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Annihilation operators and coherent states for the Jaynes-Cummings model

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An alternative way of diagonalizing the Jaynes-Cummings Hamiltonian is proposed and allows the definition of annihilation operators and coherent states for this model. Mean values and dispersions over these states are computed and interpreted. Limiting cases which are physically interesting are also examined.

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I. INTRODUCTION

The Jaynes-Cummings (JC) model [1], which is extensively used in quantum optics, describes, in its simplest version, the interaction of a cavity mode with a two-level system. The interest of this model, its solvability, and its applications have long been discussed (see Refs. [1,2] and references quoted therein). More precisely, dynamical properties have been obtained through the use of the so-called "coherent states JC model" [2–4], i.e., states which are initially harmonic oscillator coherent states, but that evolve according to the JC Hamiltonian.

The group theoretical approach to these states [5] has open the way for generalizations to supergroups [6–8] and even to quantum groups [9]. Indeed, under some hypotheses, the JC model may be seen as a generalization of the SUSY harmonic oscillator system and its exact solvability may be explained in terms of SUSY breaking [10]. Moreover, some ways to supersymmetrize this model have been investigated [11,12]. From another point of view, the JC Hamiltonian may also be interpreted as an element of a superalgebra. Due to that, it is possible to show the existence of a dynamical superalgebra, which produces results about the energy spectrum [13] and the "supercoherent states" [8]. The quantum group approach follows the same lines, defining a new q Hamiltonian and studying its properties in q-coherent states [9].

Coming back to the supergroup approach, the definition of "supercoherent states" has required the introduction of odd Grassmann numbers in the Hamiltonian as well as in the states [8]. This fact has been justified by its success for the supersymmetric (SUSY) harmonic oscillator [7] but leads, in our opinion (and we will comment on that point in Sec. III), to some difficulties in the physical interpretation of the results.

In this paper, the traditional approach to coherent states, i.e., the definition of them as eigenstates of an annihilation operator, is considered through the diagonalization of the JC Hamiltonian. The connection with the group-theoretical approach is then given. It is based on the direct product of the Weyl-Heisenberg group with SU(2), and differs essentially from the work of Kochetov [8]. The coherent states we obtain are different from the ones studied before [2–4]. In order to show their significance, we compute several physical quantities over these states, like the total number of particles, the energy and the atomic inversion. We insist on the similarities and differences with respect to the other approaches and make some comments on the exact resonance case and the weak coupling limit.

The contents of the paper are the following. In Sec. II, a brief discussion of the JC model is given. The presentation explicitly exhibits the connection with the SUSY harmonic oscillator. In Sec. III, we present an alternative way of diagonalizing the JC Hamiltonian, together with the form of the unitary operator which realizes it. We also explain the relevance of this diagonal form in the definition of suitable annihilation operators. Section IV is devoted to the definition of coherent states as eigenstates of an annihilation operator. Their time evolution is also given and comparison with other coherent states of the JC model is made. In Sec. V, mean values and dispersions of physical quantities are computed. Interesting properties are exhibited, as the importance of considering a large number of photons, the existence of minima of the energy dispersion and the presence of expected oscillations in the atomic inversion.

II. THE JAYNES-CUMMINGS MODEL AND ITS RELATION TO THE SUSY HARMONIC OSCILLATOR

Let us recall [1] that the JC model describes a spin-1/2 fermion (or equivalently a two-level system) in interaction with a one-mode magnetic field having an oscillating component along the x axis and a constant component along the z axis. Explicitly we have

$$\mathbf{B}(\mathbf{r},t) = \frac{\omega_0}{\gamma}\mathbf{e}_z + c(\kappa)[a e^{i(\kappa y-\omega t)} + a^\dagger e^{-i(\kappa y-\omega t)}]\mathbf{e}_x , \tag{2.1}$$

where $c(\kappa)$ is a constant dependent on κ and other parameters [14] and γ is the gyromagnetic ratio. In the rotating-wave approximation, it may be described by the Hamiltonian [1,3]

$$H_{\rm JC} = \omega\left(a^\dagger a + \frac{1}{2}\right)\sigma_0 + \frac{\omega_0}{2}\sigma_3 + \kappa(a^\dagger\sigma_- + a\sigma_+)\,. \quad (2.2)$$

In this expression, $a^\dagger$ and a are the photon creation and annihilation operators, $\sigma_\pm = \sigma_1 \pm i\sigma_2$, where $\{\sigma_1, \sigma_2, \sigma_3\}$ are the usual Pauli matrices and σ_0 is the identity matrix. Moreover, κ is a coupling constant, ω is the field mode frequency, and ω_0 is the atomic frequency. Let us also introduce the detuning

$$\Delta = \omega - \omega_0\,. \quad (2.3)$$

The exact solvability of this model is well known. Let us here recall the energy eigenvalues and eigenstates in order to fix the notation. We work in the Fock space

$$\mathcal{F} = \mathcal{F}_b \otimes \mathcal{F}_f = \left\{ |n,-\rangle = \begin{pmatrix} 0 \\ |n\rangle \end{pmatrix},\ |n,+\rangle = \begin{pmatrix} |n\rangle \\ 0 \end{pmatrix},\ n = 0,1,2,\ldots \right\}. \quad (2.4)$$

The energy eigenstates then take the form (for $n = 0,1,2,\ldots$)

$$|E_0^-\rangle = |0,-\rangle\,, \quad (2.5)$$

$$|E_{n+1}^-\rangle = \frac{1}{R(n+1)}\left[\kappa\sqrt{n+1}|n,+\rangle + \left(\frac{\Delta}{2} + \kappa\, r(n+1)\right)|n+1,-\rangle\right], \quad (2.6)$$

$$|E_n^+\rangle = \frac{1}{R(n+1)}\left[\left(\frac{\Delta}{2} + \kappa\, r(n+1)\right)|n,+\rangle - \kappa\sqrt{n+1}|n+1,-\rangle\right], \quad (2.7)$$

where

$$r(n) = (\delta + n)^{1/2}, \quad \delta = \left(\frac{\Delta}{2\kappa}\right)^2, \quad (2.8)$$

$$R(n) = \left[\left(\frac{\Delta}{2} + \kappa\, r(n)\right)^2 + \kappa^2 n\right]^{1/2} = \left[2\kappa\, r(n)\left(\frac{\Delta}{2} + \kappa\, r(n)\right)\right]^{1/2}. \quad (2.9)$$

In the expression of $r(n)$ we have introduced the parameter δ which will play a role in the following. The corresponding energy eigenvalues are

$$E_n^- = \omega n + \kappa\, r(n),$$
$$E_n^+ = \omega(n+1) - \kappa\, r(n+1)\,. \quad (2.10)$$

In order to make clear the connection between the JC model and the SUSY harmonic oscillator [10], we write the JC Hamiltonian as

$$H_{\rm JC} = \frac{1}{2}(\omega + \omega_0)\mathcal{N} - \frac{\Delta}{2}\mathcal{M} + \frac{i\kappa}{\sqrt{\omega}}(Q^\dagger - Q)\,, \quad (2.11)$$

where

$$\mathcal{N} = \left(a^\dagger a + \frac{1}{2}\right)\sigma_0 + \frac{1}{2}\sigma_3,$$
$$\mathcal{M} = -\left(a^\dagger + \frac{1}{2}\right)\sigma_0 + \frac{1}{2}\sigma_3\,, \quad (2.12)$$

$$Q = i\sqrt{\omega}a^\dagger\sigma_-, \quad Q^\dagger = -i\sqrt{\omega}a\sigma_+\,. \quad (2.13)$$

In the absence of the oscillating component of the magnetic field **B**, and for the exact resonance ($\Delta = 0$), we get the SUSY Hamiltonian

$$H_{\rm JC}(\Delta = \kappa = 0) = H_{\rm SUSY} = \{Q^\dagger, Q\}\,. \quad (2.14)$$

The operators Q and $Q^\dagger$ given in (2.13) become conserved supercharges. As it has already been noticed [10], the interaction term of the form $(Q^\dagger - Q)$ prevents the JC model to be supersymmetric. The exact solvability may then be explained (in the exact resonance case) as a result of SUSY breaking.

III. ANNIHILATION OPERATORS FOR $H_{\rm JC}$

The connection with the (SUSY) harmonic oscillator is very useful for our considerations. It will be used to find an annihilation operator for $H_{\rm JC}$, and the associated coherent states.

An annihilation operator for the SUSY harmonic oscillator $H_{\rm SUSY}$ may be chosen as

$$A_D = a\sigma_0\,. \quad (3.1)$$

Another possibility has been proposed in Ref. [6], but A_D will be prefered because it leads to an identification of the different definitions for the coherent states of the SUSY harmonic oscillator (see Ref. [15]). Clearly, A_D is not an annihilation operator for $H_{\rm JC}$ but we will show that it is for the associated diagonalized Hamiltonian.

The diagonalization of $H_{\rm JC}$ is easily performed by looking for the unitary operator $\mathcal{O}$ which connects the Hamiltonians. We get

$$H_D \equiv \mathcal{O}^\dagger H_{\rm JC}\mathcal{O} = \begin{pmatrix} \omega(N+1) - \kappa\, r(N+1) & 0 \\ 0 & \omega N + \kappa\, r(N) \end{pmatrix}, \quad (3.2)$$

where $N = a^\dagger a$, and the definition of $r(N)$ is given in Eq. (2.8). The operator $\mathcal{O}$ takes the form

$$\mathcal{O} = \begin{pmatrix} \frac{1}{R(N+1)}\left(\frac{\Delta}{2} + \kappa\, r(N+1)\right) & \frac{\kappa}{R(N+1)} a \\ -a^{\dagger} \frac{\kappa}{R(N+1)} & \frac{1}{R(N)}\left(\frac{\Delta}{2} + \kappa\, r(N)\right) \end{pmatrix} . \tag{3.3}$$

It can be written

$$\mathcal{O} = \exp(-Z)\,, \tag{3.4}$$

where Z is the skewhermitian operator

$$Z = a^{\dagger} f(N+1)\sigma_{-} - f(N+1)a\sigma_{+}\,. \tag{3.5}$$

The function $f(N)$ reads formally

$$f(N) = -\frac{1}{\sqrt{N}} \arctan\left(\frac{2\kappa\sqrt{N}}{\Delta + 2\kappa\, r(N)}\right) . \tag{3.6}$$

We are interested in the form (3.4) of $\mathcal{O}$ in order to compare our result with the work of Buzano *et al.* [13]. This approach is concerned with a diagonalization of the JC model based on the existence of a dynamical superalgebra. More precisely, it is shown that since H_{JC} given in (2.2) may be written as an element of a $u(1/1)$ superalgebra, it is posssible to find an inner homomorphism which diagonalizes H_{JC}, and then $u(1/1)$ turns out to be a dynamical superalgebra. Even if this result seems to be mathematically correct, two problems arise. The first one is the necessity of taking the coupling constant κ to be an odd Grassmann number loosing its physical interpretation. The second one, which is a consequence of the first, is that the spectrum of the diagonal Hamiltonian is truncated.

Let us make our remark clearer. The superalgebra $u(1/1)$ is shown [13] to be generated by the two even generators $\mathcal{N}$ and $\mathcal{M}$ given in (2.12) and the two odd ones Q and $Q^{\dagger}$ given in (2.13). The structure relations are

$$[\mathcal{N}, \mathcal{M}] = 0, \quad [\mathcal{N}, Q] = [\mathcal{N}, Q^{\dagger}] = 0,$$

$$[\mathcal{M}, Q] = -2Q\ , \quad [\mathcal{M}, Q^{\dagger}] = 2Q^{\dagger} \tag{3.7}$$

and

$$\{Q, Q^{\dagger}\} = \omega\mathcal{N}, \quad Q^2 = (Q^{\dagger})^2 = 0\,. \tag{3.8}$$

The Hamiltonian H_{JC} in (2.11) is then seen as a linear combination of the $u(1/1)$ generators. To be able to generalize the concept of dynamical algebra (or spectrum generating algebra) to superalgebra, it is necessary to take H_{JC} of the form

$$H'_{JC} = \frac{1}{2}(\omega + \omega_0)\mathcal{N} - \frac{\Delta}{2}\mathcal{M} + \frac{i}{\sqrt{\omega}}(\Gamma Q^{\dagger} + Q\overline{\Gamma})\,, \tag{3.9}$$

where Γ and $\overline{\Gamma}$ are odd Grassmann numbers. Moreover, the unitary operator

$$U = \exp Z' = \exp[-i(\psi Q^{\dagger} + Q\overline{\psi})]\,, \tag{3.10}$$

where ψ and $\overline{\psi}$ are odd Grassmann numbers, diagonalizes H'_{JC}. Such a diagonalization is easy to realize because of the presence of odd Grassmann quantities (their square equals zero and they anticommute between each other). The corresponding energy spectrum contains only $\psi\overline{\psi}$ (not its powers) and, with the identification $\psi \to \kappa$, we have the expected spectrum for H_{JC}, but truncated.

Our diagonalization does not have the problem mentioned before because we do not refer to any Grassmann element. We get the exact spectrum for H_D. Let us mention that even if Z in (3.5) is not an element of a superalgebra, it has a form which is similar to Z' in (3.10), but with the function $f(N)$ of (3.6) in place of ψ. Indeed, Z can be written

$$Z = -\frac{i}{\sqrt{\omega}}\left[f(N+1)Q^{\dagger} + Qf(N+1)\right]\,. \tag{3.11}$$

Now let us return to the question of finding annihilation operators for H_{JC}. The diagonal Hamiltonian H_D of (3.2) will be of great help. Since it only depends on N, an annihilator for it is given by A_D in (3.1). This explains *a posteriori* the introduction of the index D to remind us that we are working in the diagonal basis. Since the states depend also on the spin index $\pm$, we introduce the spinorial annihilation and creation operators

$$\Sigma_{-_D} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \Sigma_{+_D} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{3.12}$$

Note that $\Sigma_{\pm_D}$ are nothing else than the matrices $\sigma_{\pm}$. The change of notation is made in order to avoid any confusion when we work in the different bases (diagonal or not).

We have found candidates to be annihilation and creation operators for H_{JC}. The operators A_D and $A_D^{\dagger}$ give rise to

$$A = \mathcal{O}A_D\mathcal{O}^{\dagger}, \quad A^{\dagger} = \mathcal{O}A_D^{\dagger}\mathcal{O}^{\dagger}\,, \tag{3.13}$$

while the operators $\Sigma_{\pm_D}$ become

$$\Sigma_{\pm} = \mathcal{O}\Sigma_{\pm_D}\mathcal{O}^{\dagger}\,. \tag{3.14}$$

Note that the operators $\sigma_{\pm}$ really represent the physical spin of the system while the $\Sigma_{\pm}$ are annihilation and creation operators. The $\sigma_{\pm}$ and $\Sigma_{\pm}$ must not be confused. Due to the form of $\mathcal{O}$ in (3.3), the explicit expression of Σ is complicated.

IV. COHERENT STATES FOR H_{JC}

For the determination of coherent states, the situation is particularly simple when we work with H_D in (3.2), since the energy eigenstates are the Fock-space basis vectors (2.4), that is the eigenstates of the SUSY harmonic oscillator. We have then everything working in analogy with the SUSY harmonic oscillator.

Let us recall that coherent states can be defined in three ways: as minimum uncertainty states, as eigenstates of an annihilation operator or as displacement operator states. We will adopt the last way which is also known as the group theoretical approach [7]. For the second one, we will easily see that it is a consequence of the group-theoretical approach. Note that the first way would require a comparison with a classical version of the JC model.

To the annihilation and creation operators A_D, $A_D^\dagger$, and to $\Sigma_{\pm D}$, we associate the displacement operator (in analogy with the SUSY harmonic oscillator [7])

$$T(z,\beta)_D = \exp(zA_D^\dagger - \bar{z}A_D + \beta\Sigma_{+D} - \bar{\beta}\Sigma_{-D}),\quad z,\beta \in \mathbb{C}. \tag{4.1}$$

It can be written

$$T(z,\beta)_D = D(z)_D S(\beta)_D, \tag{4.2}$$

where

$$D(z)_D = \exp(zA_D^\dagger - \bar{z}A_D),\quad S(\beta)_D = \exp(\beta\Sigma_{+D} - \bar{\beta}\Sigma_{-D}). \tag{4.3}$$

The coherent states are then defined by

$$|z,\beta\rangle_D = T(z,\beta)_D|0,-\rangle. \tag{4.4}$$

Clearly, the coherent states for H_{JC} are

$$|z,\beta\rangle = \mathcal{O}|z,\beta\rangle_D = T(z,\beta)|E_0^-\rangle, \tag{4.5}$$

where

$$T(z,\beta) = \mathcal{O}T(z,\beta)_D\mathcal{O}^\dagger. \tag{4.6}$$

From the harmonic-oscillator coherent states [7], we easily get

$$|z,\beta\rangle_D = \cos(\theta/2)\begin{pmatrix} 0 \\ |z\rangle \end{pmatrix} + \sin(\theta/2)e^{i\phi}\begin{pmatrix} |z\rangle \\ 0 \end{pmatrix}, \tag{4.7}$$

with $\beta = (\theta/2)e^{i\phi}$ and $|z\rangle$ the normalized state

$$|z\rangle = e^{-|z|^2/2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle. \tag{4.8}$$

The state $|z,\beta\rangle_D$ is then a linear combination of the "fundamental coherent states"

$$|z+\rangle_D = \begin{pmatrix} |z\rangle \\ 0 \end{pmatrix},\quad |z-\rangle_D = \begin{pmatrix} 0 \\ |z\rangle \end{pmatrix}, \tag{4.9}$$

which are both eigenstates of A_D (but not of Σ_{-D}).

Let us mention that, even if we refer to the SUSY harmonic oscillator, the preceding approach is only based on a group and not on a supergroup. Indeed the group we are working with is the direct product of the Weyl-Heisenberg group and SU(2). To change to supergroups we only have to change the complex number β into an odd Grassmann number and to give a type to the generators (to graduate the vector space): the operators A and $A^\dagger$ will be taken even while the operators $\Sigma_\pm$ will be taken odd. With such a definition, the states (4.9) will become eigenstates of both A and Σ_-. The main problem will be in the interpretation of the odd Grassmann numbers since, as it has already been mentioned, the coupling constant κ will also become an odd Grassmann number (for the consistency of the theory). This is actually the reason why we have not decided to use the supergroup approach.

Let us end this section by giving the time evolution of the fundamental states (4.9); the time evolution of the general state (4.7) will be obtained easily by linear combination. We start with the diagonal case for which the evolution operator is

$$U_D(t) = e^{-itH_D} = \begin{pmatrix} \bar{e}^{it[\omega(N+1)-\kappa r(N+1)]} & 0 \\ 0 & \bar{e}^{it[\omega N+\kappa r(N)]} \end{pmatrix} \tag{4.10}$$

and we compute

$$|z,t,\pm\rangle = \mathcal{O}U_D(t)|z,\pm\rangle_D = U_{\rm JC}(t)\mathcal{O}|z,\pm\rangle_D. \tag{4.11}$$

They are explicitly given by

$$|z,t,+\rangle = e^{-|z|^2/2}e^{-i\omega t}\times\sum_{n=0}^{\infty}\frac{(ze^{-i\omega t})^n}{\sqrt{n!}}e^{it\kappa r(n+1)}|E_n^+\rangle,$$
$$|z,t,-\rangle = e^{-|z|^2/2}\sum_{n=0}^{\infty}\frac{(ze^{-i\omega t})^n}{\sqrt{n!}}e^{-it\kappa r(n)}|E_n^-\rangle. \tag{4.12}$$

These are similar to the states obtained by the evolution of the harmonic-oscillator coherent states except for the supplementary oscillation for each n in the sum. This implies that the coherent state does not evolve in time to another coherent state, on the contrary to the case of the harmonic oscillator.

Moreover, our states (4.12) are different from the ones considered in the other approaches of "coherent states JC model" [2–4]. Indeed, all these approaches deal with the states (4.9) (or a mixture of them) and their evolution is given by

$$U_{\rm JC}(t)|z,\pm\rangle_D = e^{-itH_{\rm JC}}|z,\pm\rangle_D. \tag{4.13}$$

V. RELEVANT PHYSICAL QUANTITIES

To put the emphasis on the similarities and differences between our states and the usual ones, we compute several physical quantities of the system over the states (4.12). They will be the total number of particles, the energy, and the atomic inversion.

Let now begin by some generalities and notations which will clarify the dependence of our physical quantities on the characteristics of the system. First, let us recall the introduction of the parameter δ in the expression of $r(n)$ in (2.8). It will be used as a variable in the following. It contains both the detuning Δ and the coupling parameter κ and leads to the exact resonance case when $\delta = 0$ or to the weak coupling limit when $\delta \to \infty$. The new parameter $x = |z|^2$ is also introduced and will be proved to be a good approximation of the number of photons. Second, we will deal with the function

$$G(\delta, x) = e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} r(n+1) = e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} \sqrt{\delta + n + 1}. \tag{5.1}$$

Note that it can be shown to have the integral form [16]

$$G(\delta, x) = \frac{2}{\sqrt{\pi}} e^{-x} \int_0^{\infty} e^{x e^{-t^2}} \times [(\delta+1) e^{-(\delta+1)t^2} + x e^{-(\delta+2)t^2}]\, dt. \tag{5.2}$$

A useful property is its asymptotic behavior. Using Laplace's method [17], we can see from Eq. (5.2) that for large x we have

$$G(\delta, x) \sim \sqrt{x}\,, \tag{5.3}$$

which is independent of δ. Finally, the calculations will be done explicitly over the fundamental states (4.12) since for a general state $|z, \beta, t\rangle$ we can use (4.7) and (4.12) to write the mean value and dispersion of an operator X. If we put

$$\langle X \rangle_{\pm} = \langle z, t, \pm | X | z, t, \pm \rangle, \quad \langle X \rangle_{\pm\mp} = \langle z, t, \pm | X | z, t, \mp \rangle\,, \tag{5.4}$$

the mean value of X over a general coherent state is

$$\langle X \rangle = \frac{1}{2}[(1 - \cos\theta)\langle X \rangle_+ + (1 + \cos\theta)\langle X \rangle_- + \sin\theta (e^{i\phi}\langle X \rangle_{+-} + e^{-i\phi}\langle X \rangle_{-+})]\,. \tag{5.5}$$

In the case of having $\langle X \rangle_{+-} = \langle X \rangle_{-+} = 0$ (this will be the case in what follows), the square of the dispersion is simply

$$(\Delta X)^2 \equiv \langle X^2 \rangle - \langle X \rangle^2 = \tfrac{1}{2}[(1 - \cos\theta)(\Delta X)_+^2 + (1 + \cos\theta)(\Delta X)_-^2 + \tfrac{1}{2}\sin^2\theta(\langle X \rangle_+ - \langle X \rangle_-)^2]\,. \tag{5.6}$$

A. The number of particles

The operator $\mathcal{N}$ in (2.12) corresponds to the total number of particles. It is a constant of motion and is invariant under the transformation by $\mathcal{O}$. We then get

$$\langle \mathcal{N} \rangle_+ = x + 1, \quad \langle \mathcal{N} \rangle_- = x, \quad (\Delta\mathcal{N})_+^2 = (\Delta\mathcal{N})_-^2 = x\,, \tag{5.7}$$

which are known results in connection with the SUSY harmonic oscillator.

The evaluation of the mean values of the number of photons $N = a^\dagger a$ is less trivial and gives

$$\langle N \rangle_+ = x + \frac{1}{2} - \frac{\Delta}{4\kappa} e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} \frac{1}{r(n+1)}\,, \quad \langle N \rangle_- = x - \frac{1}{2} + \frac{\Delta}{4\kappa} e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} \frac{1}{r(n)}\,. \tag{5.8}$$

Comparing with the harmonic oscillator where $\langle N \rangle = x$, we have a correction due to the interaction. Since the coupling constant κ is usually small, we can say that a good approximation of the average number of photons is x. Indeed, the contribution of the terms containing the series is, in this case, approximately 1/2.

B. The energy

We compute the mean values and dispersions of the energy in the fundamental states and study their behavior with respect to both x and δ. Note that since $\langle H_{JC} \rangle_{+-} = \langle H_{JC} \rangle_{-+} = 0$, the calculations over the general coherent states through Eqs. (5.5) and (5.6) do not give anything new with respect to the results for the fundamental states. Indeed, we can see from (5.6) that the dispersion attains its minimum only over the pure states. The mean values are easily computed and take the simple form

$$\langle H_{JC} \rangle_+ = \omega[(x+1) - \lambda G(\delta, x)], \quad \langle H_{JC} \rangle_- = \omega[x + \lambda G(\delta - 1, x)]\,, \tag{5.9}$$

while the values of the dispersion are more complicated and present interesting features

$$(\Delta H_{JC})_+^2 = \omega^2\{\lambda^2(1+\delta) + (1+\lambda^2)x + 2\lambda x[G(\delta, x) - G(\delta+1, x)] - \lambda^2[G(\delta, x)]^2\}\,, \tag{5.10}$$

$$(\Delta H_{JC})_-^2 = \omega^2\{\lambda^2\delta + (1+\lambda^2)x - 2\lambda x[G(\delta-1, x) - G(\delta, x)] - \lambda^2[G(\delta-1, x)]^2\}\,. \tag{5.11}$$

We have introduced $\lambda = \kappa/\omega$ because it will be useful to examine the dependence of the dispersions on the physical parameters.

First, when a large number of photons is considered, we can use the asymptotic behavior (5.3) of $G(\delta, x)$ to see that

$$\frac{(\Delta H_{JC})_{\pm}}{\langle H_{JC} \rangle_{\pm}} \sim \frac{1}{\sqrt{x}}\,, \tag{5.12}$$

as in the harmonic-oscillator case.

Second, we want to see how the dispersion evolves with respect to a variation of the characteristics of the system, i.e., the detuning Δ and the coupling constant κ, through δ and λ. Let us then concentrate on the form of $(\Delta H_{JC})_+^2$

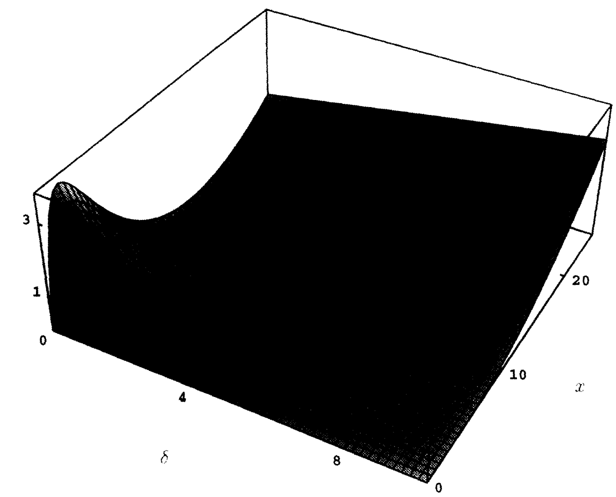

FIG. 1. Behavior of $(\Delta H_{JC})^2_+/\omega^2$ as a function of δ and x for $\lambda = 8$.

since $(\Delta H_{JC})^2_-$ shows a similar qualitative behavior. If we fix λ, the typical behavior of $(\Delta H_{JC})^2_+$ in (δ, x) is as in Fig. 1. So it can be proved that for fixed values of δ smaller than a certain δ_0, $(\Delta H_{JC})^2_+$ has a minimum for $x \neq 0$. Figure 2 shows how the minimum moves to zero when δ increases. We can now fix $\delta = 0$, which is the exact resonance case, to examine the dependence of $(\Delta H_{JC})^2_+$ in (λ, x). The surface is given in Fig. 3 and we can see that the minimum of the section λ = constant moves to zero as λ tends to zero. The sections x = constant are parabolas as we see from Eq. (5.10).

Finally, the case of weak coupling has to be treated differently. Indeed, when κ is small, it is δ^{-1} which is small and we have

$$\kappa G(\delta, x) \simeq \left(\frac{\Delta}{2}\right)^2 e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} \left(1 + \frac{1}{2}\delta^{-1}(n+1)\right)$$
$$\simeq \frac{\Delta}{2} + \frac{\kappa^2}{\Delta}(x+1) \,. \qquad (5.13)$$

Hence we can write the following limits

$$\langle H_{JC}\rangle_\pm \simeq \left(\omega \mp \frac{\kappa^2}{\Delta}\right)\left(x + \frac{1}{2} \pm \frac{1}{2}\right) \mp \frac{\Delta}{2},$$
$$[(\Delta H_{JC})^2_\pm] \simeq \omega \left(\omega \mp \frac{2\kappa^2}{\Delta}\right) x \,, \qquad (5.14)$$

and we are very close to the harmonic-oscillator case.

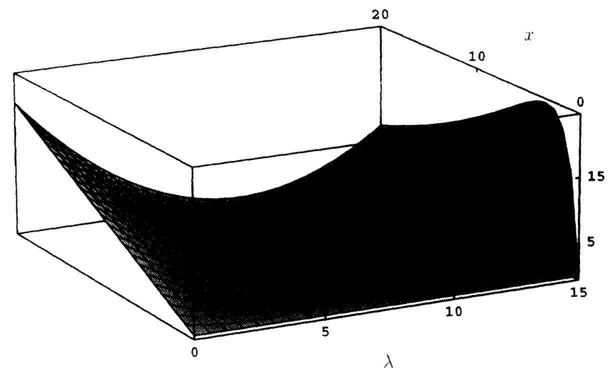

FIG. 3. Behavior of $(\Delta H_{JC})^2_+/\omega^2$ for exact resonance ($\delta = 0$). The sections x = const are parabolas whose minima move to zero as $\lambda \to 0$.

C. The atomic inversion

Let us recall that if the field mode of H_{JC} has been prepared in a usual coherent state, i.e., the states (4.13), the temporal behavior of the atomic inversion consists of Rabi oscillations [3,4]. If we start with our states (4.12), we see that the mean values of the atomic inversion are time independent. Indeed, they take the form

$$\langle\sigma_3\rangle_+ = \frac{\Delta}{2\kappa} e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} \frac{1}{r(n+1)},$$
$$\langle\sigma_3\rangle_- = -\frac{\Delta}{2\kappa} e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} \frac{1}{r(n)} \,. \qquad (5.15)$$

In fact, the Rabi oscillations still appear, but when the atomic inversion is computed over a general coherent state. The use of (5.5) shows that the time dependence comes from the value $\langle\sigma_3\rangle_{+-} = \langle\sigma_3\rangle_{-+}$ and we get

$$\langle\sigma_3\rangle = \frac{1}{2} e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} \left[\frac{\Delta}{2\kappa}\left(\frac{1-\cos\theta}{r(n+1)} - \frac{1+\cos\theta}{r(n)}\right) + 2\sin\theta \frac{\cos\varphi_n(t)}{r(n+1)}\right], \qquad (5.16)$$

where $\varphi_n(t) = \phi + 2t\kappa\, r(n+1)$. In order to compare this result with the one of Narozhny *et al.* [3], we take $\Delta = 0$ and $\theta = -\phi = \pi/2$. We then get

$$\langle\sigma_3\rangle = \frac{1}{2} e^{-x} \left[-1 + 2\sum_{n=0}^{\infty} \frac{x^n}{n!} \frac{\sin(2t\kappa\sqrt{n+1})}{\sqrt{n+1}}\right] \,. \qquad (5.17)$$

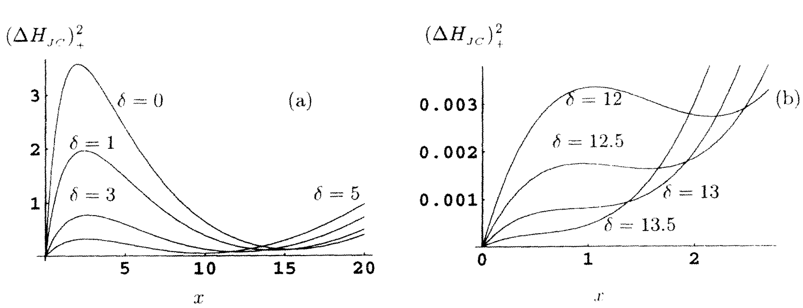

FIG. 2. Graphs showing the disappearance of the extrema of Fig. 1 for $\delta_0 \simeq 12.85$.

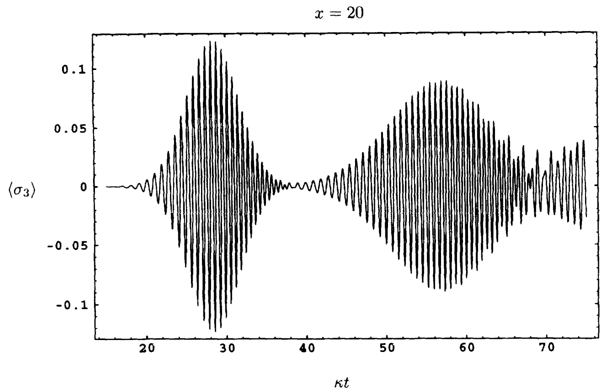

FIG. 4. The collapse and revivals of the atomic inversion in a general coherent state for $x = 20$ and $\delta = 0$.

Notice that the derivative of this function with respect to t is precisely the value obtained in Eq. (3.2) of Ref. [3] (up to a constant factor, and for $m = 1$).

In Fig. 4, we show the graph of our $\langle\sigma_3\rangle$ for $x = 20$. It is of similar form as the ones obtained in many other papers (see, for example, Refs. [3,4]) and that although the expression of $\langle\sigma_3\rangle$ is not exactly the same in all the cases. In fact, this is not surprising because of the oscillating behavior of the series and the relationship between the different forms of $\langle\sigma_3\rangle$ already mentioned. Figure 4 shows the revivals that characterize the atomic inversion in this model.

VI. CONCLUSION

A natural definition [15] of coherent states for the SUSY harmonic oscillator has lead us to the construction of corresponding states for the JC model. Our main idea has been to diagonalize the JC Hamiltonian so that the coherent states we were searching for were transformed as the ones of the SUSY harmonic oscillator. The explicit form of the JC coherent states has been given using the unitary transformation realizing the diagonalization. Our coherent states have physically interesting properties and are also advantageously compared with the harmonic-oscillator coherent states (both bosonic and SUSY).

In fact, the connection of the JC model with the SUSY harmonic oscillator has led to tentatives [8,11–13] of treating this model using "superstructures." As we have already noticed, the problem which appears deals with the introduction of odd Grassmann quantities that cannot directly be interpreted physically. We have avoided this approach in our work. Nevertheless, the consideration of a superclassical version of the JC model (close to that of the SUSY harmonic oscillator) would make possible an interpretation of the Grassmann objects. It would also possibly lead to a superclassical limit of our coherent states. This is a direction for future developments of our work.

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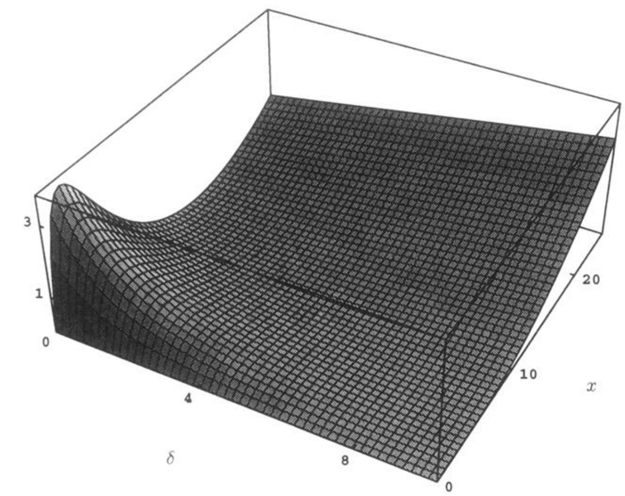

FIG. 1. Behavior of $(\Delta H_{\rm JC})^2_+/\omega^2$ as a function of δ and x for $\lambda = 8$.

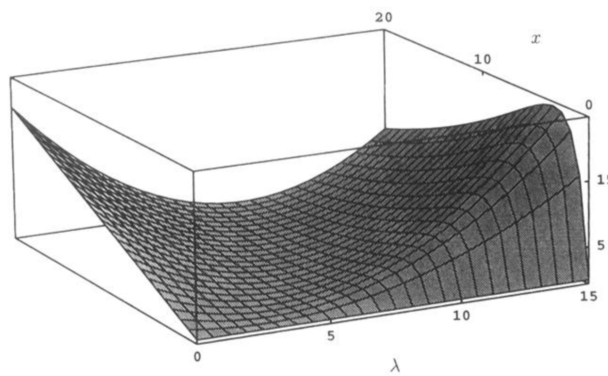

FIG. 3. Behavior of $(\Delta H_{\rm JC})^2_+/\omega^2$ for exact resonance ($\delta = 0$). The sections $x = \text{const}$ are parabolas whose minima move to zero as $\lambda \to 0$.