Bremsstrahlung near reaction thresholds

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Low-frequency approximations for both spontaneous and stimulated bremsstrahlung processes are derived that are applicable to cases in which an electron is scattered from an atom at an energy near the threshold for excitation of the target. Existing approximation procedures are modified to properly account for the strong energy dependence of the cross section, in the form of cusps or rounded steps, caused by the opening up of a new channel. These modifications preserve an essential feature of standard versions of the low-frequency approximation in that the physical, field-free scattering amplitude appears as input to the calculation. An estimate of the error in the Feshbach-Yennie approximation [Nucl. Phys. **37**, 150 (1962)] for spontaneous bremsstrahlung is obtained through an evaluation of that contribution to the matrix element corresponding to radiation by the projectile following a virtual excitation of the target. To illustrate the effect of the anomalous energy dependence in a domain containing the threshold, as well as the significance of the above-mentioned correction to the Feshbach-Yennie approximation, calculations are described for a model system; the model is defined by a parametrization of the field-free scattering amplitude that reproduces the correct threshold behavior. A closely analogous treatment is given of laser-assisted scattering near a reaction threshold. A sum rule is derived for this external-field process which, in its simplest form, reproduces a recently conjectured approximation and provides higher-order corrections to it.

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I. INTRODUCTION

Some time ago, Feshbach and Yennie [1] provided a generalization of Low's version [2] of the soft-photon approximation for spontaneous bremsstrahlung that allows for the rapid energy dependence of the scattering amplitude near a resonance. In a closely analogous development, the Kroll-Watson approximation for scattering in a low-frequency external field [3] was extended to account for resonant scattering [4,5]. All such approximations are based on the dominance of soft-photon emission and absorption in initial and final states of the scattering process, and require as input a knowledge of the physical (on-shell) field-free scattering amplitude. While it would be very useful to have estimates of the errors in these approximations, it is generally believed that this would require off-shell scattering information and hence would be difficult to obtain. These approximation techniques for treating bremsstrahlung during resonant scattering can be applied to elastic and inelastic scattering near a reaction threshold; here too the field-free amplitude shows a rapid energy dependence. This is a process that has taken on added interest recently, and has been studied, for laser-assisted excitation, both experimentally [6,7] and theoretically [8,9]. As will be demonstrated in detail in the following, the theory can be carried further for threshold scattering than was possible for scattering near a resonance. In the former case an estimate can be derived, expressed explicitly in terms of the physical field-free scattering amplitude, for the effect of radiation in those intermediate states corresponding to (virtual) excitation of the target to the newly opened channel. The derivation proceeds in two steps. We first identify the matrix element that represents the error in the earlier versions of the low-frequency approximation, and we then perform an asymptotic evaluation of this matrix element in configuration space. This latter procedure is based on the observation that for virtual excitation near threshold the dominant contribution comes from great distances, where the integrand is slowly varying; this leads to a near singularity in the matrix element that enables us to separate it cleanly from the background.

The spontaneous bremsstrahlung problem is analyzed in Sec. II. A numerical estimate of the error term is obtained in a model based on a parametrization of the field-free scattering amplitude that incorporates the correct unitarity property and threshold singularities. The closely related external-field version of the theory is taken up in Sec. III A. The low-frequency approximation thus obtained provides the basis for a sum rule, derived in Sec. III B, for the total laser-assisted scattering cross section. The form of this sum rule is very similar to one which was postulated earlier [8] as a simple extension of the Kroll-Watson formula [3], and was shown to lead to good quantitative agreement with experimental results [6,7]. The sum rule derived here provides a theoretical basis for this analysis as well as a means for including higher-order corrections. Some of the details of the calculation described in Sec. II are separated form the main text and placed in two appendices.

II. SPONTANEOUS BREMSSTRAHLUNG

A. Preliminaries: potential scattering

To provide the background for the discussion to follow, we begin with an outline of the Feshbach-Yennie

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theory and, in the course of doing so, we identify the matrix element that contains the leading correction to the Feshbach-Yennie low-frequency approximation. At this stage the structure of the target plays no essential role and, to simplify notation, we consider scattering by a static potential (taken to be of short range to avoid complications associated with the Coulomb tail). Structure effects are included in the treatment given in Sec. II B.

The amplitude for the scattering of a particle (an electron, say) of mass μ and charge e in a potential $V(r)$, accompanied by the emission of a photon of frequency ω, is represented in the dipole approximation as $\hat{\mathbf{a}}\cdot\mathbf{m}(\mathbf{p}',\mathbf{p})$, where $\hat{\mathbf{a}}$ is a (real) unit polarization vector and

$$\mathbf{m}(\mathbf{p}',\mathbf{p})=\langle u_{\mathbf{p}'}^{(-)}|\mathbf{r}|u_{\mathbf{p}}^{(+)}\rangle \ . \tag{2.1}$$

Here $u_{\mathbf{p}}^{(\pm)}$ is a field-free scattering solution corresponding to incident momentum $\mathbf{p}$ and outgoing-wave $(+)$ or incoming-wave $(-)$ boundary conditions. The wave function may be decomposed as

$$|u_{\mathbf{p}}^{(\pm)}\rangle=|\mathbf{p}\rangle+|u_{\mathbf{p};sc}^{(\pm)}\rangle \ , \tag{2.2a}$$

with the scattered wave given by

$$|u_{\mathbf{p};sc}^{(\pm)}\rangle=(E_{\mathbf{p}}\pm i\eta-H_0)^{-1}V|\mathbf{p}\rangle \ ; \tag{2.2b}$$

H_0 represents the field-free Hamiltonian, $E_{\mathbf{p}}=p^2/2\mu$, and η is a positive infinitesimal. Then, for $\mathbf{p}'\neq\mathbf{p}$, we have

$$\mathbf{m}(\mathbf{p}',\mathbf{p})=\langle\mathbf{p}'|\mathbf{r}|u_{\mathbf{p};sc}^{(+)}\rangle+\langle u_{\mathbf{p}';sc}^{(-)}|\mathbf{r}|\mathbf{p}\rangle+\langle u_{\mathbf{p}';sc}^{(-)}|\mathbf{r}|u_{\mathbf{p};sc}^{(+)}\rangle \ . \tag{2.3}$$

In the Feshbach-Yennie approximation the first two terms on the right-hand side are replaced by $\mathbf{m}_{\mathrm{FY}}$ where, in units with $\hbar=1$,

$$i\omega\mathbf{m}_{\mathrm{FY}}=-\left[\frac{\mathbf{p}'}{\mu\omega}+2(\mathbf{p}'-\mathbf{p})\frac{\partial}{\partial\tau}\right]t(E_{\mathbf{p}},\tau)+\left[\frac{\mathbf{p}}{\mu\omega}+2(\mathbf{p}'-\mathbf{p})\frac{\partial}{\partial\tau}\right]t(E_{\mathbf{p}}-\omega,\tau) \ . \tag{2.4}$$

Here $t(E_{\mathbf{p}},\tau)$ is the on-shell field-free transition amplitude expressed in terms of the energy and momentum-transfer-squared variable $\tau=(\mathbf{p}'-\mathbf{p})^2$. A term of first order in the frequency has been ignored in arriving at the approximation (2.4). For completeness, a derivation of this result (which has a different form than that given originally in Ref. [1]) is provided in Appendix A. Note that the terms involving derivatives with respect to τ need to be retained, to the required accuracy, only if the t matrix is strongly energy dependent. In many cases the third term on the right in Eq. (2.3) will be comparable in magnitude to terms neglected in the Feshbach-Yennie approximation, and it will therefore be consistent to ignore it. Exceptions arise when near singularities appear in the integral; such a situation is examined in the following.

B. Multichannel scattering

We consider a transition taking the system from channel α, corresponding to the target in its ground state with energy B_α, and the projectile incident with momentum $\mathbf{p}_\alpha$, to a final state with channel label α'. (Dynamical effects of spin are ignored for simplicity.) We need not specify at this point whether or not the collision has induced a real target excitation, but in any case virtual excitations do occur. The initial energy of the system, $E_\alpha=p_\alpha^2/2\mu+B_\alpha$, is assumed to lie close to the excited-state target energy B_β, in which case threshold singularities in either the elastic or inelastic amplitude may be expected to play a role in the analysis. These singularities are generated, in a configuration-space approach, through the slow variation of the wave function in the asymptotic domain of channel β. The form of the wave function in this region is

$$u_\alpha^{(\pm)}\sim(2\pi)^{-3/2}\chi_\beta f_{\beta\alpha}^{(\pm)}(\pm p_\beta\hat{\mathbf{r}}_\beta,\mathbf{p}_\alpha)\exp(\pm ip_\beta r_\beta)/r_\beta \ , \tag{2.5}$$

where $\mathbf{r}_\beta$ is the coordinate of the free electron in channel β, and χ_β is the excited-state target function. (The Pauli principle, ignored here, may be imposed on the transition amplitude by taking suitable combinations of direct and exchange matrix elements). The projectile momentum, satisfying $p_\beta^2/2\mu=E_\alpha-B_\beta$, will be imaginary for an initial energy lying below threshold; in that case signs are chosen so that the exponential in Eq. (2.5) is decaying. (The necessary analytic continuation of the scattering amplitude below threshold is illustrated in the model calculation described below.) The relation between the scattering amplitude and the conventionally defined t matrix is $f_{\beta\alpha}=-\mu(2\pi)^2t_{\beta\alpha}$.

In an asymptotic evaluation of the correction term, we arrive at a decomposition

$$\langle u_{\alpha';sc}^{(-)}|\mathbf{r}_\beta|u_{\alpha;sc}^{(+)}\rangle=\mathbf{C}_{\alpha'\alpha}+O(1) \ , \tag{2.6}$$

where the dominant contribution, obtained by replacing the wave functions by their asymptotic forms in channel β and integrating over the coordinates of the bound-state wave function as well as over the radial coordinate of the projectile, is readily found to be

$$\mathbf{C}_{\alpha'\alpha}(\mathbf{p}'_{\alpha'},\mathbf{p}_\alpha)=(p_\beta+p'_\beta)^{-2}(-2\pi)^{-3}\times\int d\Omega(\hat{\mathbf{r}})f_{\alpha'\beta}(\mathbf{p}'_{\alpha'},-p'_\beta\hat{\mathbf{r}})\hat{\mathbf{r}}f_{\beta\alpha}(p_\beta\hat{\mathbf{r}},\mathbf{p}_\alpha) \ . \tag{2.7}$$

The reciprocity relation $f_{\beta\alpha}^{(-)*}(\mathbf{p}',\mathbf{p})=f_{\alpha\beta}^{(+)}(\mathbf{p},\mathbf{p}')$ was used in obtaining this result, and the superscript $(+)$ on the scattering amplitude is left implicit in Eq. (2.7) and in the following. From the relation $p_\beta'^2/2\mu=E_{\alpha'}-B_\beta$, with $E_{\alpha'}=E_\alpha-\omega$, along with the similar definition of p_β given above, we see explicitly that the magnitude of the correction term is enhanced near threshold by the presence of a nearly singular factor multiplying the integral in Eq. (2.7). The fact that it is the physical scattering amplitude that appears in the angular integration suggests that deriving numerical estimates of the correction term could be facilitated by using experimental data, or results of theory, for the field-free system. The model calculation described below provides an illustration of such a procedure.

The angular integration in Eq. (2.7) may be performed with the aid of the partial-wave expansion

$$f_{\beta\alpha}(\mathbf{p}_\beta,\mathbf{p}_\alpha)=\sum_{l=0}^{\infty}(2l+1)P_l(\hat{\mathbf{p}}_\beta\cdot\hat{\mathbf{p}}_\alpha)f_{\beta\alpha;l}(E_\alpha) \ . \quad (2.8)$$

The partial-wave amplitudes near threshold may be represented as

$$f_{\beta\alpha;l}(E_\alpha)=\frac{i}{2}p_\beta^l h_{\beta\alpha;l}(E_\alpha)p_\alpha^l \ , \quad (2.9)$$

where, with the kinematic energy dependence of each channel element separated off, the remaining element $h_{\beta\alpha;l}$ is finite and in general nonzero at threshold [10]. (For simplicity the angular momentum quantum numbers have been taken to be the same for entrance and exit channels.) The relation between the scattering amplitudes and the S matrix then takes the form

$$S_{\beta\alpha;l}=\delta_{\beta\alpha}-p_\beta^{l+1/2}h_{\beta\alpha;l}p_\alpha^{l+1/2} \ . \quad (2.10)$$

Only the s and p waves will be retained; as indicated by the powers of momentum appearing in Eq. (2.9), contributions from higher partial waves are suppressed. The correction term then becomes

$$i\omega\mathbf{C}_{\alpha'\alpha}\cdot\hat{\mathbf{a}}=\frac{i\omega}{8\pi^2(p'_\beta+p_\beta)^2}[p_\beta\mathbf{p}_\alpha\cdot\hat{\mathbf{a}}h_{\alpha'\beta;0}(E_\alpha-\omega)h_{\beta\alpha;1}(E_\alpha)-p'_\beta\mathbf{p}'_{\alpha'}\cdot\hat{\mathbf{a}}h_{\alpha'\beta;1}(E_\alpha-\omega)h_{\beta\alpha;0}(E_\alpha)] \ . \quad (2.11)$$

It is consistent to retain this correction term since, by virtue of the near singularity, it is nominally of lower order than the terms neglected in the Feshbach-Yennie approximation. It would not be difficult to provide a more explicit definition of the term "order" in the present context, taking into account the existence of two small parameters, the photon frequency and the interval between the initial energy and the threshold value. Perhaps a better feeling for the significance of the correction term can come from numerical exploration, an example of which is described below.

C. A simple model

The matrix element of interest, in an approximation which accounts only for terms that contain near singularities, is of the form $\mathbf{m}_{\alpha'\alpha}\cong(\mathbf{m}_{\alpha'\alpha})_{\mathrm{FY}}+\mathbf{C}_{\alpha'\alpha}$, where the first term represents the multichannel version of the Feshbach-Yennie approximation reviewed in Sec. II. Let us suppose that the on-shell scattering amplitude is expanded in partial waves and that, for simplicity, only the first two terms are retained. The result is written out in Appendix A. A model is defined by the choice of partial-wave scattering amplitudes. To guarantee unitarity we make use of the matrix representation

$$S_l=\frac{1+iR_l}{1-iR_l} \ , \quad (2.12)$$

with the reactance matrix R_l chosen to be real and symmetric. It is known, moreover, that the matrix R_l may be put in the form

$$R_l=p^{l+1/2}K_lp^{l+1/2} \ , \quad (2.13)$$

where the elements of K_l remain finite as any of the channel momenta tend to zero and are even functions of these momenta [10]. This suggests a parameterization of the scattering matrix in the form of an effective-range expansion [11], with expansion coefficients which could, for example, be chosen to match experimental data. In the schematic model considered here we make a somewhat arbitrary choice for the leading term in the expansion for each element and discard the remaining terms. The explicit choice appears in Appendix B. Once the approximate bremsstrahlung matrix element has been chosen the doubly differential cross section, summed over photon polarization states, is obtained as

$$(r_0\lambda_C)^{-1}\omega\left[\frac{d^2\sigma}{d\omega\, d\Omega}\right]_{\alpha'\alpha}=\frac{2}{3\pi}\frac{p'_{\alpha'}}{p_\alpha}|(2\pi\mu\omega)^2\mathbf{m}_{\alpha'\alpha}|^2 \ , \quad (2.14)$$

where r_0 is the classical radius of the electron, and λ_C is its Compton wavelength.

The bremsstrahlung cross section, normalized to the dimensionless form shown in Eq. (2.14), was calculated in the present model (as defined more explicitly in Appendix B) for a range of incident momenta near threshold, and for the case of elastic scattering through an angle of 60°; the result is plotted in Fig. 1. A single cusp seen in the field-free cross section is doubled here since the photon (assumed to have an energy of 0.025 a.u.) can be radiated either before or after the collision. The Feshbach-Yennie approximation to the bremsstrahlung amplitude, the mul-

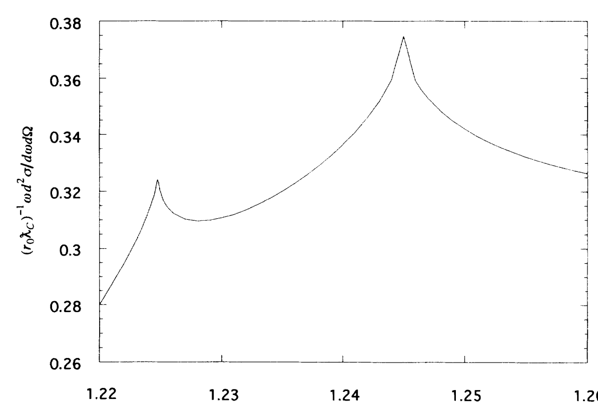

FIG. 1. Doubly differential cross section for single-photon bremsstrahlung, given by the approximation shown in Eq. (2.14) of the text as obtained in a schematic model, plotted against the momentum of the incident electron in the neighborhood of an excitation threshold. The scattering angle is 60°, the energy of the emitted photon is 0.025 a.u., and the excitation energy is 0.75 a.u. The first cusp corresponds to elastic scattering followed by photon emission, and the cusp at higher energy arises from photon emission before the collision.

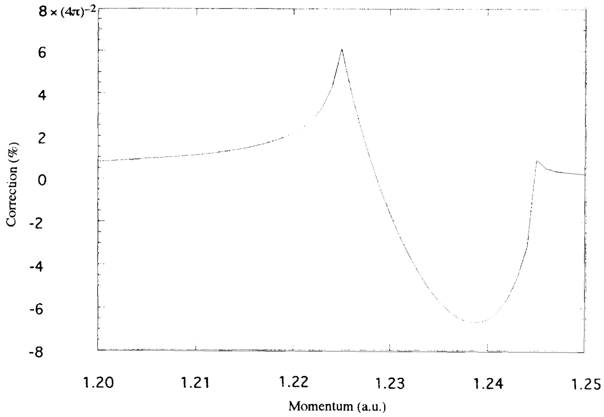

FIG. 2. Correction, expressed as a percentage, to the Feshbach-Yennie approximation to the model doubly differential cross section. The amplitude of the correction term, arising from the threshold singularity, is shown in Eq. (2.11) of the text.

tichannel version of that shown in Eq. (2.4), contains a first-order term involving a derivative of the t matrix which, for the parameters of the present model, makes a negligible contribution to the calculated cross section. Inclusion of the correction term shown in Eq. (2.11) has a small effect, though still exceeding the first-order correction generated by the standard Feshbach-Yennie approximation. A plot of the correction to the differential cross section, expressed as a percentage, appears in Fig. 2. The correction to the *total* cross section is negligible owing to the cancellation of an interference term upon angular integration.

III. EFFECT OF REACTION THRESHOLDS ON LASER-ASSISTED SCATTERING

A. Low-frequency approximation

We consider the scattering of an electron by a neutral atom in the presence of a monochromatic, linearly polarized laser field of frequency ω, with vector potential, in the dipole approximation, taken to be $\mathbf{A}(t)=\mathbf{a}\cos\omega t$. The time-dependent Schrödinger equation is of the form

$$\left[H-i\frac{\partial}{\partial t}\right]\Psi_\alpha^{(\pm)}(t)=0\ , \tag{3.1a}$$

with

$$H=\frac{1}{2\mu}\sum_{j=1}^{N}\left[-i\nabla_j-\frac{e}{c}\mathbf{A}(t)\right]^2+V\ . \tag{3.1b}$$

The wave function may be decomposed as $\Psi_\alpha^{(\pm)}=\Phi_\alpha+\Psi_{\alpha;sc}^{(\pm)}$, with the incident wave satisfying

$$\left[H-i\frac{\partial}{\partial t}\right]\Phi_\alpha=V_\alpha\Phi_\alpha\ , \tag{3.2}$$

where V_α is the interaction between the incident electron and the target in channel α. It is not possible in general to solve this equation since the state of the target in the presence of the field is not known precisely. However, if the frequency of the field is small compared to a characteristic level separation of the target, as will be assumed here, an approximate dressed-target wave function can be obtained, accurate to first order in the frequency [5,12]. It takes the form of a gauge transformation,

$$\chi_\alpha(\mathbf{A};t)\cong\exp(g_{\alpha T})\chi_\alpha\exp(-iB_\alpha t) \tag{3.3a}$$

of the field-free wave function $\chi_\alpha\exp(-iB_\alpha t)$, which we assume to be known to sufficient accuracy, from the Coulomb gauge to the field gauge. Here

$$g_{\alpha T}=\frac{ie}{c}\mathbf{A}(t)\cdot\sum_{j=1}^{N-1}\mathbf{q}_{\alpha j}\ , \tag{3.3b}$$

with the vectors $\mathbf{q}_{\alpha j}$ representing the coordinates of the target electrons in channel α. In this approximation [13] we have

$$|\Phi_\alpha(t)\rangle=\exp\left[-i\int^t\frac{p^2(\omega t')}{2\mu}dt'\right]|\mathbf{p}\rangle|\chi_\alpha(\mathbf{A};t)\rangle\ , \tag{3.4}$$

where $\mathbf{p}(\omega t)=\mathbf{p}-(e/c)\mathbf{A}(t)$. (Channel subscripts on momentum variables are omitted here to simplify notation.)

The transition amplitude for the laser-assisted collision satisfies the variational identity [12]

$$T_{\alpha'\alpha}=\int_{-\infty}^{\infty}dt\left[\langle\Phi_{\alpha'}|V_{\alpha'}|\tilde{\Psi}_\alpha^{(+)}\rangle+\langle\Psi_{\alpha';sc}^{(-)}|\left[H-i\frac{\partial}{\partial t}\right]|\tilde{\Psi}_\alpha^{(+)}\rangle\right]\ , \tag{3.5}$$

where $\tilde{\Psi}_\alpha^{(+)}$ is a trial wave function. Approximations are introduced through the replacement of the exact scattered wave $\Psi_{\alpha';sc}^{(-)}$ by a trial function $\tilde{\Psi}_{\alpha';sc}^{(-)}$. This introduces an error in the approximate transition amplitude that is bilinear in the errors in the two trial functions, and we remain with a variational approximation for the transition amplitude. We shall evaluate this approximation using a particular choice of trial function flexible enough to properly account for the strong energy dependence of the scattering near threshold. A function of this type was introduced in previous studies of resonant scattering in a low-frequency laser field [5,14]. An evaluation of the first term on the right in Eq. (3.5) was given in Ref. [5] and the results will be reviewed briefly. Of particular interest here is the contribution coming from the second term, the variational correction, which is nominally of first order in the error in the trial function. For scattering near a reaction threshold it may be expected, in analogy with the spontaneous bremsstrahlung problem analyzed in Sec. II, that a magnification of the error beyond its nominal order could be induced by near singularities in the integral representing the error. This is borne out by the explicit form of the result derived below.

To proceed, we look for trial functions of the form

$$|\tilde{\Psi}_\alpha^{(\pm)}\rangle=\sum_{n=-\infty}^{\infty}\exp[-iE_{\alpha n}t+g_\alpha]|F_{\alpha n}^{(\pm)}\rangle\ , \tag{3.6}$$

where $E_{\alpha n}=E_\alpha+n\omega+\Delta$, with the continuum level shift given by $\Delta=e^2a^2/4\mu c^2$ and with

$$g_\alpha=g_{\alpha T}+i\frac{e}{c}\mathbf{A}(t)\cdot\mathbf{r}_\alpha . \quad (3.7)$$

The function $F_{\alpha n}$ must account for the simultaneous interactions of the electron-atom system with the external field and of the electron with the target. To describe this latter interaction we introduce the off-shell wave function

$$|u_\alpha^{(\pm)}(\mathbf{p},E)\rangle=|\mathbf{p}\rangle|\chi_\alpha\rangle+|u_{\alpha;sc}^{(\pm)}(\mathbf{p},E)\rangle , \quad (3.8)$$

with

$$|u_{\alpha;sc}^{(\pm)}(\mathbf{p},E)\rangle=(E\pm i\eta-H_0)^{-1}V_\alpha|\mathbf{p}\rangle|\chi_\alpha\rangle . \quad (3.9)$$

The external-field interaction is accounted for by the familiar Volkov phase factor

$$\Theta_{\mathbf{p}n}(\phi)=n\phi+\rho_{\mathbf{p}}\sin\phi-\gamma\sin2\phi , \quad (3.10)$$

where $\rho_{\mathbf{p}}=e\mathbf{p}\cdot\mathbf{a}/\mu c\omega$ and $\gamma=\Delta/2\omega$. In terms of these quantities we define

$$|F_{\alpha n}^{(\pm)}\rangle=\int_0^{2\pi}\frac{d\phi}{2\pi}\exp[i\Theta_{\mathbf{p}n}(\phi)]|u_\alpha^{(\pm)}[\mathbf{p}(\phi),E_{\alpha n}]\rangle . \quad (3.11)$$

Before developing the approximation procedure any further, we pause to comment on several features of this trial function. We first observe that the trial function obtained from Eqs. (3.6) and (3.11) reduces to the incident wave Φ_α when the wave function $u_\alpha^{(\pm)}$ in Eq. (3.11) is replaced by its plane-wave component. Accordingly, the trial scattered wave function $\tilde{\Psi}_{\alpha;sc}^{(\pm)}$ is identified as that obtained by replacing $u_\alpha^{(\pm)}$ in Eq. (3.11) by its *scattered-wave* component. Some indication of the accuracy of the trial function can be gained by inserting this function into the Schrödinger equation; one finds that

$$\left[H-i\frac{\partial}{\partial t}\right]\tilde{\Psi}_\alpha^{(+)}(t)=-e\mathbf{E}(t)\cdot\sum_{j=1}^{N}\mathbf{r}_j\tilde{\Psi}_{\alpha;sc}^{(+)}(t) , \quad (3.12)$$

with the electric-field vector given by

$$\mathbf{E}(t)=\frac{\omega\mathbf{a}}{c}\sin\omega t . \quad (3.13)$$

The form of the remainder term on the right in Eq. (3.12) suggests that the validity of the trial function is restricted to the low-frequency domain and to moderate field intensities. Specifically, we shall assume that the parameter $\rho_{\mathbf{p}}$ defined above is of order unity or smaller and that γ is of first order; then terms appearing in the derivation that are proportional to $\omega\rho_{\mathbf{p}}$ and $\omega\gamma$ are taken to be of first and second orders, respectively [15], with terms of second order ignored. (We note that the experimental studies of laser-assisted excitation [6,7] were performed with fields in the range of "moderate" intensities as defined above.)

When the exact scattered wave appearing in the second term on the right in Eq. (3.5) is replaced by the trial function introduced above, an approximation for the transition amplitude is obtained which can be put in the form

$$T_{\alpha'\alpha}\cong2\pi\sum_{s=-\infty}^{\infty}\delta(E_{\alpha'}-E_\alpha+s\omega)(T_{\alpha'\alpha s}^{(1)}+T_{\alpha'\alpha s}^{(2)}) . \quad (3.14)$$

Each term in the sum corresponds to a process in which a net number s of photons are emitted (or absorbed for s negative). The leading term

$$T_{\alpha'\alpha s}^{(1)}=\sum_{n=-\infty}^{\infty}\int_0^{2\pi}\frac{d\phi}{2\pi}\int_0^{2\pi}\frac{d\phi'}{2\pi}\exp[-i\Theta_{\mathbf{p}'s+n}(\phi')+i\Theta_{\mathbf{p}n}(\phi)]t_{\alpha'\alpha}[\mathbf{p}'(\phi'),\mathbf{p}(\phi);E_{\alpha n}] , \quad (3.15)$$

involving the off-shell t matrix

$$t_{\alpha'\alpha}(\mathbf{p}',\mathbf{p};E)=\langle\mathbf{p}'|\langle\chi_{\alpha'}|[V_\alpha+V_{\alpha'}(E+i\eta-H_0)^{-1}V_\alpha]|\mathbf{p}\rangle|\chi_\alpha\rangle , \quad (3.16)$$

appeared in an earlier study [5]. We briefly review the procedure followed there for reducing this expression to one involving the physical t matrix. As a first step, the off-shell t matrix in Eq. (3.15) is expanded in powers of the initial and final momenta about their zero-field values. Two linear terms appear, one proportional to $\cos\phi$ and the other to $\cos\phi'$. The phase integrations may then be evaluated in terms of the generalized Bessel function

$$J_{-n}(\rho_{\mathbf{p}},\gamma)=\int_0^{2\pi}\frac{d\phi}{2\pi}\exp[i\Theta_{\mathbf{p}n}(\phi)] \quad (3.17)$$

using the relation

$$2\int_0^{2\pi}\frac{d\phi}{2\pi}\cos\phi\exp[i\Theta_{\mathbf{p}n}(\phi)]=J_{-n+1}(\rho_{\mathbf{p}},\gamma)+J_{-n-1}(\rho_{\mathbf{p}},\gamma) . \quad (3.18)$$

Since we are evaluating a first-order correction term and the argument γ of the generalized Bessel function is of first order, we may, to the required accuracy, introduce the approximation $J_{-n}(\rho,\gamma)\cong J_{-n}(\rho,0)=J_{-n}(\rho)$ on the right-hand side of Eq. (3.18), which then becomes $-(2n/\rho_{\mathbf{p}})J_{-n}(\rho_{\mathbf{p}})$ by virtue of the recursion relation satisfied by the ordinary Bessel function. Note that this remains well defined in the limit of vanishing momentum. Examining the coefficients of the terms involving the momentum gradients, and assuming for the moment that $\mathbf{p}'$ is not small, one sees that the field-free momenta have effectively been shifted to the field-dependent values

$$\mathbf{p}_n=\mathbf{p}+\frac{\mu n\omega}{\mathbf{p}\cdot\mathbf{a}}\mathbf{a} , \quad \mathbf{p}'_{s+n}=\mathbf{p}'+\frac{\mu(s+n)\omega}{\mathbf{p}'\cdot\mathbf{a}}\mathbf{a} , \quad (3.19)$$

to first order, and this places the t matrix on shell. Then, with the t matrix expressed as a function of the energy and momentum-transfer variables, the result of the calculation may, to first-order accuracy, be summarized concisely in the form

$$T^{(1)}_{\alpha'\alpha s}=\sum_{n=-\infty}^{\infty} J_{-(s+n)}(\rho_{\mathbf{p}'},\gamma)J_{-n}(\rho_{\mathbf{p}},\gamma) \times t_{\alpha'\alpha}[E_{\alpha n},(\mathbf{p}'_{s+n}-\mathbf{p}_n)^2] . \tag{3.20a}$$

To allow for greater generality, the t matrix appearing here should be written as

$$t_{\alpha'\alpha}[E_{\alpha n},(\mathbf{p}'_{s+n}-\mathbf{p}_n)^2] \to \left[1+\Delta\tau\frac{\partial}{\partial\tau}\right]t_{\alpha'\alpha}[E_{\alpha n},\tau]\Big|_{\tau=(\mathbf{p}'-\mathbf{p})^2} , \tag{3.20b}$$

with $\Delta\tau=(\mathbf{p}'_{s+n}-\mathbf{p}_n)^2-(\mathbf{p}'-\mathbf{p})^2$. The result then remains well defined and valid for an excitation process near threshold, in which case the final momentum is very small and the form shown in Eq. (3.20a) breaks down. We remark that if the t matrix is slowly varying as a function of the energy the term proportional to $\Delta\tau$ in Eq. (3.20b) may be ignored since it leads to a correction of second order. (This is evident from an examination of the more general form of the correction term, given in Eq. (3.19c) of Ref. [5].) It is readily verified that Eq. (3.20a), as amended by Eq. (3.20b), reduces in the weak-field limit to a multichannel version of the Feshbach-Yennie approximation.

The variational correction term appearing in Eq. (3.14) is evaluated, with the aid of Eq. (3.12), as

$$T^{(2)}_{\alpha'\alpha s}=i\frac{e\omega}{2c}\mathbf{a}\cdot(\mathbf{M}_{\alpha'\alpha s-1}-\mathbf{M}_{\alpha'\alpha s+1}) , \tag{3.21}$$

where

$$\mathbf{M}_{\alpha'\alpha s}=\sum_{n=-\infty}^{\infty}\int_0^{2\pi}\frac{d\phi}{2\pi}\int_0^{2\pi}\frac{d\phi'}{2\pi}\exp[-i\Theta_{\mathbf{p}'s+n}(\phi')+i\Theta_{\mathbf{p}n}(\phi)]\langle u^{(-)}_{\alpha';sc}[\mathbf{p}'(\phi'),E_{\alpha' s+n}]|\mathbf{r}_\beta|u^{(+)}_{\alpha;sc}[\mathbf{p}(\phi),E_{\alpha n}]\rangle . \tag{3.22}$$

The matrix element appearing on the right is recognized as the off-shell extension of the correction to the low-frequency approximation for spontaneous bremsstrahlung, analyzed in Sec. II. A closer correspondence is made when the amplitude is expanded in powers of the momenta about their zero-field values, in the manner described above. For the case of elastic scattering, with both initial and final momenta bounded away from zero, the shifted momenta thereby introduced are just those defined above in Eq. (3.19), and this places the free-free matrix element on the energy shell. The analysis leading to Eqs. (2.6) and (2.7) then applies directly, and we arrive at the approximation

$$\mathbf{M}_{\alpha'\alpha s\mp 1}\cong\sum_{n=-\infty}^{\infty} J_{-(s+n)}(\rho_{\mathbf{p}'},\gamma)J_{-(n\pm 1)}(\rho_{\mathbf{p}},\gamma) \times\mathbf{C}_{\alpha'\alpha}(\mathbf{p}'_{s+n},\mathbf{p}_{n\pm 1}) . \tag{3.23}$$

The physical interpretation of the correction term defined in Eq. (3.21) is apparent. That term describes a process in which the electron in the initial state exchanges n photons with the field, then either emits a photon [corresponding to the upper sign in Eq. (3.23)] or absorbs one (corresponding to the lower sign) in an intermediate state of virtual excitation; in the final state the system exchanges photons with the field of such a number to correspond to a net transfer of s photons. The matrix element describing the intermediate-state radiative interaction is just that which appears as the correction to the Feshbach-Yennie soft-photon approximation, and is evaluated, as shown in Eq. (2.7), in terms of the on-shell field-free scattering amplitude. A method for representing this amplitude in a way that properly includes the threshold singularities is described in Appendix A [16].

For inelastic scattering near threshold, Eq. (3.23) must be modified to remove the spurious near singularity appearing in the expression given in Eq. (3.19) defining the shifted final-state momentum. To do so one expresses the on-shell amplitude $\mathbf{C}_{\alpha'\alpha}\cdot\mathbf{a}$ in terms of scalar invariants and expands in powers of those invariants involving the final-state momentum, in close analogy with the procedure indicated in Eq. (3.20b). (The scalars may be taken to be the momentum transfer squared and $\mathbf{p}'_{s+n}\cdot\mathbf{a}$, for example.) Terms of second order are ignored and the first-order terms are converted, with the aid of the Bessel function recursion relation in the manner described above, to a well-defined form in which the small denominator $\mathbf{p}'\cdot\mathbf{a}$ no longer appears. Thus no difficulties are encountered in applying the result to inelastic scattering near threshold.

B. Cross-section sum rule

The total cross section summed over all final states of the field, and integrated over all directions of the emergent electron, is given by

$$\sigma_{\alpha'\alpha}=\frac{(2\pi)^4\mu}{p}\sum_{s=-\infty}^{\infty}\int d\mathbf{p}'\delta(E_{\alpha'}-E_\alpha+s\omega) \times|T^{(1)}_{\alpha'\alpha s}+T^{(2)}_{\alpha'\alpha s}|^2 . \tag{3.24}$$

This expression may be simplified considerably when the transition amplitudes are replaced by their low-frequency approximations, as derived above [17]. We write

$$\sigma_{\alpha'\alpha}\cong\frac{(2\pi)^2\mu}{p}(\sigma^{(1)}_{\alpha'\alpha}+\sigma^{(1,2)}_{\alpha'\alpha}) , \tag{3.25}$$

where the first term on the right is the result obtained by ignoring the first-order amplitude $T^{(2)}_{\alpha'\alpha s}$ in Eq. (3.24), and the cross term between this amplitude and $T^{(1)}_{\alpha'\alpha s}$ is represented by the second term in Eq. (3.25). It will now be shown that

$$\sigma^{(1)}_{\alpha'\alpha}=\sum_{n=-\infty}^{\infty} J^2_{-n}(\rho_{\mathbf{p}},\gamma)\sigma_{\alpha'\alpha n} , \tag{3.26a}$$

where

$$\sigma_{\alpha' \alpha n}=\int d\mathbf{p}'\delta\left[\frac{p'^2}{2\mu}+B_{\alpha'}-\frac{p_n^2}{2\mu}-B_\alpha\right]|t_{\alpha'\alpha}(\mathbf{p}',\mathbf{p}_n;E_{\alpha n})|^2 \quad (3.26b)$$

is proportional to the measurable field-free scattering cross section. An equation of this form (or rather a version excluding first-order corrections) was proposed in Ref. [8] as an extension of the Kroll-Watson low-frequency approximation. A verification, such as that given here, is required since the Kroll-Watson approximation is based on the assumption of smooth energy dependence and hence does not apply to the problem of scattering near reaction thresholds.

The derivation of Eq. (3.26) begins with the representation

$$T^{(1)}_{\alpha' \alpha s}=\sum_{n=-\infty}^{\infty}\int_0^{2\pi}\frac{d\phi'}{2\pi}\exp[-i\Theta_{\mathbf{p}'s+n}(\phi')]J_{-n}(\rho_{\mathbf{p}},\gamma)\times t_{\alpha'\alpha}[\mathbf{p}'(\phi'),\mathbf{p}_n;E_{\alpha n}] \ . \quad (3.27)$$

This was obtained from Eq. (3.15) by expanding the t matrix in powers of the initial momentum about its field-free value, and performing the integration over the angle ϕ in the manner described above in the derivation of Eq. (3.20). Consider the formula for the cross section $\sigma^{(1)}_{\alpha'\alpha}$ associated with this amplitude, with the absolute square expanded as a double integral over ϕ and ϕ' and a double sum over n and n'. With the substitution $s=s'-n$ the argument of the energy-conserving δ function becomes $E_{\alpha'}-p_n^2/2\mu-B_\alpha+s'\omega$. To perform the sum over s' we expand the δ function in a Taylor series in powers of $s'\omega$ and make the replacement $(\omega s')^k \rightarrow(-\omega\rho_{\mathbf{p}'}\cos\phi')^k$ for each term appearing in the expansion. This is justified using a procedure of repeated integrations by parts, with terms of second order consistently ignored. After resummation of the Taylor series the δ function appears with the argument $p'^2(\phi')/2\mu+B_{\alpha'}-p_n^2/2\mu-B_\alpha$. The relation

$$\sum_{s'=-\infty}^{\infty} e^{is'(\phi-\phi')}=2\pi\delta(\phi-\phi') \quad (3.28)$$

is now used to reduce the double integral over the phase angles to a single integral over ϕ. After the variable change $\mathbf{p}'(\phi)\rightarrow\mathbf{p}'$ the remaining phase integral is performed as

$$\int_0^{2\pi}\frac{d\phi}{2\pi}e^{i(n'-n)\phi}=\delta_{n'n} \ , \quad (3.29)$$

which leads directly to the result shown in Eq. (3.26).

The result just obtained may be interpreted along the following lines. The summation over final states has, by closure, effectively removed traces of final-state electron-field interactions. The factor J^2_{-n} in Eq. (3.26a) represents the probability for an exchange of n photons with the field in the initial state, and this is followed by a field-free scattering event at the appropriately shifted energy $E_\alpha+n\omega$. An extension of this picutre allows us to anticipate the form taken by the contribution $\sigma^{(1,2)}_{\alpha'\alpha}$. This arises from the interference between two amplitudes. One of them, $J_{-n}(\rho_{\mathbf{p}},\gamma)t(\mathbf{p}',\mathbf{p}_n;E_{\alpha n})$, represents the n-photon exchange process followed by the field-free scattering. The other [see Eqs. (3.21) and (3.23)] allows for the correction in which one of the photons is emitted or absorbed in an intermediate state of the scattering process, with the others exchanged prior to the scattering. The possibility of emission or absorption requires that one take a superposition of amplitudes of the form

$$C_{\alpha'\alpha n}\equiv\frac{ie\omega}{2c}\mathbf{a}\cdot\{J_{-(n+1)}(\rho_{\mathbf{p}},\gamma)\mathbf{C}_{\alpha'\alpha}(\mathbf{p}',\mathbf{p}_{n+1})-J_{-(n-1)}(\rho_{\mathbf{p}},\gamma)\mathbf{C}_{\alpha'\alpha}(\mathbf{p}',\mathbf{p}_{n-1})\} \ , \quad (3.30)$$

where $\mathbf{C}_{\alpha'\alpha}$ is the on-shell correction to the Feshbach-Yennie approximation for single-photon bremsstrahlung introduced earlier in Eqs. (2.6) and (2.7), with $\mathbf{p}'$ constrained by the condition $E_{\alpha'}=E_\alpha+n\omega$. (Including only the correction term at this stage avoids double counting, since the Feshbach-Yennie contribution has effectively been accounted for in the leading term.) Explicit calculation (details are omitted here) making use of methods already discussed provides us with the approximation

$$\sigma^{(1,2)}_{\alpha'\alpha}=\sum_{n=-\infty}^{\infty}\int d\mathbf{p}'\delta(E_{\alpha'}-E_\alpha-n\omega)J_{-n}(\rho_{\mathbf{p}},\gamma)\times 2\,\mathrm{Re}\{t(\mathbf{p}',\mathbf{p}_n;E_{\alpha n})C^*_{\alpha'\alpha n}\} \ . \quad (3.31)$$

In summary, we have obtained a low-frequency approximation for the total cross section, allowing for an arbitrary number of photons emitted or absorbed, which is given by Eqs. (3.25), (3.26), and (3.31), and expressed in terms of physical field-free scattering parameters.

IV. SUMMARY

As commonly understood, the validity of a low-frequency approximation for bremsstrahlung is based on the dominance of the contribution to the transition amplitude arising from soft-photon emission or absorption taking place either before or after the collision. Radiative interactions during the collision are generally of higher order in the frequency and cannot be expressed in terms of measurable field-free scattering parameters. The bremsstrahlung process in which the scattering energy lies close to an excitation threshold of the target represents an exception to the rule, and this provides an opportunity to explore and enlarge the domain of validity of the approximation procedure. Such a program takes on added interest now that laser-assisted collisions, and in particular near-threshold excitation processes [6,7], are within reach of experimental study. We have found that the amplitude for emission or absorption of a soft photon during the collision is enhanced near a reaction threshold by the appearance of a near singularity in the matrix element; the integral over the asymptotic domain of projectile coordinates in intermediate states is slowly convergent when the energy carried by the projectile is close to zero. The fact that the dominant contribution to this intermediate-state correction term comes form the asymptotic region of configuration space, where the form of the wave function is known in terms of physical scattering parameters, allows us to evaluate it in a simple and convenient manner. We began, in Sec. II, by considering the spontaneous bremsstrahlung process and derived an expression which, when added to the Feshbach-Yennie low-frequency approximation [1], leads to improved accuracy. In a schematic model adopted here for illustrative purposes, based on an effective-range analysis of the field-free scattering matrix near threshold, the improvement was found to be small. An extension of the approximation procedure to the problem of scattering in a laser field, as described in Sec. III, was based on a variational formulation along with a suitable choice of trial function that takes into account both the strong electron-field and electron-target interactions. The higher-order contribution arising from threshold singularities appears as a variational correction which we found could be expressed in terms of the analogous correction appearing in the low-frequency approximation for spontaneous bremsstrahlung. The expression obtained for the total cross section, summed over all final states of the field, takes on a particularly simple form which, in lowest order, is similar to that which had earlier been postulated [8] as an extension of the Kroll-Watson approximation and is now justified by a method that is more generally applicable. Moreover, the higher-order corrections obtained here should be useful in developing more refined comparisons between theory and experiment.

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APPENDIX A: FESHBACH-YENNIE APPROXIMATION

A straightforward procedure for deriving Eq. (2.4) of the text begins with the identity

$$\langle \mathbf{p}'|[\mathbf{r},(H_0-E_\mathbf{p})]u_{\mathbf{p};sc}^{(+)}\rangle = -\langle \mathbf{p}'|\mathbf{r}V|\mathbf{p}\rangle - \langle \mathbf{p}|V\mathbf{r}|u_{\mathbf{p};sc}^{(+)}\rangle + \omega\langle \mathbf{p}'|\mathbf{r}|u_{\mathbf{p};sc}^{(+)}\rangle \ , \tag{A1}$$

where we used the energy conservation condition $E_{\mathbf{p}'}=E_\mathbf{p}-\omega$ in obtaining the last term. With the commutator evaluated in terms of the momentum operator, the left-hand side of Eq. (A1) becomes $(i\mathbf{p}'/\mu)\langle \mathbf{p}'|u_{\mathbf{p};sc}^{(+)}\rangle$. This may be reduced further by expressing the scattered wave as

$$|u_{\mathbf{p};sc}^{(+)}\rangle = (E_\mathbf{p}+i\eta+\nabla^2/2\mu)^{-1}V|u_\mathbf{p}^{(+)}\rangle \ , \tag{A2}$$

from which we find, using the definition

$$t(\mathbf{p}',\mathbf{p};E_\mathbf{p}) = \langle \mathbf{p}'|V|u_\mathbf{p}^{(+)}\rangle \tag{A3}$$

for the off-shell field-free transition amplitude, that

$$\langle \mathbf{p}'|u_{\mathbf{p};sc}^{(+)}\rangle = \frac{1}{\omega}t(\mathbf{p}',\mathbf{p};E_\mathbf{p}) \ . \tag{A4}$$

With the replacement $\mathbf{r}\to -i\nabla_\mathbf{p}$ in the first two terms on the right in Eq. (A1), we obtain, after gathering results, the relation

$$\frac{i\mathbf{p}'}{\mu\omega}t(\mathbf{p}',\mathbf{p};E_\mathbf{p}) = -i\nabla_{\mathbf{p}'}t(\mathbf{p}',\mathbf{p};E_\mathbf{p}) + \omega\langle \mathbf{p}'|\mathbf{r}|u_{\mathbf{p};sc}^{(+)}\rangle \ . \tag{A5}$$

In a similar way we have

$$-\frac{i\mathbf{p}}{\mu\omega}t(\mathbf{p}',\mathbf{p};E_{\mathbf{p}'}) = -i\nabla_\mathbf{p}t(\mathbf{p}',\mathbf{p};E_{\mathbf{p}'}) + \omega\langle u_{\mathbf{p}';sc}^{(-)}|\mathbf{r}|\mathbf{p}\rangle \ . \tag{A6}$$

Upon addition of these two equations, off-shell contributions to the t matrix cancel. This is most easily demonstrated by expressing the t matrix, in terms of the scalar variables

$$E=p^2/2\mu \ ; \quad \tau=(\mathbf{p}'-\mathbf{p})^2 \ , \quad \xi=E-p^2/2\mu \ , \quad \xi'=E-p'^2/2\mu \ , \tag{A7}$$

as $t(E,\tau,\xi,\xi')$; the physical amplitude is then identified as $t(E,\tau)\equiv t(E,\tau,0,0)$. On the left-hand side of Eq. (A5), we set $\xi=0$ and $\xi'=E_\mathbf{p}-p'^2/2\mu=\omega$ and write

$$t(E_\mathbf{p},\tau,0,\omega) = t(E_\mathbf{p},\tau)+\omega\frac{\partial}{\partial\xi'}t(E_\mathbf{p},\tau,0,\xi')\bigg|_{\xi'=0} + O(\omega^2) \ . \tag{A8}$$

We ignore the second-order correction term. (Note that the dimensionless parameter that appears in this expansion is the product of the frequency and the logarithmic derivative of the t matrix with respect to the off-shell energy parameter. The requirement that this quantity be small does not involve the assumption that the frequency is small compared to the scattering energy.) On the right in Eq. (A5) we have

$$\nabla_{\mathbf{p}'} = (\nabla_{\mathbf{p}'}\tau)\frac{\partial}{\partial\tau} + (\nabla_{\mathbf{p}'}\xi')\frac{\partial}{\partial\xi'} = 2(\mathbf{p}'-\mathbf{p})\frac{\partial}{\partial\tau} - \frac{\mathbf{p}'}{\mu}\frac{\partial}{\partial\xi'} \ . \tag{A9}$$

When similar transformations are carried out in Eq. (A6) and the two equations are added, we arrive at the approximation shown in Eq. (2.4).

In the model calculation described in the text the Feshbach-Yennie approximation was evaluated using a field-free scattering amplitude based on representation (2.10) of the S matrix and containing only s and p waves. The multichannel version of approximation (2.4) then reduces to

$$2(2\pi\mu\omega)^2(\mathbf{m}_{\alpha'\alpha})_{\mathrm{FY}} = \mathbf{p}'_{\alpha'}[h_{\alpha'\alpha;0}(E_\alpha)+3p'_{\alpha'}p_\alpha h_{\alpha'\alpha;1}(E_\alpha)\cos\theta] - \mathbf{p}_\alpha[h_{\alpha'\alpha;0}(E_\alpha-\omega)+3p'_{\alpha'}p_\alpha h_{\alpha'\alpha;1}(E_\alpha-\omega)\cos\theta] - 3(\mathbf{p}'_{\alpha'}-\mathbf{p}_\alpha)[h_{\alpha'\alpha;1}(E_\alpha)-h_{\alpha'\alpha;1}(E_\alpha-\omega)] \ . \tag{A10}$$

APPENDIX B: EFFECTIVE-RANGE APPROXIMATION

In Ref. [11], a convenient parameterization of the reactance matrix was given that correctly displays the threshold singularities in accordance with Eq. (2.13) of the text. Three real parameters are required. A complex scattering length $a_l = A_l - iB_l$ (with B_l positive as demanded by unitarity) is chosen to represent the elastic scattering in channel β, and a parameter c_l is identified with the matrix element $R_{\alpha\alpha;l}$ for elastic scattering in the ground state. The remaining elements of the reactance matrix and the scattering matrix are then determined from Eq. (2.12). One finds that

$$\begin{aligned} R_{\beta\beta;l} &= p_\beta^{2l+1}(B_l c_l - A_l) , \\ R_{\alpha\beta;l} &= R_{\beta\alpha;l} = p_\beta^{l+1/2}(B_l[1+c_l^2])^{1/2} . \end{aligned} \tag{B1}$$

Here β labels the channel in which the target is in its excited state; the excitation energy was taken in our model to be 0.75 a.u. [Factors of $p_\alpha^{l+1/2}$ are treated as constants for simplicity and are absorbed in the definition of the matrix K_l of Eq. (2.13).] In principle, an effective-range expansion of the parameters a_l and c_l could be introduced [11], based on whatever experimental or theoretical information is available concerning the field-free scattering cross section. In this illustrative example we have simply replaced these parameters by constants, chosen somewhat arbitrarily as $c_0 = B_0 = A_0 = 1.0$, with the parameters for the p wave reduced by a factor of 10. The parametrization of the scattering matrix takes the form

$$h_{\alpha\alpha;l} = \left[\frac{2i}{p_\alpha}\right] \frac{-ic_l + p_\beta^{2l+1}(B_l + A_l c_l)}{(i+c_l)d_l} , \tag{B2}$$

$$h_{\alpha\beta;l} = h_{\beta\alpha;l} = \left[\frac{2}{p_\alpha^{l+1/2}}\right] \frac{(B_l[1+c_l^2])^{1/2}}{(i+c_l)d_l} , \tag{B3}$$

and

$$h_{\beta\beta;l} = \frac{2(iA_l + B_l)}{d_l} , \tag{B4}$$

where $d_l = 1 + ip_\beta^{2l+1} a_l$. A factor of -2 was included in Eq. (B3) to correct a typographical error in Ref. [11].

[1] H. Feshbach and D. R. Yennie, Nucl. Phys. **37**, 150 (1962).

[2] F. E. Low, Phys. Rev. **110**, 974 (1958).

[3] N. M. Kroll and K. M. Watson, Phys. Rev. A **8**, 804 (1973).

[4] H. Kruger and C. Jung, Phys. Rev. A **17**, 1706 (1978); M. H. Mittleman, *ibid.* **20**, 1965 (1979).

[5] L. Rosenberg, Phys. Rev. A **23**, 2283 (1981).

[6] B. Wallbank, J. K. Holmes, L. LeBlanc, and A. Weingartshofer, Z. Phys. D **10**, 467 (1988).

[7] N. J. Mason and W. R. Newell, J. Phys. B **20**, L323 (1987).

[8] S. Geltman and A. Maquet, J. Phys. B **22**, L419 (1989).

[9] A. Maquet and J. Cooper, Phys. Rev. A **41**, 1724 (1990).

[10] R. G. Newton, Phys. Rev. **114**, 1611 (1959).

[11] Here we follow the formulation in N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Oxford University Press, London, 1965), pp. 379–386.

[12] F. Zhou and L. Rosenberg, Phys. Rev. A **43**, 1336 (1991).

[13] Approximation (3.3a) is obtained by ignoring the $\mathbf{E}\cdot\mathbf{r}$ interaction in the field gauge. It has been demonstrated [P. Francken, Y. Attaourti, and C. J. Joachain, Phys. Rev. A **38**, 1785 (1988)] that the inclusion of this interaction, in lowest order, will in certain cases lead to a noticeable enhancement of the amplitude for scattering through very small angles. This effect may be accounted for, in the context of the low-frequency approximation, through a modification of the trial function, as discussed in Ref. [12]; it will not be considered further in what follows. The effect on the total cross section is not likely to be significant in the domain of moderate field intensities assumed here.

[14] F. Zhou and L. Rosenberg, Phys. Rev. A **48**, 505 (1993).

[15] The photon energy must be small compared to certain characteristic energies of the system; these are identified in the course of the derivation. One such energy is the separation between discrete levels of the target. Others are defined by expressing the scattering amplitude in terms of certain scalar variables and specifying the range of these variables over which the scattering amplitude changes by an appreciable fraction of itself. [See the remarks following Eq. (A8)]. It is important to note that slow variation of the scattering amplitude with respect to the total energy of the system is not assumed, nor is the kinetic energy of the electron in the channel reached by target excitation required to be large compared to the photon energy.

[16] It is assumed here, for simplicity, that the energy of the incident electron is close to an excitation threshold and only that threshold need be considered in the analysis. Estimates made in Ref. [9] indicate that this assumption is valid, in particular, for the helium–CO_2-laser system studied in the experiment of Ref. [6], where laser intensities of the order of 10^8 W/cm^2 and frequencies of the order of 0.1 eV are involved. In fact, for sufficiently strong fields the electron can absorb enough energy to raise it to still higher thresholds. The present discussion may be generalized to account for multiple thresholds, though the abovementioned estimates indicate that little improvement would thereby be gained for the fields of moderate intensity permitted in the approximation developed here.

[17] The derivation of the sum rule given here makes use of methods very similar to those introduced in Ref. [14] in the analysis of a related problem.