)[n]_q}|n-1\rangle_q$$
$$=\left[\frac{\eta[M]_q}{1+\eta}\right]^{1/2}\sum_{n=1}^{M}\left\{\frac{1}{(1+q\eta)_q^{M-1}}\frac{[M-1]_q!}{[n-1]_q![M-n]_q!}q^{n(n-1)/2}\eta^{n-1}\right\}^{1/2}|n-1\rangle_q$$
$$=\left[\frac{\eta[M]_q}{1+\eta}\right]^{1/2}\sum_{n=0}^{M-1}\left\{\frac{1}{(1+q\eta)_q^{M-1}}\begin{bmatrix}M-1\\ n\end{bmatrix}_q q^{n(n-1)/2}(q\eta)^n\right\}^{1/2}|n\rangle_q$$
$$=\left[\frac{\eta[M]_q}{\cdot 1+\eta}\right]^{1/2}\sum_{n=0}^{M-1}\sqrt{P_B(n;M-1,q\eta,q)}|n\rangle_q=\left[\frac{\eta[M]_q}{1+\eta}\right]^{1/2}|q\eta,M-1\rangle_q \tag{39}$$

and

$$a_q^n|\eta,M\rangle_q=\left\{\frac{\eta^n q^{n(n-1)/2}[M]_q!}{(1+\eta)_q^n[M-n]_q!}\right\}^{1/2}|q^n\eta,M-n\rangle_q\ . \tag{40}$$

Using the identity $(\eta+q)(1+\eta)_q^M=q(1+q^{-1}\eta)_q^{M+1}$ and $[0]_q=0$ we show

$$\begin{aligned}a_q^\dagger|\eta,M\rangle_q&=\sum_{n=0}^{M}\sqrt{P_B(n;M,\eta,q)[n+1]_q}|n+1\rangle_q\\&=\left[\frac{\eta+q}{\eta[M+1]_q}\right]^{1/2}\sum_{n=0}^{M}\left\{\frac{1}{(1+q^{-1}\eta)_q^{M+1}}\begin{bmatrix}M+1\\n+1\end{bmatrix}_q q^{n(n+1)/2}(q^{-1}\eta)^{n+1}\right\}^{1/2}[n+1]_q|n+1\rangle_q\\&=\left[\frac{\eta+q}{\eta[M+1]_q}\right]^{1/2}\sum_{n=0}^{M+1}\sqrt{P_B(n;M+1,q^{-1}\eta,q)}[N_q]_q|n\rangle_q=\left[\frac{\eta+q}{\eta[M+1]_q}\right]^{1/2}[N_q]_q|q^{-1}\eta,M+1\rangle_q\ .\end{aligned} \tag{41}$$

It then follows that

$$a_q^{\dagger n}|\eta,M\rangle_q=\{(\eta+q)_q^n[M]_q!/\eta^n[M+n]_q!\}^{1/2}a_q^{\dagger n}a_q^n|q^{-n}\eta,M+n\rangle_q\ . \tag{42}$$

As a result of Eqs. (13) and (39) we derive the mean

$${}_q\langle\eta,M|[N_q]_q|\eta,M\rangle_q=\sum_{n=1}^{M}P_B(n;M,\eta,q)[n]_q=\{\eta[M]_q/(1+\eta)\}\sum_{n=0}^{M-1}P_B(n;M-1,q\eta,q)=\eta[M]_q/1+\eta\ , \tag{43}$$

$$\begin{aligned}{}_q\langle\eta,M|[N_q]_q^2|\eta,M\rangle_q&=\sum_{n=1}^{M}P_B[n;M,\eta,q)[n]_q^2=\{\eta[M]_q/(1+\eta)\}\sum_{n=0}^{M-1}P_B(n;M-1,q\eta,q)[n+1]_q\\&=\frac{\eta[M]_q}{1+\eta}+\frac{q^2\eta^2[M]_q[M-1]_q}{(1+\eta)(1+q\eta)}\sum_{n=0}^{M-2}P_B(n;M-2,q^2\eta,q)=\frac{\eta[M]_q}{1+\eta}+\frac{q^2\eta^2[M]_q[M-1]_q}{(1+\eta)(1+q\eta)}\ .\end{aligned} \tag{44}$$

Thus the variance is

$${}_q\langle\eta,M|[N_q]_q^2|\eta,M\rangle_q-{}_q\langle\eta,M|[N_q]_q|\eta,M\rangle_q^2=\eta[M]_q(1+q^M\eta)/(1+\eta)^2(1+q\eta)\ . \tag{45}$$

The ratio of the variance to the mean is then given by

$${}_q\langle\eta,M|([N_q]_q-{}_q\langle\eta,M|[N_q]_q|\eta,M\rangle_q)^2|\eta,M\rangle_q/{}_q\langle\eta,M|[N_q]_q|\eta,M\rangle_q=(1+q^M\eta)/(1+\eta)(1+q\eta)>0\ , \tag{46}$$

indicating the sub-Poisson nature of the q-binomial state $|\eta,M\rangle_q$. Moreover, the bunching parameter for the q-binomial state is

$${}_q\langle\eta,M|([N_q]_q-{}_q\langle\eta,M|[N_q]_q|\eta,M\rangle_q)^2|\eta,M\rangle_q-{}_q\langle\eta,M|[N_q]_q|\eta,M\rangle_q=-\frac{\eta^2[M]_q(1+q-q^M+\eta q)}{(1+\eta)^2(1+q\eta)}<0\ . \tag{47}$$

By calculating

$$\begin{aligned}{}_q\langle\eta,M|a_q^{\dagger 2}a_q^2|\eta,M\rangle_q&={}_q\langle\eta,M|[N_q]_q[N_q-1]_q|\eta,M\rangle_q=\sum_{n=2}^{M}P_B(n;M,\eta,q)[n]_q[n-1]_q\\&=\{\eta^2[M]_q[M-1]_q/(1+\eta)(1+q\eta)\}\sum_{n=2}^{M}\frac{1}{(1+q^2\eta)_q^{M-2}}\frac{[M-2]_q!}{[n-2]_q![M-n]_q!}q^{n(n-1)/2}\eta^{n-2}\\&=q\eta^2[M]_q[M-1]_q/(1+\eta)(1+q\eta)\ ,\end{aligned} \tag{48}$$

we obtain the second-order correlation function

$$G^{(2)}\equiv{}_q\langle\eta,M|a_q^{\dagger 2}a_q^2|\eta,M\rangle_q/{}_q\langle\eta,M|a_q^\dagger a_q|\eta,M\rangle_q^2=q(1+\eta)[M-1]_q/(1+q\eta)[M]_q<1\ , \tag{49}$$

which shows that the q-binomial state $|\eta,M\rangle_q$ is antibunched.

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