PHYSICAL REVIEW A VOLUME 50, NUMBER 2 AUGUST 1994

Spectral analysis of the degenerate optical parametric oscillator as a noiseless amplifier

I. E. Protsenko* and L. A. Lugiato
Dipartimento di Fisica, Università di Milano, 20133 Milano, Italy

C. Fabre
Laboratoire Kastler Brossel, Université Pierre et Marie Curie, Case 74, 75252 Paris Cedex 05, France
(Received 26 January 1994)

We consider a degenerate optical parametric oscillator driven by a stationary field of frequency 2Ω and by a modulated field of frequency Ω with a small modulation (signal) of frequency $\omega' \ll \Omega$. By using the eigenvectors of an appropriate matrix we identify the conditions to obtain maximum amplification when a single quadrature component is modulated; the amplification coefficient is calculated analytically as a function of the frequency ω' and of the parameters. Using a method of quantum Langevin equations, we express in analytic form the noise spectrum for a generic quadrature component of the field of frequency Ω. By combining the results concerning amplification and noise we compare the signal-to-noise ratio in the output with that in the input, and determine the range of parameters in which the performance of the system is better than that of the phase-insensitive linear amplifier. In particular we identify the cases in which noiseless amplification is possible. The approach developed in this paper can be applied to a generic cavity-based phase-sensitive amplifier.

PACS number(s): 42.50.Dv, 42.50.Lc, 42.65.Ky

I. INTRODUCTION

It is well known that phase independent linear amplifiers introduce unavoidably at least 3 dB of noise in the output, whereas phase-sensitive amplification can be noiseless, in the sense that the signal-to-noise ratio in the output is equal to that in the input [1–8]. Experiments which approach the noiseless amplification performance have been reported recently [9,10].

In this paper, we provide a theoretical description of noiseless amplification considering both the frequency dependence of the signal amplification, and the full spectral composition of the noise in the output as it is measured, for example, in any squeezing experiment. Thus, we are able to study the full frequency dependence of the signal-to-noise ratio in the output compared with that in the input. Precisely, we focus on the case of degenerate parametric oscillators. A nonlinear $\chi^{(2)}$ medium, which converts the frequency 2Ω into the frequency Ω and vice versa, is contained in an optical one-ended cavity, close to resonance with both frequencies 2Ω and Ω. The system is driven by two coherent fields with frequencies 2Ω and Ω, respectively. The first driving field is stationary, whereas the second is composed by a stationary part and by a contribution modulated with a frequency ω'; the second term corresponds to the signal, which carries the information.

We describe this system using the quantum model of [11], [12]. First, we analyze the semiclassical model in absence of modulation; our calculation of the stationary solution of the system generalizes the results of [11–13]. Second, still in the semiclassical model we compare the intensity of the modulated part of the output with that of the input. By using the eigenvalues of an appropriately defined matrix $[M]$ which governs the amplification of the signal, we determine the conditions for best amplification. This step, once added to the request that the modulation in the input occurs in a single quadrature component, identifies the input quadrature component that must be modulated to achieve optimum amplification, and ensures that the modulation in the output occurs in the same quadrature component. The two conditions of best amplification and single quadrature component modulation can be simultaneously satisfied only in three well-defined cases, which are analyzed in details in the following. In each of these cases, the amplification coefficient of the small signal is calculated analytically for all values of frequency ω'.

Next we consider the fully quantum-mechanical version of the model linearized around a stationary solution. For each of the three cases the spectrum of squeezing in the output field is calculated analytically for an arbitrary quadrature component, and the squeezing properties of the quantum fluctuations in the output field are described as a function of the parameters. Finally, in order to analyze the signal-to-noise ratio with the best amplification of the signal, we focus on the modulated quadrature component. The quantum noise in this quadrature also displays, of course, amplification, or "antisqueezing," i.e., the contrary to the squeezing. We compare the signal-to-noise ratio at the output and at the input for all values of ω'. In this way, we are able to determine the values of the system parameters, which allow to preserve best the signal-to-noise ratio.

In Sec. II we analyze the stationary solution of the nonlinear semiclassical equations over the parameter space of the system and, in particular, we identify the cases in which the system exhibits a bistable response. In

*Permanent address: Lebedev Physics Institute, Leninsky prospect 53, 117924, Moscow, Russia.

this section, we restrict our investigation to the case $\Delta_0=-\Delta_1$, where Δ_0 and Δ_1 are the detuning parameters for the two fields with frequencies 2Ω and Ω, respectively; this limitation no longer applies to the following part of the paper. Section III is devoted to the analysis of the amplification of a weak monochromatic signal in the mode of frequency Ω; we identify the three cases in which optimum amplification and single quadrature modulation are simultaneously possible. In Sec. IV we calculate the noise spectrum in the three cases and, on this basis, we analyze in Sec. V the transfer coefficient between input and output signal-to-noise ratios. The results for the optical parametric oscillator are systematically compared with the performance of the linear phase-insensitive amplifier. Section VI summarizes the main results of the paper.

II. THE DYNAMICAL EQUATIONS AND THE STATIONARY SOLUTIONS

A. The dynamical model

The cavity-based parametric amplifier that we study is presented in Fig. 1. The χ^2 nonlinear medium located inside the signal-ended cavity causes the parametric down-conversion and second-harmonic generation processes:

$$\Omega_0 \rightarrow \Omega_1+\Omega_1=\Omega_0 \quad \text{and} \quad \Omega_1+\Omega_1 \rightarrow \Omega_0 , \tag{1}$$

respectively, between the quasimonochromatic fields $\mathcal{E}_j$, $j=0,1$ of optical frequencies Ω_j:

$$\mathcal{E}_j=\mathrm{Re}[\alpha_j \exp(-i\Omega_j t)] , \tag{2}$$

where α_j is the slowly varying amplitude. The intracavity fields (2) originate from the input fields $\mathcal{E}_j^{(\mathrm{in})}$, and produce the output fields $\mathcal{E}_j^{(\mathrm{out})}$ ($j=0,1$):

$$\begin{aligned} \mathcal{E}_j^{(\mathrm{in})}&=\mathrm{Re}[\alpha_j^{(\mathrm{in})}\exp(-i\Omega_j t)] , \\ \mathcal{E}_j^{(\mathrm{out})}&=\mathrm{Re}[\alpha_j^{(\mathrm{out})}\exp(-i\Omega_j t)] . \end{aligned} \tag{3}$$

We assume that $\Omega_0=2\Omega_1$ and the slowly varying envelope approximation:

$$|\dot{\alpha}_j/\alpha_j| \ll \Omega_j . \tag{4}$$

The time evolution of the amplitudes α_j, $j=0,1$ is governed by the model formulated in [11]:

$$\dot{\alpha}_0=-\gamma_0(i\Delta_0+1)\alpha_0-(g/2)\alpha_2^2+2\gamma_0^{1/2}\alpha_0^{(\mathrm{in})} , \tag{5}$$

$$\dot{\alpha}_1=-\gamma_1(i\Delta_1+1)\alpha_1+g\alpha_1^*\alpha_0+2\gamma_1^{1/2}\alpha_1^{(\mathrm{in})} , \tag{6}$$

where the differentiation is made with respect to dimensionless time

$$\tau \equiv t/(2\tau_R) , \tag{7}$$

and τ_R is the round trip time of the cavity. The parameters γ_j denote the transmission coefficients of mirror 1 (Fig. 1) for the intensity of the field $j=0,1$; g is the coupling constant of the two-photon process (1) multiplied by $2\tau_R$, and finally

$$\Delta_j \equiv \frac{2\tau_R(\Omega_{Rj}-\Omega_j)}{\gamma_j} , \tag{8}$$

where Ω_{Rj} is the frequency of the cavity mode closest to Ω_j. The factor 2 in the last term of Eqs. (5), (6) arises from the fact that the cavity is one ended.

In order to simplify Eqs. (5), (6) we introduce the normalized variables:

$$A_0 \equiv g\alpha_0/\gamma_1, \quad A_1 \equiv g\alpha_1/(2\gamma_0\gamma_1)^{1/2} , \tag{9}$$

and normalized parameters

$$E \equiv \frac{2g\alpha_0^{(\mathrm{in})}}{\gamma_1\gamma_0^{1/2}}, \quad e \equiv \frac{\sqrt{2}g\alpha_1^{(\mathrm{in})}}{\gamma_1\gamma_0^{1/2}}, \quad \gamma \equiv \gamma_1/\gamma_0 . \tag{10}$$

Then, Eqs. (5), (6) become

$$\gamma_0^{-1}\dot{A}_0=E-(1+i\Delta_0)A_0-A_1^2 , \tag{11}$$

$$\gamma_1^{-1}\dot{A}_1=e-(1+i\Delta_1)A_1-A_1^*A_0 . \tag{12}$$

Equations (11), (12) are the same as in [13], apart from the term e describing the input for field 1. We can safely assume, that the input field E for the field 0 is real and non-negative, while e is in general complex:

$$e \equiv |e|\exp(i\varphi), \quad E \geq 0 . \tag{13}$$

B. The stationary solutions

Now we analyze the stationary solutions of Eqs. (11), (12) by setting $\dot{A}_j=0$:

$$0=E-(i\Delta_0+1)A_0-A_1^2 , \tag{14}$$

$$0=e-(i\Delta_1+1)A_1+A_1^*A_0 . \tag{15}$$

In order to reduce the number of parameters in Eqs. (14), (15) one can introduce the notations:

$$A_1 \equiv [(1+\Delta_1^2)(1+\Delta_0^2)]^{1/4}\overline{A}_1 , \tag{16}$$

$$A_0 \equiv (1+\Delta_1^2)^{1/2}\exp[-i(\phi_1-\phi_0)/2]\overline{A}_0 , \tag{17}$$

$$\begin{aligned} e \equiv\ & (1+\Delta_1^2)^{3/4}(1+\Delta_0^2)^{1/4} \\ & \times\exp[-i(\phi_1-\phi_0)/2]\overline{e}\times\exp(i\overline{\varphi}) , \end{aligned} \tag{18}$$

$$E \equiv [(1+\Delta_1^2)(1+\Delta_0^2)]^{1/2}\overline{E} , \tag{19}$$

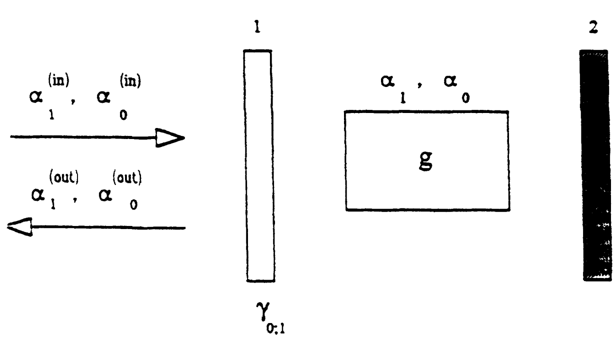

FIG. 1. Scheme of parametric amplifier with single-ended cavity. Mirror 2 is assumed to be completely reflecting.

where $\bar{e}$ is real and positive and

$$\cos\phi_j=(1+\Delta_j^2)^{-1/2},\quad \sin\phi_j=\Delta_j(1+\Delta_j^2)^{-1/2}\ , \tag{20}$$

so that Eqs. (14), (15) become

$$0=\bar{E}-e^{i\psi}\bar{A}_0-\bar{A}_1^2\ , \tag{21}$$

$$0=\bar{e}\exp(i\bar{\varphi})-e^{i\psi}\bar{A}_1+\bar{A}_1^*\bar{A}_0\ , \tag{22}$$

where

$$\psi\equiv(\tfrac{1}{2})(\phi_0+\phi_1)\ . \tag{23}$$

There are four real parameters in the set (21), (22): ψ, $\bar{E}$, $\bar{e}$, and $\bar{\varphi}$. In order to reduce the difficulty of analyzing the stationary solutions in the parameter space, we assume in the following, that $\phi_0=-\phi_1$, i.e.,

$$\psi=0\ , \tag{24}$$

which is true when

$$\Delta_0=-\Delta_1\equiv\Delta\ , \tag{25}$$

so that Eqs. (16)–(19) reduce to

$$A_1=(1+\Delta^2)^{1/2}\bar{A}_1,\quad A_0=(1-i\Delta)\bar{A}_0\ ,$$

$$E=(1+\Delta^2)\bar{E},\quad e=(1+\Delta^2)^{1/2}(1-i\Delta)\bar{e}\exp(i\varphi) \tag{26}$$

and Eqs. (14), (15) become

$$0=\bar{E}-\bar{A}_0-\bar{A}_1^2\ , \tag{27}$$

$$0=\bar{e}\exp(i\bar{\varphi})-\bar{A}_1+\bar{A}_1^*\bar{A}_0\ . \tag{28}$$

From (27) we have

$$\bar{A}_0=\bar{E}-\bar{A}_1^2\ , \tag{29}$$

so that, introducing the notation

$$|\bar{A}_1|^2\equiv\bar{I}_1\ , \tag{30}$$

we have from (28)

$$0=\bar{e}\exp(i\bar{\varphi})-\bar{A}_1(1+\bar{I}_1)+\bar{A}_1^*\bar{E}\ , \tag{31}$$

$$0=\bar{e}\exp(-i\bar{\varphi})-\bar{A}_1^*(1+\bar{I}_1)+\bar{A}_1\bar{E}\ . \tag{32}$$

The analysis of the cases $\bar{\varphi}=0,\pi$ was already done in [14]; in the following we consider general value for $\bar{\varphi}$. By solving (31), (32) with respect to $\bar{A}_1$ we find

$$\bar{A}_1=\bar{e}\frac{\exp(i\bar{\varphi})(1+\bar{I}_1)+\exp(-i\bar{\varphi})\bar{E}}{(1+\bar{I}_1)^2-\bar{E}^2} \tag{33}$$

from which, taking the square modulus of both sides, we obtain the equation for $\bar{I}_1$:

$$F(\bar{I}_1)\equiv\bar{I}_1[(1+\bar{I}_1)^2-\bar{E}^2]^2-\bar{e}^2[(1+\bar{I}_1)^2+\bar{E}^2+2\bar{E}(1+\bar{I}_1)\cos(2\bar{\varphi})]=0\ . \tag{34}$$

Equation (34) is of fifth order, so it may have up to five different real roots. Let us find the regions in the parameter space $(\bar{e},\bar{E},\bar{\varphi})$, where there is more than one real and positive root. The simplest procedure to perform this analysis is to fix two of the parameters and to vary the third. If we fix, for example, $\bar{E}$ and $\bar{\varphi}$, it is convenient to solve Eq. (34) with respect to $\bar{e}^2$:

$$\bar{e}^2(\bar{I}_1)=\frac{\bar{I}[(1+\bar{I}_1)^2-\bar{E}^2]^2}{(1+\bar{I}_1)^2+\bar{E}^2+2\bar{E}(1+\bar{I}_1)\cos(2\bar{\varphi})}\ . \tag{35}$$

Some typical curves $\bar{I}_1(\bar{e}^2)$ in Figs. 2(a), 2(b) show that there is coexistence of three stationary solutions in the region

$$0=[\bar{e}^{(t)}]^2<\bar{e}^2<[\bar{e}^{(b)}]^2\ . \tag{36}$$

In correspondence with the boundary values $[\bar{e}(t)]^2\equiv\bar{e}^2(\bar{I}_1^{(t)})$ and $[\bar{e}^{(b)}]^2\equiv\bar{e}^2(\bar{I}_1^{(b)})$ (turning points of

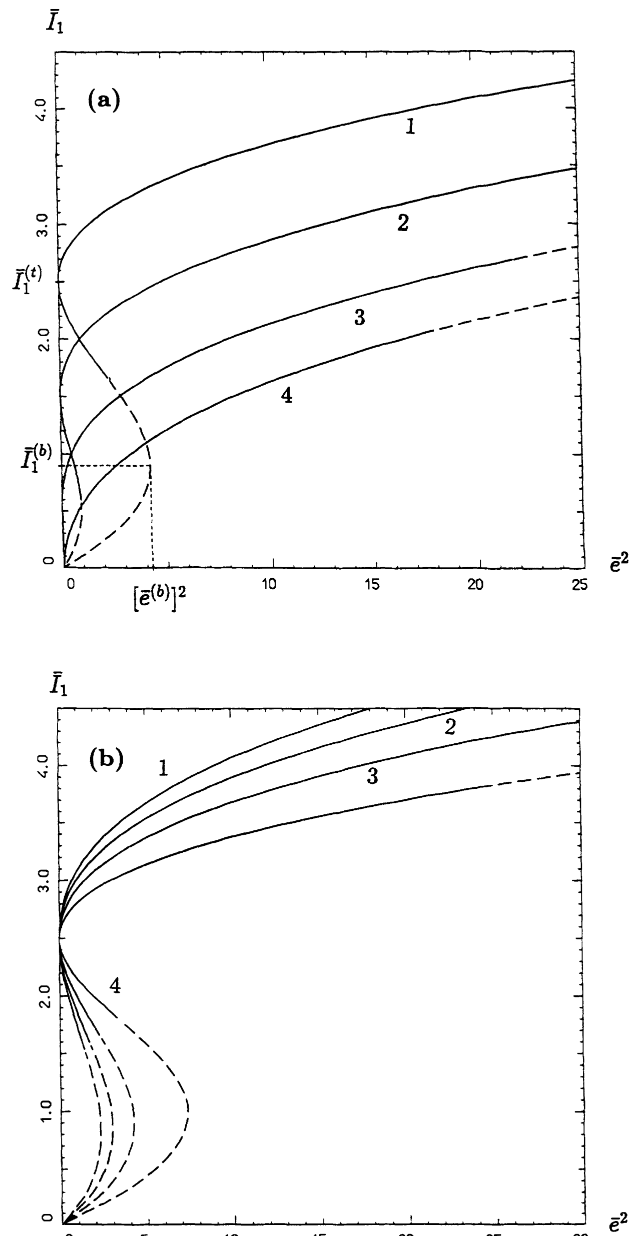

FIG. 2. Stationary intensity $\bar{I}_1$ in the output as a function of the input intensity $\bar{e}^2$ of field 1. Unstable segements for $\gamma=1$ and $\Delta_0=-\Delta_1\equiv\Delta=0.2$ are indicated by broken lines. (a) $\bar{\varphi}=\pi/4$. Curves 1, 2, 3, and 4 correspond to $\bar{E}=3.5$, 2.5, 1.5, 0.5, respectively.

steady-state curve) one has

$$\frac{\partial \bar{e}^2(\bar{I}_1)}{\partial \bar{I}_1}=0 . \tag{37}$$

Another example is obtained by fixing $\bar{e}^2$ and $\bar{E}$. By solving Eq. (34) with respect to $\cos(2\bar{\varphi})$ we have

$$\cos(2\varphi)=\frac{(\bar{I}_1/\bar{e}^2)[(1+\bar{I}_1)^2-\bar{E}^2]^2-(1+\bar{I}_1)^2-\bar{E}^2}{2\bar{E}(1+\bar{I}_1)} . \tag{38}$$

Figure 3 shows some steady-state curves as a function of $\bar{\varphi}$. Also, for this case we find a boundary value $\bar{\varphi}^{(b)}\equiv\bar{\varphi}(\bar{I}_1^{(b)})$, which corresponds to the turning point of the steady-state curve, identified by the equation

$$\frac{\partial \cos(2\bar{\varphi})}{\partial \bar{I}_1}=0 . \tag{39}$$

The curve $\bar{\varphi}(\bar{I}_1)$ presents a peculiarity at $\bar{I}_1=\bar{I}_1^{(t)}$, $\bar{\varphi}=\pi/2$, where the derivative does not exist. To understand what happens in this case one can consider the curve $\bar{I}_1(\bar{e}^2)$ for $\bar{\varphi}$ close to $\pi/2$ (Fig. 4). When $\bar{\varphi}\to\pi/2$ two branches of the curve approach each other and coalesce for $\bar{\varphi}=\pi/2$. This feature is a consequence of the limitation (25), and disappears for $\Delta_0\neq-\Delta_1$.

As one can see from Figs. 2, 3, and 4, in the case (25) that we consider here there is at most coexistence of three stationary solutions. One can find [15] the boundary of the domain of coexistence in the space of the parameters $\bar{\varphi},\bar{e},\bar{E}$ and the corresponding values $\bar{I}_1^{(t)}$ and $\bar{I}_1^{(b)}$, by coupling Eq. (34) with the equation

$$\frac{\partial F}{\partial \bar{I}_1}=0 . \tag{40}$$

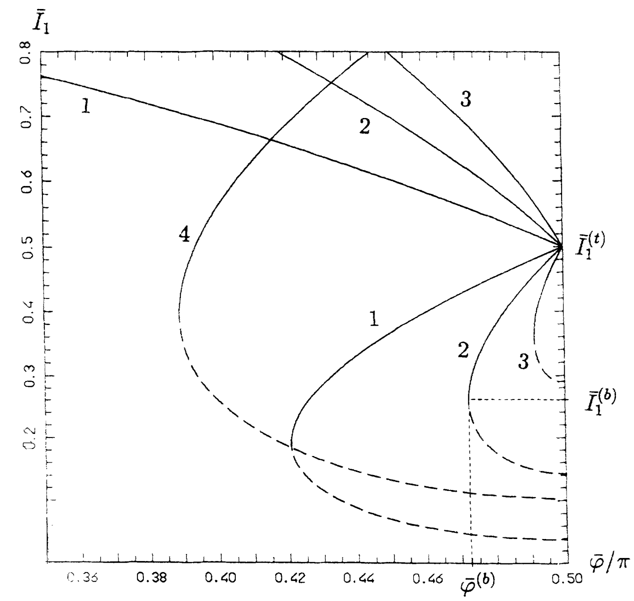

FIG. 3. Stationary value of $\bar{I}_1$ as a function of $\bar{\varphi}$ for (1) $\bar{E}=1.5$, $\bar{e}=0.5$; (2) $\bar{E}=1.5$, $\bar{e}=1$; (3) $\bar{E}=1.5$, $\bar{e}=1.5$; (4) $\bar{E}=2$, $\bar{e}=1$. Unstable portions for $\gamma=1$ and $\Delta=0$ are indicated by broken lines.

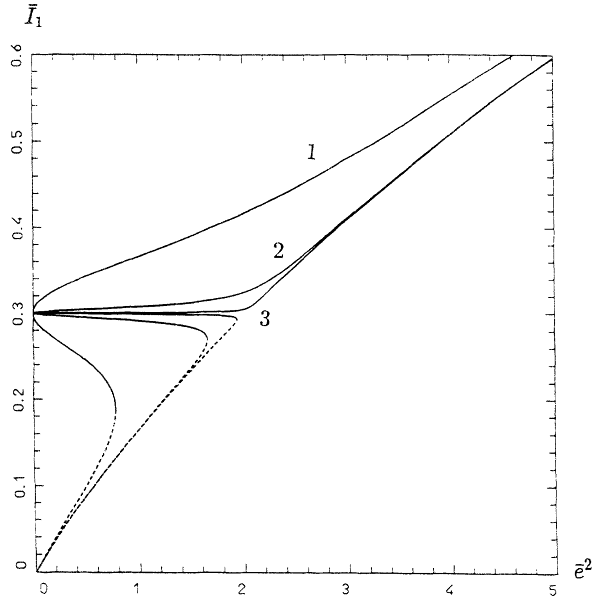

FIG. 4. Stationary dependence of $\bar{I}_1(\bar{e}^2)$, when $\bar{\varphi}$ is close to $\pi/2$: $\bar{E}=1.3$, $\bar{\varphi}=0.49\pi(1)$, $0.499\pi(2)$, $0.4999\pi(3)$. Unstable segments for $\gamma=1$ and $\Delta=0$ are indicated by broken lines.

In terms of the variable

$$x\equiv\bar{I}_1+1 , \tag{41}$$

Eqs. (34), (40) read

$$x^5-x^4-2P_1x^3+P_2x^2+P_3x-[3P_1^2-P_1P_2]=0 , \tag{42}$$

$$5x^4-4x^3-6P_1x^2+2P_2x+P_3=0 , \tag{43}$$

where

$$P_1\equiv\bar{E}^2,\quad P_2\equiv2\bar{E}^2-\bar{e}^2,\quad P_3\equiv\bar{E}^4-2\bar{E}\,\bar{e}^2\cos(2\bar{\varphi}) . \tag{44}$$

By multiplying Eq. (43) by x and subtracting Eq. (42) from the result we obtain

$$4x^5-3x^4-4P_1x^3+P_2x^2+3P_1^2-P_1P_2 \equiv(x^2-P_1)(4x^3-3x^2-3P_1+P_2)=0 \tag{45}$$

which, using Eq. (41), gives the non-negative solutions

$$\bar{I}_1^{(t)}=\bar{E}-1 , \tag{46}$$

$$\bar{I}_1^{(b)}=\tfrac{1}{4}\{[2J^{1/2}+(1+4J)^{1/2}]^{2/3} +[2J^{1/2}-(1+4J)^{1/2}]^{2/3}\} , \tag{47}$$

where

$$J\equiv\bar{e}^2+\bar{E}^2 . \tag{48}$$

By inserting the expressions (46) and (47) into Eqs. (42) and (41), we can find the boundary in the parameter space. More straightforwardly, we substitute Eq. (46) into (35), which is equivalent to Eq. (42) and obtain the first boundary

$$[\bar{e}(t)]^2\equiv\bar{e}^2(\bar{I}_1^{(t)})=0 \tag{49}$$

for arbitrary $\bar{E}$, $\bar{\varphi}$. Similarly, the second boundary equation is conveniently obtained by introducing Eq. (47) into Eq. (38):

$$\cos(2\bar{\varphi})=\frac{(\bar{I}_1^{(b)}/\bar{e}^2)[(1+\bar{I}_1^{(b)})^2-\bar{E}^2]^2-(1+\bar{I}_1^{(b)})^2-\bar{E}^2}{2\bar{E}(1+\bar{I}_1^{(b)})}, \tag{50}$$

where $\bar{I}_1^{(b)}$ is the function of $\bar{e}^2$ and $\bar{E}$ defined by Eqs. (47) and (48). Because $\bar{I}_1>0$ by definition, we see from Eq. (46), that a necessary condition to have coexistence of multiple stationary solutions is that

$$\bar{E}>1 . \tag{51}$$

On the other hand, the expression of $\bar{I}_1^{(b)}$ given by Eq. (47) is always positive for $\bar{E}>1$. The domain of coexistence is shown in Fig. 5 in the plane of variables $(\bar{\varphi},\bar{e})$ for different values of $\bar{E}$.

To complete the analysis of the stationary solution we calculate the phase $\bar{\varphi}_1$ of the complex amplitude $\bar{A}_1$ using Eq. (33), and obtain

$$\exp(i\bar{\varphi}_1)=\pm\frac{(1+\bar{I}_1)\exp(i\bar{\varphi})+\bar{E}\exp(-i\bar{\varphi})}{[(1+\bar{I}_1)^2+\bar{E}^2+2\bar{E}(1+\bar{I}_1)\cos(2\bar{\varphi})]^{1/2}}, \tag{52}$$

where $\bar{I}_1$ is the solution of Eq. (34); the sign "+" refers to the case $\bar{I}_1>\bar{E}-1$, and the sign "−" to the case $0<\bar{I}_1<\bar{E}-1$, respectively, as it follows from Eq. (33). Therefore, in correspondence to the turning point $\bar{I}_1^{(b)}$ in Fig. 2 the two coalescing branches have the same intensity $\bar{I}_1^{(b)}$ but phases different in π. As a matter of fact, it is not possible to pass from one to the other branch by varying only the intensity $\bar{e}^2$ of the input field; in order to obtain the transition it is necessary to control also the plane $\bar{\varphi}$. This is seen clearly by considering simultaneously the case $\bar{\varphi}=0$ and $\bar{\chi}=0$, and $\bar{\varphi}=\pi$, in which the stationary values of $\bar{A}_1$ are real. Introducing the variable $\tilde{e}$ such that

$$\tilde{e}=\begin{cases}\bar{e} & \text{for } \bar{\varphi}=0\\ -\bar{e} & \text{for } \bar{\varphi}=\pi ,\end{cases} \tag{53}$$

one obtains the plot of $\bar{A}_1$ as a function of $\tilde{e}$ shown in Fig. 6 (see also [14]). Clearly, transitions can be obtained only by changing the sign of $\tilde{e}$, as indicated by the arrows.

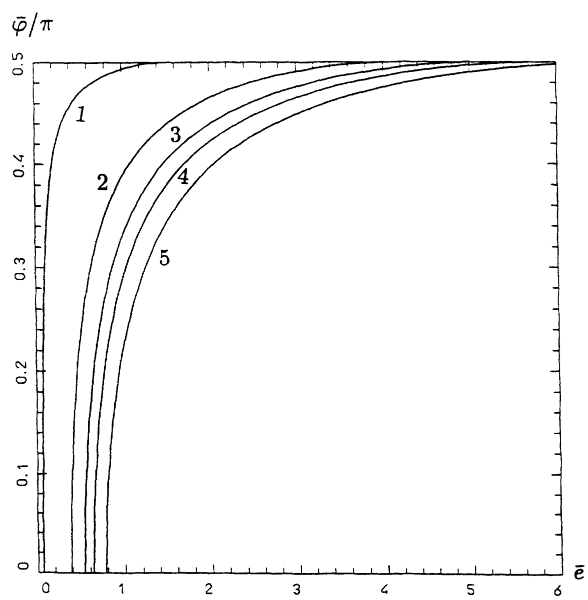

FIG. 5. Domains of coexistence of three stationary solutions in the space of the parameters $\bar{\varphi}$ and $\bar{e}$ for $\bar{E}=1.3$ (curve 1), 2(2), 2.25(3), 2.4(4), 2.6(5). Coexistence occurs in the region located on the left of the curve $\varphi(\bar{e})$.

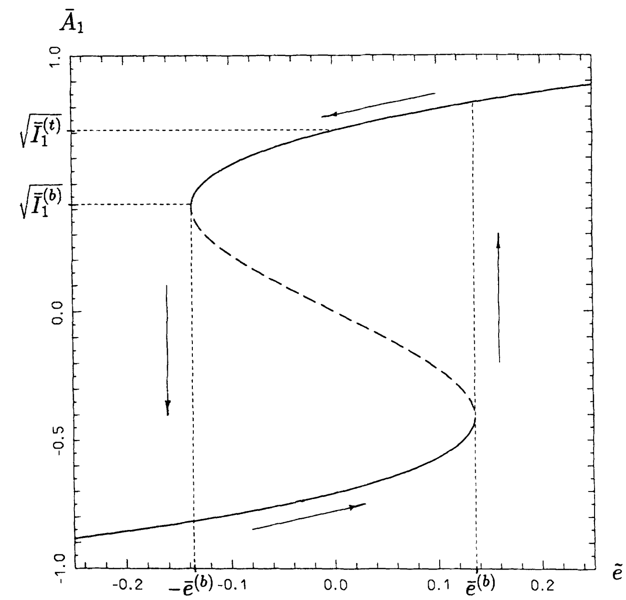

FIG. 6. Steady-state amplitude $\bar{A}_1$ as a function of $\tilde{e}$ for the cases $\bar{\varphi}=0$ (positive $\tilde{e}$) and $\bar{\varphi}=\pi$ (negative $\tilde{e}$). The values of γ and Δ are assumed such that the entire positive-slope parts of this steady-state curve are stable, $\bar{E}=1.5$.

Essential is to control the linear stability of the stationary solutions. For some particular cases we calculate explicit expressions for the four eigenvalues λ of the set of Eqs. (11), (12) linearized around the steady state:

$$\lambda=-1\pm\frac{1}{\sqrt{2}}\left\{I_0-4\left[I_1-\frac{\Delta^2}{2}\right]\pm\left[I_0^2-8I_0\left[I_1+\frac{\Delta^2}{2}\right]+4I_0\Delta^2\right]^{1/2}\right\}^{1/2} \tag{54}$$

for $\gamma_0=\gamma_1$, $\Delta_0=-\Delta_1\equiv\Delta$; $I_i(i=0,1)$ are the stationary values of $|A_i|^2$. For $\Delta=0$ and $\gamma=\gamma_1/\gamma_0$ arbitrary we obtain

$$\lambda=-1\pm\frac{1}{2}\{1-1/\gamma+I_0^{1/2}\pm[(1-1/\gamma+I_0^{1/2})^2-8I_1/\gamma]^{1/2}\} . \tag{55}$$

The four eigenvalues λ are obtained by taking all possible combinations of + and − signs in Eqs. (54), (55). In the general case we carry out the stability analysis numerically using the Routh-Hurwitz criterion [16]. The regions of instability are indicated by broken lines in Figs. 2–4, and 6 and in the figures below. The branch $\bar{I}_1 < \bar{I}_1^{(b)}$ of the stationary solution is always unstable.

III. AMPLIFICATION OF A WEAK MONOCHROMATIC SIGNAL IN THE MODE 1

A. Linearized equations for a weak signal

We assume in this section that the amplitude $\alpha_1^{(\text{in})}$ in the input mode 1 consists of the following contributions. The first term $\alpha_{c1}^{(\text{in})}$ is constant, and the remaining ones represent a monochromatic "signal" of frequency ω', precisely,

$$\alpha_1^{(\text{in})} = \alpha_{c1}^{(\text{in})} + [\alpha_{s1}^{(\text{in})}(\omega')e^{-i\omega'\tau} + \alpha_{s1}^{(\text{in})}(-\omega')e^{i\omega'\tau}] . \quad (56)$$

The notation $\alpha_{s1}^{(\text{in})}(\omega')$ is used for the amplitude of the signal $\sim e^{-i\omega'\tau}$, and the notation $\alpha_{s1}^{(\text{in})}(-\omega')$—for the amplitude of the signal $\sim e^{i\omega'\tau}$, as it is customary for Fourier components. On the other hand, the amplitude of mode 0 is assumed to be constant: $\alpha_0^{(\text{in})} \equiv \alpha_{c0}^{(\text{in})}$. Consistently with Eq. (7) we normalize all frequencies to $(2\tau_R)^{-1}$, e.g.,

$$\Omega'_j \equiv (2\tau_R)\Omega_j \quad (j=0,1) .$$

We assume that the signal varies slowly with respect to the optical frequencies, i.e.,

$$\omega' \ll \Omega'_0, \Omega'_1 , \quad (57)$$

and that the signal amplitudes $\alpha_{s1}^{(\text{in})}(\pm\omega')$ are so small, that they are amplified linearly. Hence the amplitudes α_j and $\alpha_j^{(\text{out})}$ $(j=0,1)$ of the fields in the cavity and of the output fields, respectively, have the form

$$\alpha_j = \alpha_{cj} + [\alpha_{sj}(\omega')e^{-i\omega'\tau} + \alpha_{sj}(-\omega')e^{i\omega'\tau}] \quad (j=0,1) , \quad (58)$$

$$\alpha_j^{(\text{out})} = \alpha_{cj}^{(\text{out})} + [\alpha_{sj}^{(\text{out})}(\omega')e^{-i\omega'\tau} + \alpha_{sj}^{(\text{out})}(-\omega')e^{i\omega'\tau}] \quad (j=0,1) . \quad (59)$$

The amplitudes α_{cj} are stationary solutions of Eqs. (5), (6). The amplitudes $\alpha_j^{(\text{out})}$ are related to $\alpha_j^{(\text{in})}$, α_j as it follows:

$$\alpha_j^{(\text{out})} = \gamma_j^{1/2}\alpha_j - \alpha_j^{(\text{in})} , \quad (60)$$

as imposed the boundary conditions at the partially transparent mirror of the single-ended cavity (see, e.g., [17]) in the high reflectivity limit.

Let us now calculate the amplification coefficient for the small input signal of frequency ω'.

In order to obtain the equations for $\alpha_{sj}(\pm\omega')$ we substitute Eqs. (56), (58) into the set of Eqs. (5), (6), we linearize Eqs. (5), (6) with respect to $\alpha_{sj}(\pm\omega')$ and equate separately the terms $\sim e^{-i\omega'\tau}$. The result of this procedure is

$$[i(\gamma_1\Delta_1 - \omega') + \gamma_1]\bar{\alpha}_1 - g\alpha_{c1}^*\bar{\alpha}_0 - g\alpha_{c0}\bar{\alpha}_{1*} = 2\gamma_1^{1/2}\bar{\alpha}_1^{(\text{in})} , \quad (61)$$

$$[-i(\gamma_1\Delta_1 + \omega') + \gamma_1]\bar{\alpha}_{1*} - g\alpha_{c1}\bar{\alpha}_{0*} - g\alpha_{c0}^*\bar{\alpha}_1 = 2\gamma_1^{1/2}\bar{\alpha}_{1*}^{(\text{in})} , \quad (62)$$

$$[i(\gamma_0\Delta_0 - \omega') + \gamma_0]\bar{\alpha}_0 + g\alpha_{c1}\bar{\alpha}_1 = 0 , \quad (63)$$

$$[-i(\gamma_0\Delta_0 + \omega') + \gamma_0]\bar{\alpha}_{0*} + g\alpha_{c1}^*\bar{\alpha}_{1*} = 0 , \quad (64)$$

where we introduce the notations

$$\bar{\alpha}_1^{(\text{in})} \equiv \alpha_{s1}^{(\text{in})}(\omega'), \quad \bar{\alpha}_{1*}^{(\text{in})} \equiv \alpha_{s1}^{(\text{in})*}(-\omega') \quad (65)$$

and similar definitions also for $\bar{\alpha}_j$ and $\bar{\alpha}_{j*}$, $\bar{\alpha}_j^{(\text{out})}$, and $\bar{\alpha}_{j*}^{(\text{out})}$ $(j=0,1)$. We find from Eqs. (63), (64), that

$$\bar{\alpha}_0 = \frac{g\alpha_{c1}\bar{\alpha}_1}{i\omega' - \gamma_0(1+i\Delta_0)} , \quad (66)$$

$$\bar{\alpha}_{0*} = \frac{g\alpha_{c1}^*\bar{\alpha}_{1*}}{i\omega' - \gamma_0(1-i\Delta_0)} . \quad (67)$$

Upon substitution into Eqs. (61), (62) we obtain

$$\bar{R}(\omega')\bar{\alpha}_1 - g\alpha_{c0}\bar{\alpha}_{1*} = 2\gamma_1^{1/2}\bar{\alpha}_1^{(\text{in})} , \quad (68)$$

$$\bar{R}_*(\omega')\bar{\alpha}_{1*} - g\alpha_{c0}^*\bar{\alpha}_1 = 2\gamma_1^{1/2}\bar{\alpha}_{1*}^{(\text{in})} , \quad (69)$$

where

$$\bar{R}(\omega') \equiv -i\omega' + \gamma_1(1+i\Delta_1) + \frac{g^2|\alpha_{c1}|^2}{-i\omega' + \gamma_0(1+i\Delta_0)} , \quad (70)$$

$$\bar{R}_*(\omega') \equiv \bar{R}^*(-\omega') . \quad (71)$$

It is convenient now to introduce two-component vectors $\boldsymbol{\alpha}_1^{(\text{in})}$, $\boldsymbol{\alpha}_1^{(\text{out})}$, and $\boldsymbol{\alpha}_1$:

$$\boldsymbol{\alpha}_1^{(\text{in})} = (\bar{\alpha}_1^{(\text{in})}; \bar{\alpha}_{1*}^{(\text{in})}) , \quad (72)$$

$$\boldsymbol{\alpha}_1^{(\text{out})} = (\bar{\alpha}_1^{(\text{out})}; \bar{\alpha}_{1*}^{(\text{out})}) , \quad (73)$$

$$\boldsymbol{\alpha}_1 = (\bar{\alpha}_1; \bar{\alpha}_{1*}) , \quad (74)$$

and to write the set (68), (69) in matrix form

$$[M]\boldsymbol{\alpha}_1 = 2\gamma_1^{1/2}\boldsymbol{\alpha}_1^{(\text{in})} , \quad (75)$$

where $[M]$ is the 2×2 complex matrix:

$$[M] \equiv \begin{bmatrix} \bar{R}(\omega') & -g\alpha_{c0} \\ -g\alpha_{c0}^* & \bar{R}_*(\omega') \end{bmatrix} . \quad (76)$$

The relations (60) give the expression for $\boldsymbol{\alpha}_1^{(\text{out})}$ through $\boldsymbol{\alpha}_1^{(\text{in})}$ and $\boldsymbol{\alpha}_1$:

$$\boldsymbol{\alpha}_1^{(\text{out})} = \gamma_1^{1/2}\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_1^{(\text{in})} . \quad (77)$$

Thus, the output amplitudes of the signal field can be found from Eqs. (72)–(77).

B. General conditions for maximum amplification coefficient with minimum noise

1. Maximum amplification coefficient

Let us select the parameters of the input signal so that we achieve maximum amplification in the output of mode 1. For that we use the fact that any arbitrary input vector (72) can be represented as a linear combination of the eigenvectors of $[M]$. Thus, for given parameters of the amplifier, maximum amplification occurs in the case, when the input vector (72) coincides with the eigenvector of the matrix $[M]$, which corresponds to appropriate eigenvalue. Precisely, we assume

$$[M]\boldsymbol{\alpha}_1^{(\mathrm{in})}=\lambda\boldsymbol{\alpha}_1^{(\mathrm{in})} , \tag{78}$$

where λ is the eigenvalue which depends on ω'. Hence we obtain from Eqs. (75), (78)

$$\boldsymbol{\alpha}_1=(2\gamma_1^{1/2}/\lambda)\boldsymbol{\alpha}_1^{(\mathrm{in})} , \tag{79}$$

and, by using Eq. (77),

$$\boldsymbol{\alpha}_1^{(\mathrm{out})}=(2\gamma_1/\lambda-1)\boldsymbol{\alpha}_1^{(\mathrm{in})} . \tag{80}$$

Let us now define the phases $\varphi_{\pm\omega}$ and φ_λ as follows:

$$2\gamma_1/\lambda-1\equiv|2\gamma_1/\lambda-1|\exp(i\varphi_\lambda) , \tag{81}$$

$$\bar{\alpha}_1^{(\mathrm{in})}=|\bar{\alpha}_1^{(\mathrm{in})}|\exp(i\varphi_\omega) ,\quad \bar{\alpha}_{1*}^{(\mathrm{in})}=|\bar{\alpha}_{1*}^{(\mathrm{in})}|\exp(-i\varphi_{-\omega}) ; \tag{82}$$

then we can represent the signal part of the output field amplitude (59) in the form

$$\alpha_{s1}^{(\mathrm{out})}(\omega')e^{-i\omega'\tau}+\alpha_{s1}^{(\mathrm{out})}(-\omega')e^{i\omega'\tau} =|2\gamma_1/\lambda-1|[|\bar{\alpha}_1^{(\mathrm{in})}|\exp(-i\omega'\tau+i\varphi_s) +|\bar{\alpha}_{1*}^{(\mathrm{in})}|\exp(i\omega'\tau-i\varphi_s)]\exp(i\delta) , \tag{83}$$

with the notations

$$\delta\equiv\frac{\varphi_\omega+\varphi_{-\omega}}{2}, \quad \varphi_s\equiv\frac{\varphi_\omega-\varphi_{-\omega}}{2}+\varphi_\lambda . \tag{84}$$

Now let us calculate the total output and input fields $\mathcal{E}_1^{(\mathrm{out})}$ and $\mathcal{E}_1^{(\mathrm{in})}$ for the mode 1. To find $\mathcal{E}_1^{(\mathrm{out})}$ we substitute Eq. (83) into expression (59) for the output field amplitude, and we insert the result in Eq. (3). Thus, we obtain

$$\mathcal{E}_l^{(\mathrm{out})}=\mathcal{E}_{c1}^{(\mathrm{out})}+\mathcal{E}_{s1}^{(\mathrm{out})} , \tag{85}$$

where

$$\mathcal{E}_{c1}^{(\mathrm{out})}\equiv\mathrm{Re}[\alpha_{c1}^{(\mathrm{out})}\exp(-i\Omega_1'\tau)] \tag{86}$$

and

$$\mathcal{E}_{s1}^{(\mathrm{out})}\equiv|2\gamma_1/\lambda-1|\mathrm{Re}\{[|\bar{\alpha}_1^{(\mathrm{in})}|\exp(-i\omega'\tau+i\varphi_s) +|\bar{\alpha}_{1*}^{(\mathrm{in})}|\exp(i\omega'\tau-i\varphi_s)] \times\exp(-i\Omega_1'\tau+i\delta)\} , \tag{87}$$

which gives

$$\mathcal{E}_{s1}^{(\mathrm{out})}=|2\gamma_1/\lambda-1| \times[(|\bar{\alpha}_1^{(\mathrm{in})}|+|\bar{\alpha}_{1*}^{(\mathrm{in})}|)\cos(\omega'\tau-\varphi_s) \times\cos(-\Omega_1'\tau+\delta)+(|\bar{\alpha}_1^{(\mathrm{in})}|-|\bar{\alpha}_{1*}^{(\mathrm{in})}|) \times\sin(\omega'\tau-\varphi_s)\sin(-\Omega_1'\tau+\delta)] . \tag{88}$$

Similarly, we obtain $\mathcal{E}_1^{(\mathrm{in})}$ using Eq. (56)

$$\mathcal{E}_1^{(\mathrm{in})}=\mathcal{E}_{c1}^{(\mathrm{in})}+\mathcal{E}_{s1}^{(\mathrm{in})} , \tag{89}$$

where

$$\mathcal{E}_{c1}^{(\mathrm{in})}\equiv\mathrm{Re}[\alpha_{c1}^{(\mathrm{in})}\exp(-i\Omega_1'\tau)] \tag{90}$$

and

$$\mathcal{E}_{s1}^{(\mathrm{in})}\equiv(|\bar{\alpha}_1^{(\mathrm{in})}|+|\bar{\alpha}_{1*}^{(\mathrm{in})}|)\cos(\omega'\tau-\varphi_s^{(\mathrm{in})}) \times\cos(-\Omega_1'\tau+\delta)+(|\bar{\alpha}_1^{(\mathrm{in})}|-|\bar{\alpha}_{1*}^{(\mathrm{in})}|) \times\sin(\omega'\tau-\varphi_s^{(\mathrm{in})})\sin(-\Omega_1'\tau+\delta) ; \tag{91}$$

here the phase $\varphi_s^{(\mathrm{in})}$ is

$$\varphi_s^{(\mathrm{in})}=\frac{\varphi_\omega-\varphi_{-\omega}}{2} . \tag{92}$$

By sending the output field (85) or the input field (89) into a photodetector and making the resulting photocurrent pass through a spectrum analyzer one can measure the intensities $I_{s1}^{(\mathrm{out})}(\omega')$ and $I_{s1}^{(\mathrm{in})}(\omega')$ of the signal components in the input and in the output fields. They are

$$I_{s1}^{(\mathrm{out})}(\omega')\equiv\overline{(\mathcal{E}_{s1}^{(\mathrm{out})})^2}=|2\gamma_1/\lambda-1|^2(\tfrac{1}{2})[(|\bar{\alpha}_1^{(\mathrm{in})}|+|\bar{\alpha}_{1*}^{(\mathrm{in})}|)^2\cos^2(\omega'\tau-\varphi_s)+(|\bar{\alpha}_1^{(\mathrm{in})}|-|\bar{\alpha}_{1*}^{(\mathrm{in})}|)^2\sin^2(\omega'\tau-\varphi_s)] , \tag{93}$$

$$I_{s1}^{(\mathrm{in})}(\omega')\equiv\overline{(\mathcal{E}_{s1}^{(\mathrm{in})})^2}=(\tfrac{1}{2})[(|\bar{\alpha}_1^{(\mathrm{in})}|+|\bar{\alpha}_{*}^{(\mathrm{in})}|)^2\cos^2(\omega'\tau-\varphi_s^{(\mathrm{in})})+(|\bar{\alpha}_1^{(\mathrm{in})}|-|\bar{\alpha}_{1*}^{(\mathrm{in})}|)^2\sin^2(\omega'\tau-\varphi_s^{(\mathrm{in})})] . \tag{94}$$

Here, the bar indicates an average over fast oscillations with optical frequency Ω_1'.

One can see that expressions (93) and (93) are different from each other by an immaterial phase factor (when λ is not real, $\varphi_s\neq\varphi_s^{(\mathrm{in})}$), and by the multiplier

$$\Gamma_1(\omega')\equiv|2\gamma_1/\lambda-1|^2 , \tag{95}$$

which is the amplification coefficient for the intensity of the weak signal of frequency ω' in the mode 1. This expression is the essential result of this section.

The analysis of Eq. (95) shows that

$$\Gamma_1(\omega')>1 \quad \text{if } \mathrm{Re}(\lambda)<\gamma_1 , \tag{96}$$

and that $\Gamma_1(\omega')>1$ is maximum for the eigenvalue λ which has the minimum modulus.

The expressions (88) and (91) show, that, in general, one modulates in a different way simultaneously two quadrature components of the electric field.

2. *Minimum noise*

In the general case, when in Eq. (88) $|\bar{\alpha}_1^{(\mathrm{in})}|\neq|\bar{\alpha}_{1*}^{(\mathrm{in})}|$, we have modulation in two quadrature components. For practical purposes it is not convenient to carry the information in both components. The advantage of modulating a single quadrature is that by applying the homodyne detection scheme one can reduce the output noise by cutting away the unmodulated quadrature components.

As one can see from Eq. (91), the single quadrature condition can be fulfilled when the eigenvector of the matrix $[M]$ is such that $|\bar{\alpha}_1^{(\text{in})}|=|\bar{\alpha}_{1*}^{(\text{in})}|$, i.e., the ratio

$$t_1 \equiv |\bar{\alpha}_1^{(\text{in})}|/|\bar{\alpha}_{1*}^{(\text{in})}| \tag{97}$$

is equal to one. From Eq. (88) we see that also in the output a single quadrature component (in fact, the same component as in the input) is modulated. The modulated quadrature component is identified by the value of δ given by Eq. (84); note that this value does not depend on the arbitrary phase factor $\varphi_s^{(\text{in})}$ which affects the eigenvector of $[M]$.

By the following procedure we indicate how the parameters of the amplifier must be selected in order to fulfill the condition $t_1=1$.

Let us consider a general matrix $[M]$:

$$[M]\boldsymbol{\alpha}_1 \equiv \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \bar{\alpha}_1 \\ \bar{\alpha}_{1*} \end{bmatrix} = \lambda \begin{bmatrix} \bar{\alpha}_1 \\ \bar{\alpha}_{1*} \end{bmatrix} \equiv \lambda \boldsymbol{\alpha}_1 \; . \tag{98}$$

By extracting the phases of the components of $\mathbf{a}_1$ as in Eq. (82), we obtain the two equations:

$$(a_{11}-\lambda)\exp(i\varphi_\omega)|\bar{\alpha}_1| + a_{12}\exp(-i\varphi_{-\omega})|\bar{\alpha}_{1*}|=0 \; , \tag{99}$$

$$a_{21}\exp(i\varphi_\omega)|\bar{\alpha}_1| + (a_{22}-\lambda)\exp(-i\varphi_{-\omega})|\bar{\alpha}_{1*}|=0 \; . \tag{100}$$

Assuming that $|\alpha_{1*}|\neq 0$ we can write these equations in terms of t_1 and δ, defined by Eqs. (97) and (84):

$$(a_{11}-\lambda)t_1 + a_{12}e^{-2i\delta}=0 \; , \tag{101}$$

$$a_{21}t_1 + (a_{22}-\lambda)e^{-2i\delta}=0 \; . \tag{102}$$

By eliminating λ and assuming $a_{21}\neq 0$ one arrives at the following equation for t_1:

$$t_1^2 - [(a_{11}-a_{22})/a_{21}]e^{-2i\delta}t_1 - (a_{12}/a_{21})e^{-4i\delta}=0 \; . \tag{103}$$

We want at least one root of Eq. (103) be equal to 1. The requirement $t_1=1$ gives the following conditions for coefficients a_{ij} and the phase δ:

$$a_{21}e^{2i\delta} - a_{12}e^{-2i\delta} = a_{11}-a_{22} \; . \tag{104}$$

The formula (104) provides the criterion, which guarantees that $|\bar{\alpha}_1^{(\text{in})}|=|\bar{\alpha}_{1*}^{(\text{in})}|$.

C. Minimum noise and maximum amplification coefficient for parametric amplifier

1. Criteria for minimum noise

Let us apply the criterion (104) to the matrix (76). With the matrix elements of (76), condition (104) becomes

$$(g\alpha_{c0}e^{-2i\delta} - g\alpha_{c0}^* e^{2i\delta}) = \bar{R}(\omega') - \bar{R}_*(\omega') \; . \tag{105}$$

It is convenient to introduce the normalized frequency

$$\omega \equiv \frac{\omega'}{\gamma_1} \; , \tag{106}$$

and represent $\bar{R}(\omega')$ as follows:

$$\bar{R}(\omega') \equiv \gamma_1 R(\omega) \; , \tag{107}$$

where

$$R(\omega) \equiv 1+\Theta(\omega) - i[\omega - \Delta_1 + (\Delta_0 - \gamma\omega)\Theta(\omega)] \; , \tag{108}$$

with

$$\Theta(\omega) \equiv \frac{2|A_1|^2}{1+(\Delta_0-\gamma\omega)^2} \; , \tag{109}$$

where we used Eq. (9).

As before, we define

$$R_*(\omega) \equiv R^*(-\omega). \tag{110}$$

By inserting Eq. (107) into (105) and using Eq. (9), we have

$$R(\omega) - R_*(-\omega) = 2i|A_0|\sin(\varphi_0 - 2\delta) \; , \tag{111}$$

where φ_0 is the phase of A_0:

$$A_0 \equiv |A_0|\exp(i\varphi_0) \; . \tag{112}$$

As one can verify, one can find a solution of Eq. (111) with respect to δ only if the two following conditions are satisfied together:

$$\text{Re}[R_*(\omega) - R(\omega)] = 0 \; , \tag{113}$$

$$|R_*(\omega) - R(\omega)| \le 2|A_0| \; . \tag{114}$$

The first condition, Eq. (113), with the explicit expressions (108), (109), and (110) becomes

$$\Theta(-\omega) - \Theta(\omega) = 2|A_1|^2\{[1+(\Delta_0+\gamma\omega)^2]^{-1} - [1+(\Delta_0-\gamma\omega)^2]^{-1}\} = 0 \; , \tag{115}$$

which is true if any one of the following conditions is satisfied:

$$|A_1|^2 = 0 \; , \tag{116}$$

$$\Delta_0 = 0 \; , \tag{117}$$

$$\omega = 0 \; . \tag{118}$$

Note that the case (116) is verified, when $\alpha_{c1}^{(\text{in})}=0$ and, in addition to the absence of the input field $\mathcal{E}_{c1}^{(\text{in})}$, the parametric amplifier is below threshold, i.e.,

$$E^2 < (1+\Delta_1^2)(1+\Delta_0^2) \; , \tag{119}$$

or, equivalently [13],

$$|A_0|^2 < 1+\Delta_0^2 \; . \tag{120}$$

Now let us examine the condition (114) for each of the cases (116), (117), and (118). Using Eq. (108), in case (116), $R(\omega)$ becomes

$$R(\omega) \equiv R^{(1)}(\omega) = 1 - i\omega + i\Delta_1 \; , \tag{121}$$

so that the condition (114) reduces to

$$|\Delta_1| \le |A_0| \; . \tag{122}$$

In the case (117) we have

$$R(\omega)\equiv R^{(2)}(\omega)$$
$$=1+\Theta^{(2)}(\omega)+i\{\Delta_1-\omega[1-\gamma\Theta^{(2)}(\omega)]\}\ , \quad (123)$$

where

$$\Theta^{(2)}(\omega)\equiv\frac{2|A_1|^2}{1+(\gamma\omega)^2}\ , \quad (124)$$

so that condition (114) is again given by Eq. (122).

In the last case (118) we obtain

$$R(\omega)=R_0\equiv 1+\Theta_0+i[\Delta_1-\Delta_0\Theta_0]\ , \quad (125)$$

with

$$\Theta_0\equiv 2|A_1|^2/(1+\Delta_0^2)\ , \quad (126)$$

hence the restriction (114) now becomes

$$|\Theta_0\Delta_0-\Delta_1|\le|A_0|\ . \quad (127)$$

In conclusion, one can keep the signal in a single quadrature, when one of the two conditions (116) or (117) is satisfied together with (122), or when (118) is satisfied together with (127).

2. *Maximum amplification coefficient*

Now let us calculate the amplification coefficient $\Gamma_1(\omega)$ and the phase δ of the amplified quadrature for each of the cases (116), (117), and (118).

First, consider the case (116). The two eigenvalues of the matrix $[M]$ are given by

$$\lambda_1=\gamma_1\lambda_+^{(1)}(\omega)\ , \quad (128)$$

$$\lambda_2=\gamma_1\lambda_1^{(1)}(\omega)\ , \quad (129)$$

where

$$\lambda_\pm^{(1)}(\omega)\equiv 1-i\omega\pm(|A_0|^2-\Delta_1^2)^{1/2}\ . \quad (130)$$

To obtain the amplification coefficient Γ_1 we substitute the expression of λ_2, which has a smaller absolute value than λ_1, into Eq. (95) and obtain

$$\Gamma_1(\omega)=\left|\frac{2}{1-(|A_0|^2-\Delta_1^2)^{1/2}-i\omega}-1\right|^2\ . \quad (131)$$

One has then

$$\Gamma_1(\omega)=\Gamma_1^{(1)}(\omega)\equiv 1+\frac{4(|A_0|^2-\Delta_1^2)^{1/2}}{[1-(|A_0|^2-\Delta_1^2)^{1/2}]^2+\omega^2}\ . \quad (132)$$

As one can see from Eqs. (131), (132), $\Gamma_1^{(1)}(\omega)\ge 1$ when condition (122) is true, and $\Gamma_1^{(1)}(\omega)=1$ for $|\Delta_1|=|A_0|$, hence Eq. (122) ensures amplification of the weak signal. Maximum amplification

$$\Gamma_1^{(1)}(\omega)=1+4/\omega^2 \quad (133)$$

occurs for $|A_0|^2=\Delta_1^2+1$, which corresponds to the threshold of the parametric amplifier in absence of the field $\mathcal{E}_{1c}^{(\mathrm{in})}$ [13].

The phase δ of the modulated quadrature can be found from Eq. (102) with $t_1=1$. The result is

$$\delta=\frac{1}{2}\left\{-\arctan\left[\frac{\Delta_1}{(|A_0|^2-\Delta_1^2)^{1/2}}\right]+\varphi_0\right\}\ , \quad (134)$$

where φ_0 is defined by Eq. (112). Note that for $\Delta_1=0$ one has $\delta=\varphi_0/2$.

Now we consider the case (117). The eigenvalues are given by $\lambda_{1,2}\equiv\gamma_1\lambda_\pm^{(2)}(\omega)$, with

$$\lambda_\pm^{(2)}(\omega)=1+\Theta^{(2)}(\omega)\pm(|A_0|^2-\Delta_1^2)^{1/2}$$
$$-i\omega[1-\gamma\Theta^{(2)}(\omega)]\ , \quad (135)$$

where $\Theta^{(2)}(\omega)$ is given by Eq. (124). The amplification coefficient is

$$\Gamma_1(\omega)=\Gamma_1^{(2)}(\omega)\equiv 1+\frac{4[(|A_0|^2-\Delta_1^2)^{1/2}-\Theta^{(2)}(\omega)]}{[1+\Theta^{(2)}(\omega)-(|A_0|^2-\Delta_1^2)^{1/2}]^2+\omega^2[1-\gamma\Theta^{(2)}(\omega)]^2}\ . \quad (136)$$

We have $\Gamma_1^{(2)}(\omega)>1$ in the region

$$|A_0|^2>[\Theta^{(2)}(\omega)]^2+\Delta_1^2\ . \quad (137)$$

$\Gamma_1^{(2)}$ reaches its maximum value

$$\Gamma_1^{(2)}(\omega)=1+\frac{4}{\omega^2[1-\gamma\Theta^{(2)}(\omega)]^2}\ , \quad (138)$$

for $|A_0|^2=[1+\Theta^{(2)}(\omega)]^2+\Delta_1^2$. In the case of Eq. (138) $\Gamma_1^{(2)}(\omega)\to\infty$ not only with $\omega\to 0$ but also for $\gamma\Theta^{(2)}(\omega)\to 1$. The value of δ is given by Eq. (134) also in this case.

In the last case (118) the eigenvalues are $\lambda_{1,2}\equiv\gamma_1\lambda_\pm^{(0)}$, where

$$\lambda_\pm^{(0)}\equiv 1+\Theta_0\pm[|A_0|^2-(\Delta_1-\Delta_0\Theta_0)^2]^{1/2} \quad (139)$$

and are real, due to the condition (127). The amplification coefficient for $\omega=0$ is

$$\Gamma_1(0)=1+\frac{4\{[|A_0|^2-(\Delta_1-\Delta_0\Theta_0)^2]^{1/2}-\Theta_0\}}{\{1+\Theta_0-[|A_0|^2-(\Delta_1-\Delta_0\Theta_0)^2]^{1/2}\}^2} \quad (140)$$

and is larger than 1 if

$$|A_0|^2>(\Delta_1-\Delta_0\Theta_0)^2+\Theta_0^2\ ; \quad (141)$$

$\Gamma_1(0)\to\infty$ if $|A_0|^2$ approaches the value $(\Delta_1-\Delta_0\Theta_0)^2+(1+\Theta_0)^2$.

The phase δ of the amplified quadrature now reads

$$\delta=\frac{1}{2}\left\{-\arctan\left[\frac{\Delta_1-\Delta_0\Theta_0}{[|A_0|^2-(\Delta_1-\Delta_0\Theta_0)^2]^{1/2}}\right]+\varphi_0\right\}\ . \quad (142)$$

IV. NOISE SPECTRUM IN THE SIGNAL OUTPUT QUADRATURE OF THE PARAMETRIC AMPLIFIER

A. The Hamiltonian and the equations for operators

The Hamiltonian for the field inside the single-ended cavity of the parametric amplifier shown in Fig. 1 is the following:

$$H=\hbar\Omega_{R0}\tilde{a}_0^\dagger\tilde{a}_0+\hbar\Omega_{R1}\tilde{a}_1^\dagger\tilde{a}_1 +i\hbar\frac{\tilde{g}}{2}(\tilde{a}_0-\tilde{a}_0^\dagger)(\tilde{a}_1-\tilde{a}_1^\dagger)^2+V_1+V_0 . \quad (143)$$

Here $\tilde{a}_0$ and $\tilde{a}_1$ are Bose operators; the first two terms in Eq. (143) describe the free oscillation of the field modes 1 and 0 involved in the parametric amplification, the third term describes the parametric biphoton interaction between modes 1 and 0, the last two terms are responsible for the exit of photons in the cavity and for input field through the partially transparent mirror. We suppose that there are two monochromatic input fields of frequencies Ω_1 and Ω_0, which are close to the cavity modes Ω_{R1}, Ω_{R0}, and in exact biphotonic resonance, i.e.,

$$\Omega_1\approx\Omega_{R1},\quad \Omega_0\approx\Omega_{R0},\quad \Omega_0=2\Omega_1 . \quad (144)$$

The terms V_j, $j=0,1$ depend, on the Bose operators $\tilde{a}_j^{(\mathrm{in})}$ of the input fields, and on the cavity field operators. The equations of motion for the cavity mode operators $\tilde{a}_0$, $\tilde{a}_1$ are

$$i\hbar\frac{da_0}{dt}=\hbar\Omega_{R0}\tilde{a}_0-i\hbar\frac{\tilde{g}}{2}(\tilde{a}_1-\tilde{a}_1^\dagger)^2+[\tilde{a}_0,V_0] , \quad (145)$$

$$i\hbar\frac{d\tilde{a}_1}{dt}=\hbar\Omega_{R1}\tilde{a}_1-i\hbar\tilde{g}(\tilde{a}_0-\tilde{a}_0^\dagger)(\tilde{a}_1-\tilde{a}_1^\dagger)+[\tilde{a}_1,V_1] . \quad (146)$$

Now we transform to slowly varying amplitude operators:

$$\tilde{a}_j=a_j\exp(-i\Omega_jt);\quad \tilde{a}_j^{(\mathrm{in})}=a_j^{(\mathrm{in})}\exp(-i\Omega_jt) ;$$
$$\tilde{a}_j^{(\mathrm{out})}=a_j^{(\mathrm{out})}\exp(-i\Omega_jt);\quad j=0,1 ; \quad (147)$$

introducing the dimensionless time (7) one can obtain the set of equations for the slowly varying amplitude operators a_j:

$$\dot{a}_0=-i\gamma_0\Delta_0a_0-\frac{g}{2}a_1^2+\frac{2\tau_R}{i\hbar}[a_0,V_0] , \quad (148)$$

$$\dot{a}_1=-i\gamma_1\Delta_1a_1-ga_0a_1^\dagger+\frac{2\tau_R}{i\hbar}[a_1,V_1] , \quad (149)$$

where

$$g\equiv2\tau_R\tilde{g} , \quad (150)$$

and Δ_j is defined by Eq. (8). The output field operators are expressed through intracavity and input field operators by the same relation (60), which is valid for classical mean values.

We find the explicit form of the commutators in Eqs. (148), (149) by following a method essentially identified to the quantum Langevin equations method developed in [18,19].

Here, we are interested in the output field, and restrict ourselves to the linearized analysis of quantum noise. In this case, the explicit form for the commutators in (148), (149) can be found from the following conditions.

(1) The expressions for $[a_j,V_j^{(\mathrm{in})}]$ must be linear with respect to the field operators.

(2) After quantum averaging these expressions one must obtain the correct terms in the semiclassical Eqs. (5), (6).

(3) The output field operators must obey the Bose commutation relations.

The two first conditions are satisfied, if one sets

$$\frac{2\tau_R}{i\hbar}[a_j,V_j]=-\gamma_ja_j+2\gamma_j^{1/2}a_j^{(\mathrm{in})},\quad j=0,1 . \quad (151)$$

It can be verified [18] that with the choice (151) also the third condition is satisfied provided $a_j^{(\mathrm{in})}$, $j=0,1$ are Bose operators.

Next we linearize Eqs. (148), (149), by expressing a_j, $a_j^{(\mathrm{in})}$, and $a_j^{(\mathrm{out})}$ as sums of average and of quantum noise parts:

$$a_j=\langle a_j\rangle+\bar{a}_j , \quad (152)$$

$$a_j^{(\mathrm{in})}=\langle a_j^{(\mathrm{in})}\rangle+\bar{a}_j^{(\mathrm{in})} , \quad (153)$$

$$a_j^{(\mathrm{out})}=\langle a_j^{(\mathrm{out})}\rangle+\bar{a}_j^{(\mathrm{out})} . \quad (154)$$

The linearized equations read

$$\dot{\bar{a}}_0=-i\gamma_0\Delta_0\bar{a}_0-g\alpha_{c1}\bar{a}_1-\gamma_0\bar{a}_0+2\gamma_0^{1/2}\bar{a}_0^{(\mathrm{in})} , \quad (155)$$

$$\dot{\bar{a}}_1=-i\gamma_1\Delta_1\bar{a}_1+g\alpha_{c0}\bar{a}_1^\dagger +g\alpha_{c1}^*\bar{a}_0-\gamma_1\bar{a}_1+2\gamma_1^{1/2}\bar{a}_1^{(\mathrm{in})} \quad (156)$$

and will be used below for the calculations of the noise spectrum of the parametric amplifier.

B. Equations for Fourier-component operators

Let us calculate the noise power spectrum in the output quadrature of mode 1, for each of the cases considered in Sec. III.

We introduce Fourier-component operators for the quantum electromagnetic field ($c^\dagger$=adjoint of c):

$$c(\omega'),\quad c_+(\omega')\equiv c^\dagger(-\omega') , \quad (157)$$

which are obtained for time-dependent field operators $\bar{c}$, $\bar{c}^\dagger$ by the relations

$$\bar{c}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}c(\omega')e^{-i\omega'\tau}d\omega' ,$$
$$\bar{c}^\dagger=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}c_+(\omega')e^{-i\omega'\tau}d\omega' . \quad (158)$$

It is meant in Eqs. (157) and (158) that $\bar{c}$ is one of the operators: $\{\bar{a}_j^{(\mathrm{in})},\ \bar{a}_j^{(\mathrm{out})},\bar{a}_j\}$, and $c(\omega')$ is one of the operators $\{a_j(\omega'),a_j^{(\mathrm{in})}(\omega'),a_j^{(\mathrm{out})}(\omega')\}$. The commutation relations for the Fourier-component operators

$$[a_j^{(\mathrm{in})}(\omega'),a_i^{(\mathrm{in})}(\omega'')]=0\ ,$$
$$[a_j^{(\mathrm{in})}(\omega'),a_{i+}^{(\mathrm{in})}(\omega'')]=(\tfrac{1}{2})\delta_{ij}\delta(\omega'+\omega'')\ ,\quad i,j=0,1\ , \tag{159}$$

follow from Bose commutation relations for the time-dependent operators:

$$[\bar{a}_j^{(\mathrm{in})}(\tau),\bar{a}_i^{(\mathrm{in})}(\tau')]=0\ ,$$
$$[\bar{a}_j^{(\mathrm{in})}(\tau),\bar{a}_i^{(\mathrm{in})\dagger}(\tau')]=(\tfrac{1}{2})\delta_{ij}\delta(\tau-\tau'),\quad i,j=0,1\ . \tag{159'}$$

The same relations hold for the output field Fourier-component operators $a_j^{(\mathrm{out})}(\omega')$ and $a_{j+}^{(\mathrm{out})}(\omega')$. Note that the factor $(\frac{1}{2})$ in Eqs. (159), (159′), as well as the factor 2 in Eq. (170) appear because the time variable has been normalized to $2\tau_R$ instead of τ_R.

By substitution of Eq. (158) for $\bar{a}_j$ and $\bar{a}_j^{(\mathrm{in})}$ into Eqs. (155), (156) one obtains the following set of algebraic equations for $a_j(\omega')$:

$$[i(\gamma_1\Delta_1-\omega')+\gamma_1]a_1(\omega')-g\alpha_{c1}^*a_0(\omega')-g\alpha_{c0}a_{1+}(\omega')=2\gamma_1^{1/2}a_1^{(\mathrm{in})}(\omega')\ , \tag{160}$$

$$[i(\gamma_0\Delta_0-\omega')+\gamma_0]a_0(\omega')+g\alpha_{c1}a_1(\omega')=2\gamma_0^{1/2}a_0^{(\mathrm{in})}(\omega')\ . \tag{161}$$

In terms of the normalized quantities (9), (106) these equations become

$$[i(\Delta_1-\omega)+1]a_1(\omega)-(2/\gamma)^{1/2}A_1^*a_0(\omega)-A_0a_{1+}(\omega)=2\gamma_1^{-1/2}a_1^{(\mathrm{in})}(\omega)\ , \tag{162}$$

$$[i(\Delta_0-\gamma\omega)+1]a_0(\omega)+(2\gamma)^{1/2}A_1a_1(\omega)=2\gamma_0^{-1/2}a_0^{(\mathrm{in})}(\omega)\ . \tag{163}$$

The equations for $a_{j+}(\omega)$ can be found from (162), (163) using the second of relations (157). From Eq. (60) the expressions for $a_j^{(\mathrm{out})}(\omega)$ are

$$a_j^{(\mathrm{out})}(\omega)=\gamma_j^{1/2}a_j(\omega)-a_j^{(\mathrm{in})}(\omega)\ ,$$
$$a_{j+}^{(\mathrm{out})}(\omega)=\gamma_j^{1/2}a_{j+}(\omega)-a_{j+}^{(\mathrm{in})}(\omega)\ . \tag{164}$$

Thus, $a_j^{(\mathrm{out})}(\omega)$ can be expressed in terms of the input Fourier-component operators $a_j^{(\mathrm{in})}(\omega)$ by using Eqs. (162), (163).

C. Noise spectrum of the parametric amplifier below threshold

Let us calculate the noise spectrum for the case (116). For $A_1=0$ Eqs. (162), (163) become independent of each other; in particular, for $a_1(\omega)$, $a_{1+}(\omega)$ we have

$$[i(\Delta_1-\omega)+1]a_1(\omega)-A_0a_{1+}(\omega)=2\gamma_1^{-1/2}a_1^{(\mathrm{in})}(\omega)\ , \tag{165}$$

$$[-i(\Delta_1+\omega)+1]a_{1+}(\omega)-A_0^*a_1(\omega)=2\gamma_1^{-1/2}a_{1+}^{(\mathrm{in})}(\omega)\ , \tag{166}$$

which gives, using notations (121),

$$a_1(\omega)=(2/\gamma_1^{1/2})\frac{R_*^{(1)}(\omega)a_1^{(\mathrm{in})}(\omega)+A_0a_{1+}^{(\mathrm{in})}(\omega)}{R^{(1)}(\omega)R_*^{(1)}(\omega)-|A_0|^2} \tag{167}$$

and, with the help of Eq. (164),

$$a_1^{(\mathrm{out})}(\omega)=2\frac{R_*^{(1)}(\omega)a_1^{(\mathrm{in})}(\omega)+A_0a_{1+}^{(\mathrm{in})}(\omega)}{R^{(1)}(\omega)R_*^{(1)}(\omega)-|A_0|^2}-a_1^{(\mathrm{in})}(\omega)\ . \tag{168}$$

In order to calculate the noise in the signal quadrature, we introduce the output quadrature operator:

$$\bar{a}_{1\delta}^{(\mathrm{out})}\equiv\bar{a}_1^{(\mathrm{out})}e^{-i\delta}+\bar{a}_1^{(\mathrm{out})\dagger}e^{i\delta}\ , \tag{169}$$

where δ is the phase of the quadrature (the same that we used previously in the paper).

Here we restrict ourselves to the case of perfect homodyne photodetection, i.e., the quantum efficiency of the photodetector is 1 and the losses in the detected beam are negligibly small. The signal-to-noise ratio for the phase-insensitive linear amplifier in the case of imperfect detection of the signal has been discussed in [20], [21].

For the sake of convenience in the following calculations we represent the formula for the noise spectrum $N_{1\delta}(\omega)$ (see, for example, [22], [23]) in the form

$$N_{1\delta}(\omega)=2\int_{-\infty}^{+\infty}d\tau' e^{i\omega\tau'}\langle\bar{a}_{1\delta}^{(\mathrm{out})}(0)\bar{a}_{1\delta}^{(\mathrm{out})}(\tau')\rangle\ , \tag{170}$$

where $\tau'=\tau\gamma_1$, τ is defined by Eq. (7). Because Eq. (170) ,with $\bar{a}_{1\delta}^{(\mathrm{out})}$ given by Eq. (169), incorporates only Bose operators, we do not demand in Eq. (170) the time ordering [19]. Since we include in Eq. (170) the shot noise contribution, we do not demand the normal ordering of Bose operators either.

The following standard procedure is applied for the calculation of $N_{1\delta}(\omega)$. We substitute Eq. (169) into Eq. (170):

$$N_{1\delta}(\omega)=2\int_{-\infty}^{+\infty}d\tau' e^{i\omega\tau'}[\langle\bar{a}_1^{(\mathrm{out})}(0)\bar{a}_1^{(\mathrm{out})}(\tau')\rangle e^{-2i\delta}+\langle\bar{a}_1^{(\mathrm{out})}(0)\bar{a}_1^{(\mathrm{out})\dagger}(\tau')\rangle+\langle\bar{a}_1^{(\mathrm{out})\dagger}(0)\bar{a}_1^{(\mathrm{out})}(\tau')\rangle+\langle\bar{a}_1^{(\mathrm{out})\dagger}(0)\bar{a}_1^{(\mathrm{out})\dagger}(\tau')\rangle e^{2i\delta}]\ , \tag{171}$$

and we express the operators in terms of their Fourier components, for example,

$$\langle\bar{a}_1^{(\mathrm{out})}(0)\bar{a}_1^{(\mathrm{out})}(\tau')\rangle=\frac{1}{2\pi}\int_{-\infty}^{+\infty}d\omega'\int_{-\infty}^{+\infty}d\omega'' e^{-i\omega''\tau'}\langle a_1^{(\mathrm{out})}(\omega')a_1^{(\mathrm{out})}(\omega'')\rangle\ . \tag{172}$$

Next, we substitute Eq. (168) and calculate the averages taking into account that in the case of coherent input field normal products of fluctuation operators give zero contributions, e.g.,

$$\begin{aligned}\langle a_1^{(\mathrm{in})}(\omega')a_{1+}^{(\mathrm{in})}(\omega'')\rangle &\equiv \langle [a_1^{(\mathrm{in})}(\omega'),a_{1+}^{(\mathrm{in})}(\omega'')]+a_{1+}^{(\mathrm{in})}(\omega')a_1^{(\mathrm{in})}(\omega'')\rangle \\ &=\langle [a_1^{(\mathrm{in})}(\omega'),a_{1+}^{(\mathrm{in})}(\omega'')]\rangle \\ &=\delta(\omega'+\omega'') \ . \end{aligned} \tag{173}$$

Proceeding in this way we obtain, using Eq. (168),

$$\begin{aligned}\langle \bar a_1^{(\mathrm{out})}(0)\bar a_1^{(\mathrm{out})}(\tau')\rangle &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega'' e^{-i\omega''\tau'}2A_0 \\ &\quad\times[2R_*^{(1)}(\omega'')-Q^*(\omega'')]/|Q(\omega'')|^2 \ , \end{aligned} \tag{174}$$

where

$$Q(\Omega)\equiv R^{(1)}(\omega)R_*^{(1)}(\omega)-|A_0|^2 \ . \tag{175}$$

Similar calculations give

$$\begin{aligned}\langle \bar a_1^{(\mathrm{out})\dagger}\bar a_1^{(\mathrm{out})\dagger}(\tau')\rangle &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega'' e^{-i\omega''\tau'}2A_0^* \\ &\quad\times[2R^{(1)}(\omega'')-Q(\omega'')]/|Q(\omega'')|^2 \ , \end{aligned} \tag{176}$$

$$\begin{aligned}\langle \bar a_1^{(\mathrm{out})}(0)\bar a_1^{(\mathrm{out})\dagger}(\tau')\rangle &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega'' e^{-i\omega''\tau'} \\ &\quad\times|[2R^{(1)}(\omega'')-Q(\omega'')]/Q(\omega'')|^2 \ , \end{aligned} \tag{177}$$

$$\langle \bar a_1^{(\mathrm{out})\dagger}(0)\bar a_1^{(\mathrm{out})}(\tau')\rangle = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega'' e^{-i\omega''\tau'}|2A_0/Q(\omega'')|^2 \ . \tag{178}$$

We insert Eqs. (174), (176)–(178) into Eq. (171) and obtain finally

$$N_{1\delta}(\omega)\equiv N_{1\delta}^{(1)}(\omega) = \left| \frac{R_*^{(1)}(\omega)R^{(1)*}(\omega)+|A_0|^2+2A_0^*e^{2i\delta}}{R_*^{(1)}(\omega)R^{(1)}(\omega)-|A_0|^2} \right|^2 \ . \tag{179}$$

1. Noise spectrum for best squeezed quadrature

To compare this result with the previous results reported in [13], [24] and obtained by other methods, let us calculate from Eq. (179) the phase $\delta_{\mathrm{sq}}(\omega)$ of the most squeezed quadrature and the noise spectrum in this quadrature. With the explicit expression (121) for $R^{(1)}(\omega)$ one has

$$N_{1\delta}^{(1)}(\omega)=\frac{|1+\omega^2+|A_0|^2-\Delta_1^2-2i\Delta_1+2A_0^*e^{2i\delta}|^2}{(1-\omega^{2+}\Delta_1^2-|A_0|^2)^2+4\omega^2} \ . \tag{180}$$

Maximum squeezing is obtained by selecting

$$\exp(-2i\delta_{\mathrm{sq}}) = -\frac{A_0^*}{|A_0|}\frac{1+\omega^2+|A_0|^2-\Delta_1^2+2i\Delta_1}{[(1+\omega^2+|A_0|^2-\Delta_1^2)^2+4\Delta_1^2]^{1/2}} \ , \tag{181}$$

which gives

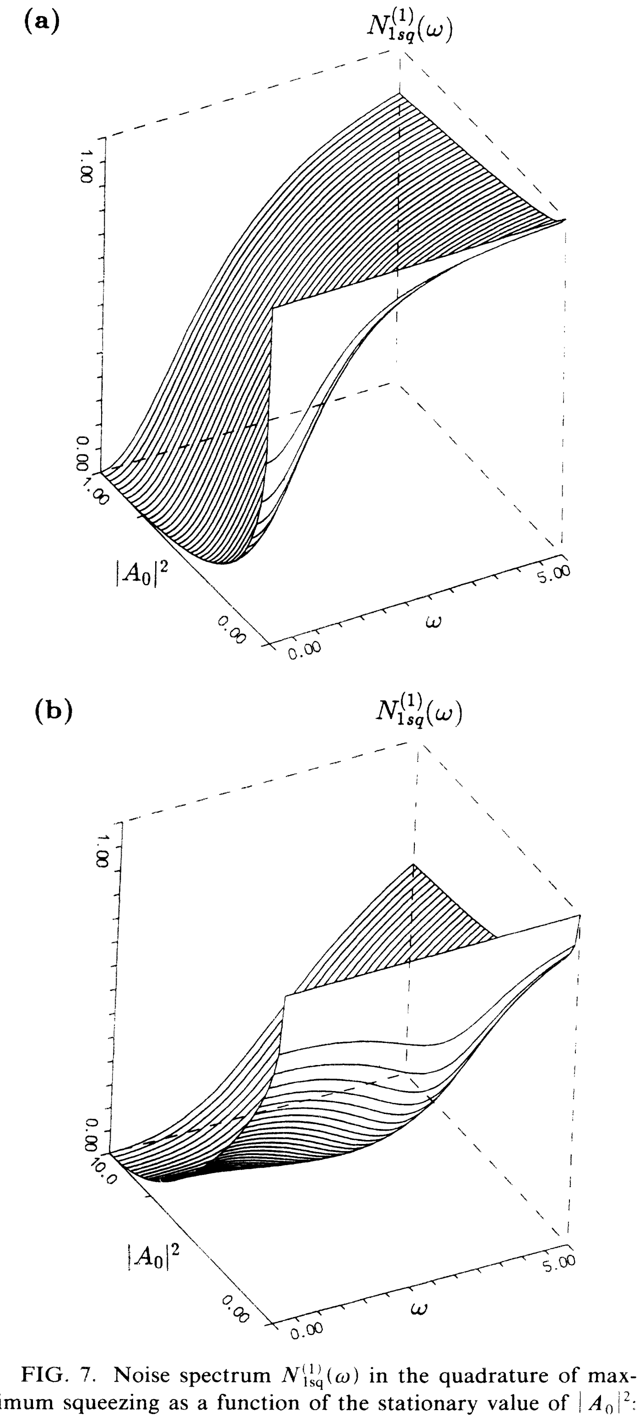

FIG. 7. Noise spectrum $N_{1\mathrm{sq}}^{(1)}(\omega)$ in the quadrature of maximum squeezing as a function of the stationary value of $|A_0|^2$: $\Delta_1=0$; $\Delta_1=3$ (b).

$$N_{1\mathrm{sq}}^{(1)}(\omega)=1-\frac{4|A_0|}{2|A_0|+[(1-\Delta_1^2+|A_0|^2+\omega^2)^2+4\Delta_1^2]^{1/2}} . \tag{182}$$

In the special case $\omega=0$ this coincides with the result of [13], whereas for $\omega\neq 0$ it is different, because [13] is considered the quadrature of maximum squeezing for $\omega=0$, while Eq. (181) gives the phase of the best squeezed quadrature for arbitrary frequency. Some examples of $N_{1\mathrm{sq}}^{(1)}(\omega)$ are presented in Figs. 7(a) and 7(b). Ideal squeezing $N_{1\mathrm{sq}}^{(1)}(\omega)=0$ is approached at threshold $|A_0|^2=1+\Delta_1^2$ of parametric generation. With $\Delta_1\neq 0$ there may be maximum squeezing at some frequency $\omega\neq 0$ for some value of $|A_0|$—Fig. 7(b).

2. *Noise spectrum in the modulated quadrature*

To obtain the noise spectrum for the modulated quadrature component we must take in Eq. (179) the phase δ defined by Eq. (134). The result is

$$N_{1\delta}^{(1)}(\omega)=1+\frac{4(|A_0|^2-\Delta_1^2)^{1/2}}{[1-(|A_0|^2-\Delta_1^2)^{1/2}]^2+\omega^2}+\frac{16\Delta_1^2}{(1+\Delta_1^2-|A_0|^2-\omega^2)^2+4\omega^2} . \tag{183}$$

Using the expression (132) for the amplification coefficient $\Gamma_1^{(1)}(\omega)$ and the definitions (130) of the eigenvalues one can represent Eq. (183) in the compact form:

$$N_{1\delta}^{(1)}(\omega)=\Gamma_1^{(1)}(\omega)+\delta N^{(1)}(\omega) ,$$

where

$$\delta N^{(1)}(\omega)\equiv\frac{16\Delta_1^2}{|\lambda_+^{(1)}(\omega)\lambda_-^{(1)}(\omega)|^2} \tag{184}$$

is the extra noise, that the parametric amplifier operating below threshold adds to the output of the mode 1. For the case $\Delta_1=0$, it follows from Eq. (184) that $\delta N^{(1)}=0$, so that parametric amplifier is noiseless: it does not add any extra noise in the output of mode 1, but just amplifies the input noise as well as the input signal. Note that in the case $\Delta_1=0$, δ coincides with $\delta_{\mathrm{sq}}+\pi/2$.

D. Noise spectrum for the cases $\Delta_0=0$ and $\omega=0$

Next, we consider the case identified by the conditions (117) and (122). By proceeding as in the first case, one arises at the formula

$$N_{1\delta}(\omega)\equiv N_{1\delta}^{(2)}(\omega) =\frac{|2R^{(2)}(\omega)-R_*^{(2)}(\omega)R^{(2)}(\omega)+|A_0|^2+2A_0e^{-2i\delta}|^2+4\Theta^{(2)}(\omega)|A_0e^{-2i\delta}+R^{(2)}(\omega)|^2}{|R^{(2)}(\omega)R_*^{(2)}(\omega)-|A_0|^2|^2} , \tag{185}$$

where $\Theta^{(2)}(\omega)$ is given by Eq. (124), and the phase δ of the signal quadrature is defined by Eq. (134).

In the case $\Delta_1=0$ we obtain from Eq. (134) that $\delta=\varphi_0/2$, where φ_0 is the phase of A_0. It follows from (185) and Eq. (123) that for the quadrature $\delta=\varphi_0/2$, the spectrum $N_{1\delta}(\omega)$ displays maximum noise, i.e., there is perfect antisqueezing:

$$N_{1\delta}(\omega)=1+\frac{4|A_0|}{[1+\Theta^{(2)}(\omega)-|A_0|]^2+\omega^{2[1-}\gamma\Theta^{(2)}(\omega)]^2} . \tag{186}$$

On the other hand, the quadrature $\delta_{\mathrm{sq}}=(\varphi_0+\pi)/2$ corresponds to maximum squeezing with a noise spectrum given by

$$N_{1\delta}(\omega) \equiv N_{1\mathrm{sq}}^{(2)}(\omega) =1-\frac{4|A_0|}{[1+\Theta^{(2)}(\omega)+|A_0|]^2+\omega^2[1-\gamma\Theta^{(2)}(\omega)]^2} . \tag{187}$$

Some examples of $N_{1\mathrm{sq}}^{(2)}(\omega)$ are presented in Fig. 8 for different values of the parameters.

More in general, by some tedious algebra the spectrum (185) can be written in the form:

$$N_{1\delta}^{(2)}(\omega)=\Gamma_1^{(2)}(\omega)+\frac{16\Delta_1^2[1+\Theta^{(2)}(\omega)]}{|\lambda_+^{(2)}(\omega)\lambda_-^{(2)}(\omega)|^2}+\frac{4\Theta^{(2)}(\omega)}{|\lambda_-^{(2)}(\omega)|^2} , \tag{188}$$

where $\lambda_+^{(2)}$ and $\lambda_-^{(2)}$, are given by Eq. (135), $\Gamma_1^{(2)}(\omega)$ is given by Eq. (136), and $\Theta^{(2)}(\omega)$ is defined by Eq. (124).

The two last terms in Eq. (188) describe the extra noise in the output signal. By setting $\Delta_1=0$ one can approach the case of noiseless amplification in the limit $\Gamma_1^{(2)}\gg 4\Theta^{(2)}/|\lambda_-^{(2)}|^2$. By using Eqs. (135), (136) one can verify that the last condition is the same as

$$\Theta^{(2)}(\omega)\equiv\frac{2|A_1|^2}{1+(\gamma\omega)^2}\ll|A_0| . \tag{189}$$

One can satisfy condition (189) for $\omega\neq 0$, for example, by selecting a sufficiently large γ, provided $\gamma\omega\ll 1$, and keeping $|A_0|$ near unity. In this way one can make the amplifier almost noiseless, but with a quite high amplification coefficient, because, according to Eq. (136), $\Gamma_1^{(2)}(\omega)$ becomes large for $\Delta_1=0$, ω close to zero and $\Theta^{(2)}(\omega)\ll|A_0|\approx 1$.

Finally, let us focus on the case (118) assuming that also inequality (127) is true. By following the same procedure as in the previous cases one obtains the noise at $\omega=0$:

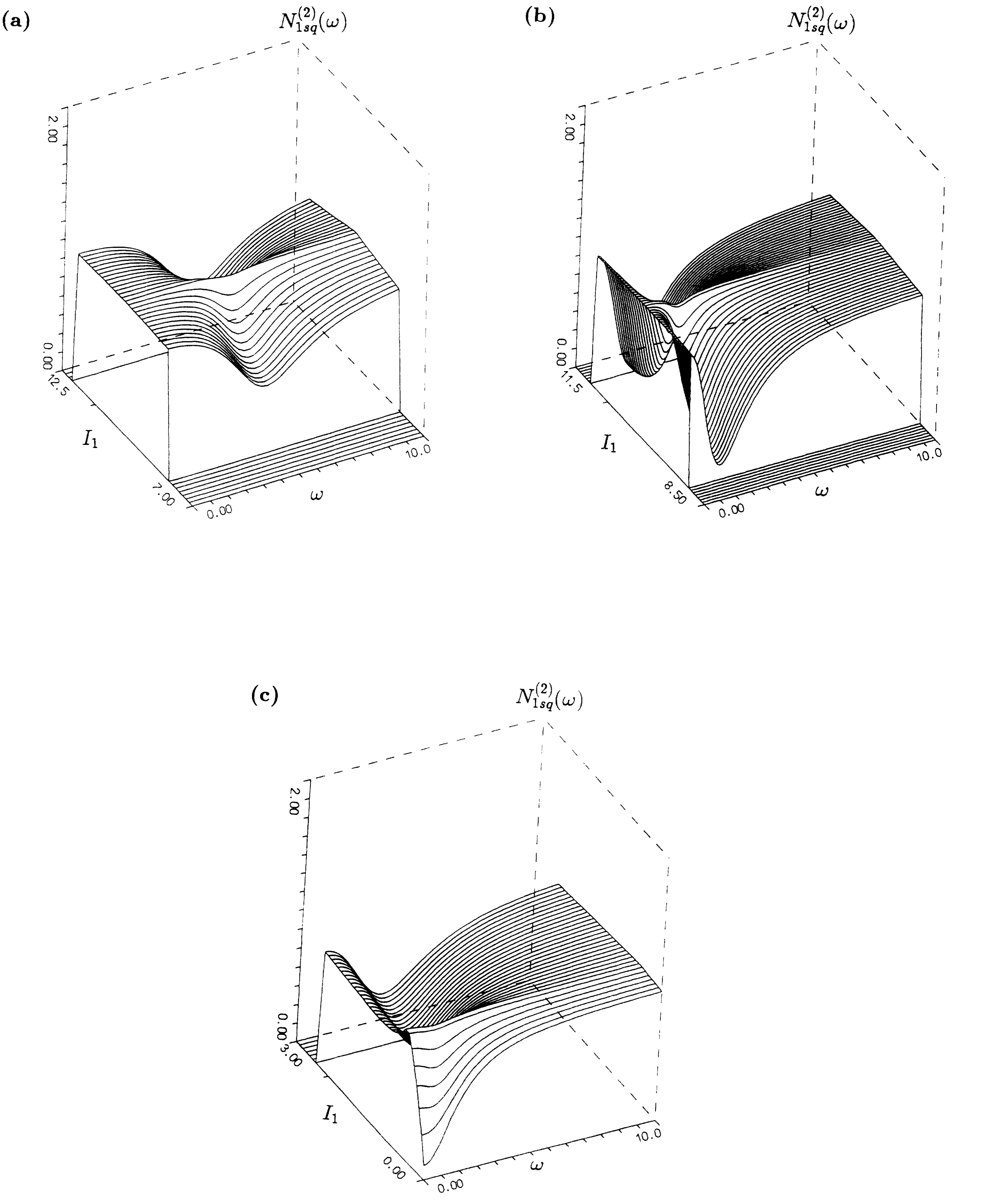

FIG. 8. Noise spectrum $N^{(2)}_{1\text{sq}}(\omega)$ in the best squeezed quadrature for the resonant case: $\Delta_1=\Delta_0=0$, $\varphi=0$. The phase δ_{sq} of this quadrature is the same for all ω. The parameters are $E=10$, $\gamma=1$ (a), $E=10$, $\gamma=10$ (b), $E=0.5$, $\gamma=1$ (c). In the regions of parameters which are unstable we set $N^{(1)}_{1\text{sq}}(\omega)=0$. The quantity I_1 is defined as $|A_1|^2$.

$$N_{1\delta}(0)=\frac{|2(A_0e^{-2i\delta}+R_0)-|R_0|^2+|A_0|^2|^2+4\Theta_0|A_0e^{-2i\delta}+R_0|^2}{(|R_0|^2-|A_0|^2)^2}, \tag{190}$$

where R_0 is given by Eq. (125), Θ_0 is defined by Eq. (126), and the phase δ of the signal quadrature is given by (142). With some algebra $N_{1\delta}(0)$ can be written in the form:

$$N_{1\delta}(0)=\Gamma_1(0)+16(\Delta_1-\Delta_0\Theta_0)^2\frac{1+\Theta_0}{(\lambda_+^{(0)}\lambda_-^{(0)})^2}+\frac{4\Theta_0}{(\lambda_-^{(0)})^2}, \tag{191}$$

where $\Gamma_1(0)$ is defined by Eq. (140), and the eigenvalues $\lambda_+^{(0)}$, $\lambda_-^{(0)}$ [which are real because of condition (127)] are given by Eq. (139).

As in the previous case, there are two terms of extra noise introduced by the amplifier. The first term can be eliminated by taking $\Delta_1=\Delta_0=0$, or by setting

$$\Delta_1=\Delta_0\Theta_0 . \tag{192}$$

The second extra noise term is negligible, if $\Gamma_1(0)\gg 4\Theta_0/(\lambda_-^{(0)})^2$, which with help of Eqs. (139), (140) can be recast in the form

$$\Theta_0\equiv\frac{2|A_1|^2}{1+\Delta_0^2}\ll|A_0| . \tag{193}$$

Thus, in this section, we have calculated the noise spectrum in the signal output quadrature of the parametric amplifier for all of the cases considered in Sec. III.

V. TRANSFER COEFFICIENT BETWEEN INPUT AND OUTPUT SIGNAL-TO-NOISE RATIOS

To characterize quantitatively the amount of extra noise added by the parametric amplifier into the output signal, it is convenient to introduce the transfer coefficient $T_{SN}(\omega)$ between input and output signal-to-noise ratios.

We define input and output signal-to-noise ratios $R_{SN}^{(in)}(\omega)$ and $R_{SN}^{(out)}(\omega)$, respectively:

$$R_{SN}^{(in)}(\omega)\equiv\frac{S_1^{(in)}(\omega)}{N_1^{(in)}(\omega)},\quad R_{SN}^{(out)}(\omega)\equiv\frac{S_1^{(out)}(\omega)}{N_{1\delta}(\omega)}, \tag{194}$$

where $S_i^{(in)}(\omega)$, $(i=0,1)$, and $S_i^{(out)}(\omega)$ are the signal intensities in the input and in the output, i.e., the coefficients of the $\cos^2$ term in Eqs. (93) and (94), respectively; all the quantities refer to the same quadrature component. We define $T_{SN}(\omega)$ as the ratio between the input and the output signal-to-noise ratios:

$$T_{SN}(\omega)\equiv R_{SN}^{(in)}(\omega)/R_{SN}^{(out)}(\omega) . \tag{195}$$

By taking the input quadrature operator similar to Eq. (169)

$$\bar a_{1\delta}^{(in)}\equiv\bar a_1^{(in)}e^{-i\delta}+\bar a_1^{(in)\dagger}e^{i\delta}, \tag{196}$$

and taking into account that $\bar a_1^{(in)}$ is a Bose operator with zero average in a coherent state, one has

$$N_1^{(in)}(\omega)=\int_{-\infty}^{+\infty}d\tau' e^{i\omega\tau'}\langle\bar a_{1\delta}^{(in)}(0)\bar a_{1\delta}^{(in)}(\tau')\rangle=1 . \tag{197}$$

On the other hand, by definition,

$$\frac{S_1^{(in)}(\omega)}{S_1^{(out)}(\omega)}\frac{1}{\Gamma_1(\omega)}, \tag{198}$$

where $\Gamma_1(\omega)$ is amplification coefficient of the signal, given by Eq. (95). Hence we obtain

$$T_{SN}(\omega)=\frac{N_{1\delta}(\omega)}{\Gamma_1(\omega)} . \tag{199}$$

Because any linear amplifier cannot inject more information into the signal, but can add more noise, so that the input signal-to-noise ratio cannot be improved [2], it must be $T_{SN}(\omega)\geq 1$. In the case $T_{SN}(\omega)=1$, the amplifier is noiseless, i.e., it preserves the input signal-to-noise ratio.

For the case defined by Eqs. (116), (122), $T_{SN}(\omega)\equiv T_{SN}^{(1)}(\omega)$ can be calculated by substitution of the noise spectrum (183) and of the amplification coefficient (132) into Eq. (199):

$$T_{SN}^{(1)}(\omega)=1+\frac{16\Delta_1^2}{\{[1+(|A_0|^2-\Delta_1^2)^{1/2}]^2+\omega^2\}^2}\equiv 1+\frac{16\Delta_1^2}{|\lambda_+^{(1)}(\omega)|^4}, \tag{200}$$

where $\lambda_+^{(1)}(\omega)$ is given by Eq. (130). The dependence of $T_{SN}^{(1)}$ on $\Gamma_1^{(1)}$ obtained by eliminating $|A_0|^2$ between Eqs. (200) and (132) is presented in Fig. 9 for different values of Δ_1 and ω. For comparison there is also the quantity T_{SN} for the case of the phase-insensitive linear amplifier (calculated in the Appendix) indicated in dots. If $\Delta_1\neq 0$, for given $\Gamma_1^{(1)}$, $T_{SN}^{(1)}$ decreases with the increase of ω, because when $\omega\to\infty$ we have $[T_{SN}^{(1)}-1]\sim\omega^{-4}$, while, according to Eq. (132), the amplification coefficient $[\Gamma_1^{(1)}-1]\sim\omega^{-2}$.

For the case, defined by Eqs. (117) and (122), we find $T_{SN}(\omega)\equiv T_{SN}^{(2)}(\omega)$ by inserting into Eq. (199) the expressions (188) and (95) with λ given by Eq. (135):

$$T_{SN}^{(2)}(\omega)=1+\frac{16\Delta_1^2[1+\Theta^{(2)}(\omega)]}{|\lambda_+^{(2)}(\omega)[2-\lambda_-^{(2)}(\omega)]|^2}+\frac{4\Theta^{(2)}(\omega)}{|2-\lambda_-^{(2)}(\omega)|^2} . \tag{201}$$

Figure 10 shows the behavior of $T_{SN}^{(2)}$ as a function of $\Gamma_1^{(2)}$ at some fixed values of the parameters E, $\gamma\equiv\gamma_1/\gamma_0$, ω of parametric amplifier (see Sec. I), while $\Delta_0=\Delta_1=0$

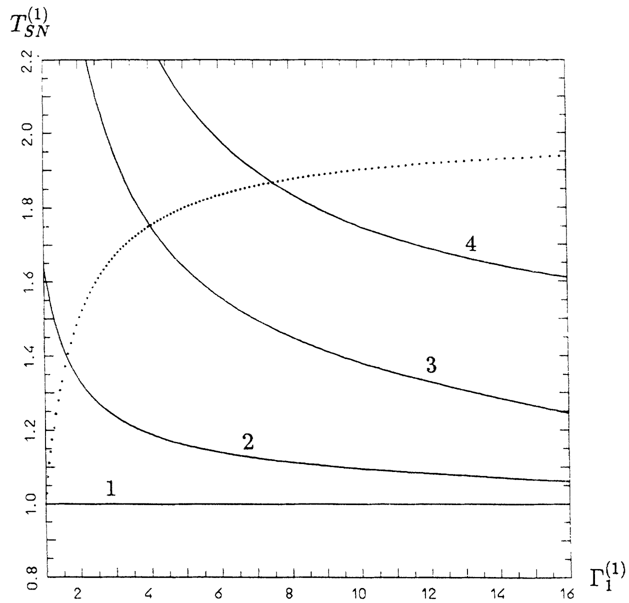

FIG. 9. Dependence of the signal-to-noise ratio transfer coefficient $T_{\rm SN}^{(1)}$ on the amplification coefficient $\Gamma_1^{(1)}$ in the case $|A_1|=0$ [see Eqs. (116), (122)]. The parameters are [$\Delta_1=0$] (1), [$\Delta_1=0.25$, $\omega=0.5$] (2), [$\Delta_1=0.5$, $\omega=0.5$] (3), [$\Delta_1=0.5$, $\omega=0$] (4). The dependence of $T_{\rm SN}$ on Γ for the phase-insensitive linear amplifier is shown in dots.

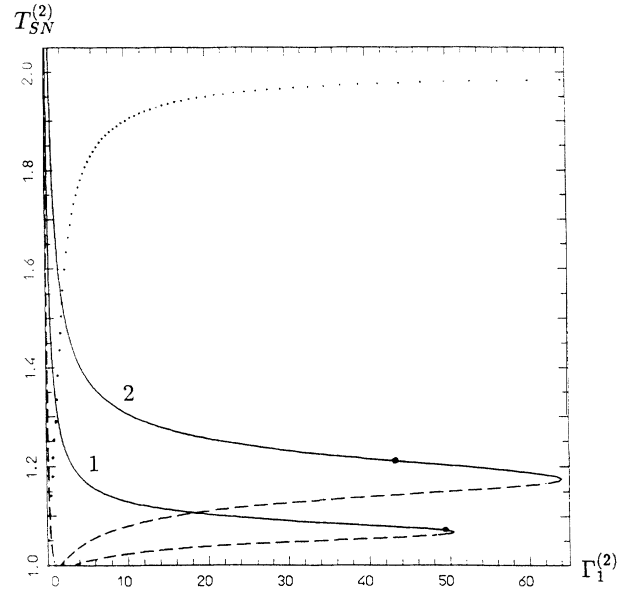

FIG. 10. Dependence of $T_{\rm SN}^{(2)}$ on $\Gamma_1^{(2)}$ for the resonant case: $\Delta_1=\Delta_0=0$, and for $\varphi=0$, $I_1>0$ [see Eqs. (117), (122)]. The part of the graph in solid line corresponds to a stable stationary solution, the part in broken line to an unstable one. The solid dot indicates the values of $T_{\rm SN}^{(2)}$ and $\Gamma_1^{(2)}$ in correspondence to the point $I_1=I_1^{(b)}$ (see Fig. 2). The dependence of $T_{\rm SN}$ on Γ for the phase-insensitive amplifier is shown by the dotted line. The parameters are $E=3.5$, $\gamma=10$, $\omega=0.85$ (line 1) $E=10$, $\gamma=5$, $\omega=2$ (line 2).

and $\varphi=0$. Figure 10 is obtained by eliminating $|A_1|^2$ between Eqs. (201) and (136), [using the expressions (124), (29), and (26)], so that $T_{\rm SN}$ and Γ are actually parametrized by the input carrier field amplitude e of field 1. The signal frequencies considered in Fig. 10 have been chosen numerically to provide the minimum possible $T_{\rm SN}^{(2)}$ for the given set of parameters. One can clearly see, that there can be a substantial decrease in $T_{\rm SN}^{(2)}$ with respect to the case of linear phase-insensitive amplifier when the amplification coefficient is large enough. By comparison of line 1 with line 2 in Fig. 10, one can find that in accordance with the condition $\Theta^{(2)}(\omega)\ll|A_0|$ (see Sec. IV D), the quantity $T_{\rm SN}^{(2)}$ decreases rapidly with the increase of the amplification coefficient $\Gamma_1^{(2)}$ for large γ. The behavior of $T_{\rm SN}^{(2)}$ and $\Gamma_1^{(2)}$ in the plane of parameters $I_1\equiv|A_1|^2$, ω is presented in Figs. 11(a)–11(c). Figures 11(a), and 11(b) refer to the case where there is bistability in the stationary solution, Fig. 11(c)—where there is no bistability. The stationary solution is unstable in the lower part of the figures separated by dotted/dashed line.

In the last case, defined by Eqs. (118) and (127), we find $T_{\rm SN}(0)$ from Eqs. (199), (191), (95), and (139):

$$T_{\rm SN}(0)=1+16(\Delta_1-\Delta_0\Theta^{(0)})^2\frac{1+\Theta_0}{[(2-\lambda_-^{(0)})\lambda_+^{(0)}]^2}+\frac{4\Theta^{(0)}}{(2-\lambda_-^{(0)})^2}\ . \tag{202}$$

Some plots of $T_{\rm SN}(0)$ as a function of $\Gamma_1(0)$, obtained by eliminating $|A_1|^2$ between Eqs. (202) and (140) and taking into account Eqs. (126), (29), and (26), are shown in Fig. 12 in comparison with the result valid for the linear phase-insensitive amplifier.

One can represent Eqs. (200)–(202) by one simple expression

$$T_{\rm SN}(\omega)=1+8P(\Delta_1,\Delta_0)\frac{\lambda_+(\omega)+\lambda_-^*(\omega)}{|\lambda_+(\omega)[2-\lambda_+(\omega)]|^2}+2\frac{\lambda_+(\omega)-\lambda_-(\omega)-2}{|2-\lambda_-(\omega)|^2}\ , \tag{203}$$

where $P(\Delta_1,\Delta_0)=\Delta_1^2$ for the cases (116), (117), and $P(\Delta_1,\Delta_0)=(\Delta_1-\Delta_0\Theta_0)^2$ for the case (118); $\lambda_\pm(\omega)$ are the respective eigenvalues normalized to γ_1.

VI. CONCLUSION

We have analyzed the optical parametric oscillator with two input fields in its semiclassical and quantum aspects. We have considered both linear and nonlinear regimes, including pump depletion in our description.

We showed that this cavity-based phase-sensitive amplifier is noiseless for all frequencies when the following conditions are simultaneously satisfied:

(1) The stationary part of the driving field with frequency Ω vanishes.

(2) The intensity of the stationary driving field of frequency 2Ω is such, that the parametric oscillator is below threshold.

(3) The cavity is on resonance with the field of frequency Ω.

When the input intensities are increased with respect to the conditions defined by (1) and (2), the performance of the system is degraded, but still there are extended ranges of the control parameters, in which the behavior with respect to the signal-to-noise ratio remains definitely better than in the phase-insensitive linear amplifier. Two especially interesting cases are

(i) when the input field of frequency 2Ω is resonant with the cavity;

(ii) when the modulation frequency ω' is much smaller than all relaxation rates of the system.

We finish the paper with two remarks.

(a) There are situations in which the signal modulation is a pure amplitude modulation; for example, when conditions (1) and (3) are satisfied, but (2) is not, and the cavity is in resonance also with the field of frequency 2Ω. In this case, it is not necessary to use a homodyne detection and one can consider the total intensity of the modulated

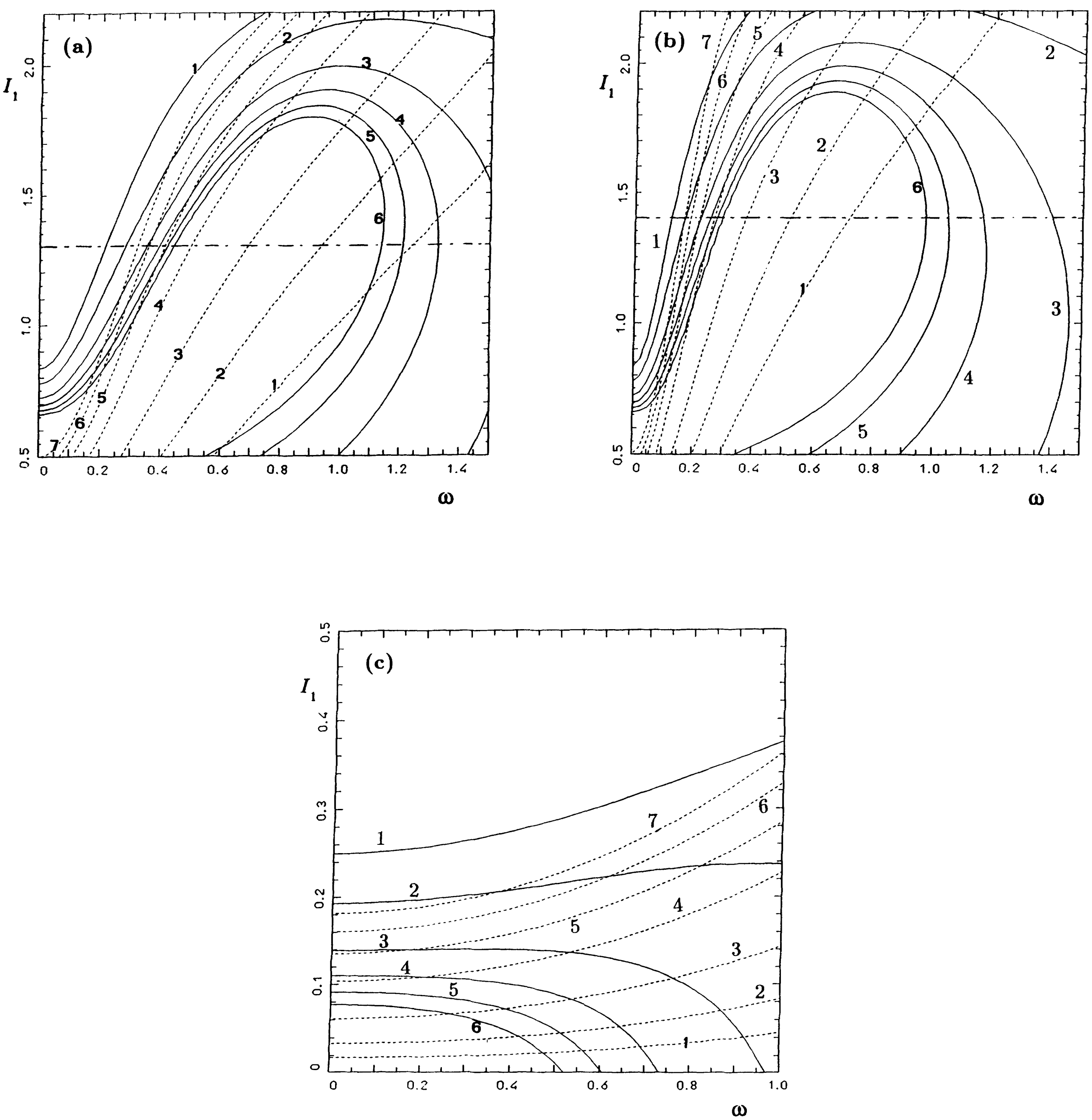

FIG. 11. Lines of constant $\Gamma_1^{(2)}$ (solid lines) and constant $T_{SN}^{(2)}$ (dashed lines) in the plane of parameters (I_1,ω) for the resonant case: $\Delta_1=\Delta_0=0$, and with $\varphi=0$, for $E=2.5$, $\gamma=5$ (a); $E=2.5$, $\gamma=10$ (b); $E=0.75$, $\gamma=1$ (c). For the solid lines $\Gamma_1^{(2)}=1$ (line 1), 2 (line 2), 4 (line 3), 6 (line 4), 8 (line 5), 10 (line 6). For the dashed lines $T_{SN}^{(2)}=1.05$ (line 1), 1.1 (line 2), 1.2 (line 3), 1.4 (line 4), 1.6 (line 5), 1.8 (line 6), 2 (line 7). The unstable regions (if any) are located below the dashed/dotted line.

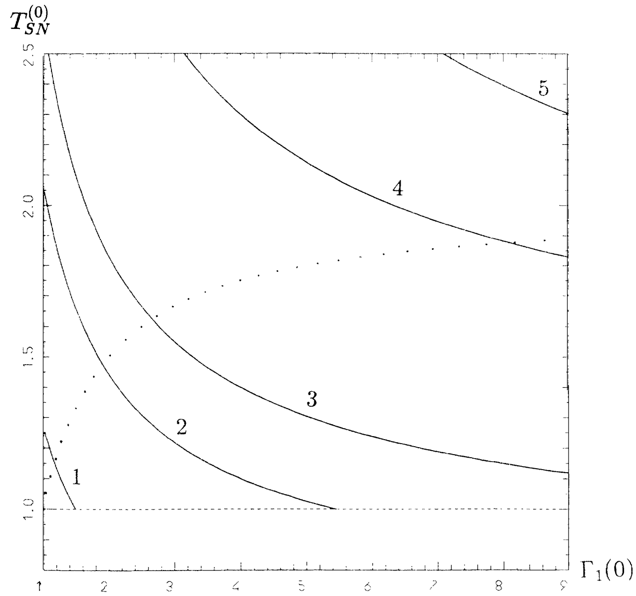

FIG. 12. Dependence of T_{SN} on Γ for zero frequency under condition (127). The values of the parameters are $\varphi=0$, $\gamma=1$; $\Delta_0=\Delta_1=0$, $E=0.1$ (curve 1), $E=0.4$ (curve 2), $E=0.6$ (curve 3), $E=1.2$ (curve 4), $\Delta_0=-\Delta_1=0.4$. $E=1.2$ (curve 5). The dependence of T_{SN} on Γ for the phase-insensitive linear amplifier is shown by the dotted line, the value $T_{SN}=1$ is indicated by the broken line.

signal and the noise in the total intensity of the field Ω.

(b) The approach that we devised in this paper, by defining the maximum amplification, minimum noise conditions, is general and can be applied to other cavity-based phase-sensitive amplifiers different from the optical parametric oscillator. This will be done in subsequent work.

ACKNOWLEDGMENTS

We are grateful to E. Giacobino and J. Mlynek for suggesting the problem studied in this paper. Helpful conversations with J.-Ph. Poizat are acknowledged. This work was carried out in the framework of the ESPRIT Basic Research Project 6934 "Quantum Information Technology" and of the Human Capital and Mobility network 920887 "Nonclassical Light." The Laboratoire Kastler Brossel is Laboratoire de l'Université P. M. Curie, et de l'Ecole Normale Supérieure, associé au CNRS.

APPENDIX: THE FUNCTION $T_{SN}(\Gamma)$ FOR THE LINEAR PHASE-INSENSITIVE AMPLIFIER

As an example of linear phase-insensitive amplifier we consider the single-ended cavity laser operated below threshold. We suppose that the input field contains only a small signal [as the one given by Eq. (56) with $\alpha_{c1}^{(\text{in})}=0$]. The optical frequency of the input field is in exact resonance with the cavity mode.

The amplification coefficient Γ is given, for example, by the formula (4.11) of Ref. [20]. For the case of the single-ended cavity and in terms of the frequency ω' normalized to the double round-trip time $2\tau_R$ this formula can be represented as

$$\Gamma(\omega')=\frac{(\omega')^2+\gamma^2(1+C)^2}{(\omega')^2+\gamma^2(1-C)^2}\equiv 1+\frac{4\gamma^2 C}{(\omega')^2+\gamma^2(1-C)^2}\ , \tag{A1}$$

where γ is the field intensity transitivity coefficient of the semitransparent mirror, C is the normalized pump parameter.

Now we calculate the noise in an output quadrature on the basis of Ref. [25]. Let $\delta\alpha$ be the fluctuation of the intracavity mode caused by quantum noise; the Langevin equation for $\delta\alpha$ is

$$\delta\dot{\alpha}(\tau)=-\gamma(1-C)\delta\alpha(\tau)+2\gamma(C)^{1/2}\xi(\tau)\ , \tag{A2}$$

where the correlation functions for the Langevin force $\xi(\tau)$ are

$$\begin{aligned}&\langle \xi(\tau)\xi(\tau')\rangle=\langle \xi^*(\tau)\xi^*(\tau')\rangle=0\ ,\\ &\langle \xi^*(\tau)\xi(\tau')\rangle=\delta(\tau-\tau')\ ,\end{aligned} \tag{A3}$$

and the factor 2 in the last term in Eq. (A2) reflects the fact that the cavity is single ended. The fluctuation $\delta\alpha_\delta$ of the field quadrature is given by

$$\delta\alpha_\delta(\tau)=\delta\alpha(\tau)e^{i\delta}+\delta\alpha^*(\tau)e^{-i\delta}\ . \tag{A4}$$

The noise spectrum $N(\omega')$ for the quadrature of the output field is given by the formula

$$N(\omega')=1+\gamma S(\omega')\ , \tag{A5}$$

where unity is the shot noise contribution and

$$S(\omega')=\int_{-\infty}^{+\infty} d\tau\, e^{i\omega'\tau}\langle \delta\alpha_\delta(0)\delta\alpha_\delta(\tau)\rangle\ . \tag{A6}$$

We calculate the Fourier components of $\delta\alpha(\tau)$ from Eq. (A2), substitute the Fourier decomposition of $\delta\alpha(\tau)$ in Eqs. (A4), (A6) taking into account the relations (A3); the result is

$$S(\omega')=\frac{8\gamma C}{(\omega')^2+\gamma^2(1-C)^2}\ . \tag{A7}$$

By combining Eqs. (A5), (A7), (A1), and (199) we finally obtain $T_{SN}(\Gamma)$ for the linear phase-insensitive amplifier

$$T_{SN}(\Gamma)=\frac{2\Gamma-1}{\Gamma}\ . \tag{A8}$$

The relation (A8) is true for any frequency.

[1] G. L. Mander, R. Loudon, and T. G. Shepherd, in *Photons and Quantum Fluctuations,* edited by R. E. Pike and H. Walther (Hilger, Bristol, 1988).

[2] M. Caves, Phys. Rev. D **26**, 1817 (1982).

[3] B. Yurke and J. S. Denker, Phys. Rev. A **29**, 1419 (1984).

[4] G. J. Milburn, M. L. Steyn-Ross, and D. F. Walls, Phys. Rev. A **35**, 4443 (1987); M. J. Collett and D. F. Walls, Phys. Rev. Lett. **61**, 2442 (1988).

[5] C. R. Gilson, S. M. Barnett, and S. Stenholm, J. Mod. Opt. **34**, 898 (1987).

[6] J. A. Vaccaro and D. J. Pegg, J. Mod. Opt. **34**, 855 (1987).

[7] H. J. Kimble, in *Systèmes Fondamentaux en Optique Quantique,* edited by J. Dalibard, J. M. Raimond, and J. Zinn-Justin (Les Houches, Session LIII, 1990, Fundamental Systems in Quantum Optics (Elsevier, Amsterdam, 1982).

[8] Y. Yamamoto, S. Machida, S. Saito, N. Imoto, T. Yanagawa, M. Kitagawa, and G. Bjork, in *Progress in Optics XXVIII,* edited by E. Wolf (Elsevier, Amsterdam, 1990).

[9] J. A. Levenson, J. Abram, T. Rivera, P. Fayolle, J. C. Garreau, and P. Granier, Phys. Rev. Lett. **70**, 267 (1993).

[10] Z. Y. Ou, S. F. Pereira, and H. J. Kimble, Phys. Rev. Lett. **70**, 3239 (1993).

[11] P. D. Drummond, K. J. McNeil, and D. F. Walls, Opt. Acta **27**, 321 (1990).

[12] P. D. Drummond, K. J. McNeil, and D. F. Walls, Opt. Acta **28**, 211 (1981).

[13] C. Fabre, E. Giacobino, A. Heidmann, L. A. Lugiato, S. Reynaud, M. Vadacchino, and Wang Kaige, Quantum Opt. **2**, 159 (1990).

[14] P. D. Drummond, K. J. McNeil, and D. F. Walls, Opt. Commun. **28**, 255 (1979).

[15] T. Poston and I. Stewart, in *Catastrophe Theory and its Applications,* edited by A. Jeffrey (Pitman, London, 1978).

[16] H. Haken, *Synergetics. An Introduction,* 2nd ed., Springer Series in Synergetics Vol. 1 (Springer-Verlag, Berlin, 1977), p. 123.

[17] R. Bonifacio and L. A. Lugiato, Lett. Nuovo Cimento **21**, 505 (1978).

[18] M. J. Collet and C. W. Gardiner, Phys. Rev. A **30**, 1386 (1984).

[19] C. W. Gardiner and M. J. Collet, Phys. Rev. A **31**, 3761 (1985).

[20] R. Loudon, M. Harris, T. G. Shepherd, and J. M. Vaughan, Phys. Rev. A **48**, 681 (1993).

[21] M. Harris, R. Loudon, T. G. Shepherd, and J. M. Vaughan, J. Mod. Opt. **39**, 1195 (1992).

[22] H. G. Carmichael, J. Opt. Soc. Am. B **4**, 1588 (1987).

[23] P. Galatola, L. A. Lugiato, M. G. Poirreca, P. Tombesi, and G. Leuchs, Opt. Commun. **85**, 95 (1991).

[24] M. J. Collett and D. F. Walls, Phys. Rev. A **32**, 2887 (1985).

[25] M. Sargent III, M. O. Scully, and W. E. Lamb, Jr., *Laser Physics* (Addison-Wesley, Reading, MA, 1974).

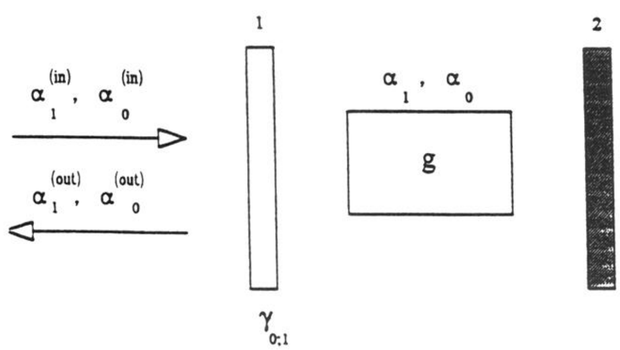

FIG. 1. Scheme of parametric amplifier with single-ended cavity. Mirror 2 is assumed to be completely reflecting.