Wave function in the invariant representation and squeezed-state function of the time-dependent harmonic oscillator

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The two quantum invariant operators are found from the time-dependent Hamiltonian of the harmonic oscillator with an auxiliary condition. The solution of the Schrödinger equation for the system, such as the eigenfunctions, eigenvalues, and minimum uncertainty, is derived by utilizing these invariant operators. The coherent states of this system are not the squeezed states, and the eigenfunction of the invariant operator is not the eigenfunction of the Hamiltonian of the system unless it is in the invariant representation. The squeezing function, which is an eigenfunction of the Hamiltonian of the system in the invariant representation and which also gives the minimum uncertainty, is obtained by a set of unitary transformed operators, i.e., squeezing operators.

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I. INTRODUCTION

Over the past several decades, much attention has been paid to obtaining the exact solution to the Schrödinger equation for the time-dependent system [1]. Among several techniques to treat these systems, the quantum invariant operator method [2] and propagator method [3] are particularly well known. Since Lewis and Riesenfeld [4] have derived the relation between the dynamical invariant and solution of the Schrödinger equation for the time-dependent oscillators, its generalization offers wide applications to various fields [5].

After Glauber's investigation of photon statistics in radiation fields [6], coherent states and squeezed states have become important in many areas [7], especially quantum optics [8–10]. Walls [11] carried out pioneering work on squeezed states of light. Various authors have obtained the coherent states for the damped [12] or damped driven harmonic oscillator [13], harmonic oscillator with time-dependent frequency [14], and squeezed states for general symmetric systems [15].

Recently, through the path-integral method, we have obtained wave functions, energy expectation values, the uncertainty relation and transition amplitudes for a quantum damped driven harmonic oscillator [16], coupled forced harmonic oscillator [17] and forced time-dependent harmonic oscillator [18], and coherent states for the damped harmonic oscillator [19] and harmonic oscillator with time-dependent frequency [20]. We have also derived the relation between the wave function and a dynamical invariant, which determines whether or not the system is bound for a time-dependent quadratic Hamiltonian [21]. In this present paper, a squeezed state, which is caused by the asymmetry in the uncertainties of the two observables of a system, is obtained using a set of unitary transformed operators or squeezing operators in the invariant representation.

In this paper, in Sec. II we derive the two quantum (creation and annihilation) invariant operators for the time-dependent harmonic oscillator. Then in Sec. III we solve the Schrödinger equation in the invariant representation with the auxiliary condition as the classical solution. In Sec. IV we show the squeezing function of the system using the unitary transformed creation and annihilation operators and Sec. V summarizes our overall results.

II. CLASSICAL TREATMENT APPROACH

The Hamiltonian of the harmonic-oscillator system is given by

$$H=\frac{p^2}{2M}+\tfrac{1}{2}M\omega^2(t)q^2 \ , \tag{1}$$

where q and p are canonical variables and $\omega^2(t)$ is a real positive function, and the classical equation of motion is given as

$$\ddot{q}+\omega(t)^2q=0 \ . \tag{2}$$

Although the differential equation (2) does not have an easy solution, it can be expressed in the form

$$q=\rho(t)e^{i\gamma(t)} \ , \tag{3}$$

where the functions $\rho(t)$ and $\gamma(t)$ are determinable from Eq. (2). These functions are real and depend only on time, and substitution of Eq. (3) into (2) yields two equations from the real and imaginary parts as

$$\ddot{\rho}-\rho\dot{\gamma}^2+\omega(t)^2\rho=0 , \tag{4}$$

$$\rho\ddot{\gamma}+2\dot{\rho}\dot{\gamma}=0 . \tag{5}$$

One invariant quantity can be found from Eq. (5) in the form

$$\Omega=M\rho^2\dot{\gamma} , \tag{6}$$

with an auxiliary condition given by the classical solution. Substituting Eq. (6) into (4), it becomes

$$\ddot{\rho}-\frac{\Omega^2}{M^2\rho^3}+\omega(t)^2\rho=0 . \tag{7}$$

Another time invariant quantity can be evaluated from

$$\frac{dI}{dt}=\frac{\partial I}{\partial t}+\frac{\partial I}{\partial x}\frac{\partial H}{\partial p}-\frac{\partial I}{\partial p}\frac{\partial H}{\partial x}=0 . \tag{8}$$

From Eqs. (1) and (8), we obtain the classical invariant quantity as

$$I=\frac{1}{2}\left[\frac{\Omega^2}{\rho^2}q^2+(\rho p-M\dot{\rho}q)^2\right] . \tag{9}$$

These invariant quantities Ω and I are the measure of the bound system. If Ω is real, Eq. (9) is an elliptic equation in phase space. Thus, as the values of q and p in the system are limited in some region, it is a bound system. However, if Ω is imaginary, Eq. (9) is a hyperbola in phase space and q can take any value in the space, hence making the system unbound.

III. QUANTUM-MECHANICAL TREATMENT

To treat the system quantum mechanically, by replacing the canonical variables of the classical system by quantum operators in the Hamiltonian, one gets the quantum invariant quantity. In order to obtain eigenfunctions and eigenvalues of the invariant operator, we re-express it in terms of creation and annihilation operators. To do this, we define the time-dependent canonical annihilation and creation operators a and $a^\dagger$ by the relations

$$\begin{aligned} a&=\frac{1}{\sqrt{2M\hbar\Omega}}\left[\frac{\Omega}{\rho}q+i(\rho p-M\dot{\rho}q)\right]\\ &=\frac{1}{\sqrt{2M\hbar\dot{\gamma}}}\left[M\dot{\gamma}\left[1-\frac{\dot{\rho}}{\rho\dot{\gamma}}\right]q+ip\right] , \end{aligned}\tag{10}$$

$$\begin{aligned} a^\dagger&=\frac{1}{\sqrt{2M\hbar\Omega}}\left[\frac{\Omega}{\rho}q-i(\rho p-M\dot{\rho}q)\right]\\ &=\frac{1}{\sqrt{2M\hbar\dot{\gamma}}}\left[M\dot{\gamma}\left[1+\frac{\dot{\rho}}{\rho\dot{\gamma}}\right]q-ip\right] , \end{aligned}\tag{11}$$

with the auxiliary condition of Eqs. (4) and (5). These operators satisfy the canonical commutation rule

$$[a,a^\dagger]=1 \tag{12}$$

and obey the usual properties of creation and annihilation operators. The invariant operator Eq. (9) can be rewritten in terms of a and $a^\dagger$ as

$$I=\hbar\Omega(a^\dagger a+\tfrac{1}{2}) , \tag{13}$$

and the eigenstates of the invariant operator have the same form as the normalized eigenstates $|n\rangle$ of $a^\dagger a$,

$$a^\dagger a|n\rangle=n|n\rangle , \quad n=0,1,2,\ldots . \tag{14}$$

The eigenvalue spectrum of I is obtained by

$$\lambda=\Omega\hbar(n+\tfrac{1}{2}) , \quad n=0,1,2,\ldots . \tag{15}$$

The ground state, denoted by u_0, beyond which the lowering (i.e., annihilation) ends, must satisfy the condition

$$au_0=0 . \tag{16}$$

In q space, Eq. (16) is written as

$$\frac{\partial u_0}{\partial q}+\frac{M\dot{\gamma}}{\hbar}(1-\dot{\rho}\rho\dot{\gamma})q=0 , \tag{17}$$

whose normalized solution is

$$u_0=\left[\frac{M\dot{\gamma}}{\pi\hbar}\right]^{1/4}\exp\left[-\frac{M\dot{\gamma}}{2\hbar}(1-i\dot{\rho}\rho\dot{\gamma})q^2\right] . \tag{18}$$

The excited eigenfunctions are given as

$$u_n=\frac{1}{\sqrt{n!}}(a^\dagger)^n u_0 , \tag{19}$$

where their explicit forms can be found by the solution of the differential equation

$$e^{\xi q^2/2}\frac{d^n}{dq^n}e^{-\xi q^2/2}=\left[\frac{d}{dq}-\xi q\right]^n , \tag{20}$$

which is a Hermite polynomial of order n:

$$H_n(\sqrt{\xi}q)=(-\sqrt{\xi})^{-n}e^{\xi q^2}\frac{d^n}{dq^n}(e^{-\xi q^2}) . \tag{21}$$

Hence, the normalized eigenfunction of an excited state n can be expressed as

$$u_n(q,t)=\left[\frac{1}{2^n n!}\right]^{1/2}\left[\frac{M\dot{\gamma}}{\pi\hbar}\right]^{1/4}\exp\left[-\frac{M\dot{\gamma}}{2\hbar}\left[1-i\frac{\rho}{\dot{\gamma}\rho}\right]q^2\right]H_n\left[\left[\frac{M\dot{\gamma}}{\hbar}q\right]^{1/2}\right] . \tag{22}$$

This is an eigenfunction of the invariant operator obeying the auxiliary conditions of Eqs. (4) and (5). However, we would like to point out that the eigenfunction of the invariant operator I [Eq. (22)] is not the wave function of the system, but the wave function of the system of which the Schrödinger equation is given as

$$i\hbar\frac{\partial}{\partial t}\psi=-\frac{\hbar^2}{2M}\frac{\partial^2}{\partial q^2}\psi+\frac{M}{2}\omega(t)^2q^2\psi \ . \tag{23}$$

To obtain the eigenfunction of the system, let us assume that the form of the wave function is given as

$$\psi_n(q,t)=e^{i\alpha_n}u_n(q,t) \ . \tag{24}$$

Substitution of Eq. (24) into (23) gives

$$\alpha_n=-(\tfrac{1}{2}+n)\gamma \ , \tag{25}$$

so that we obtain the exact wave function of the nth state of the system as

$$\psi_n(q,t)=\left[\frac{1}{2^n n!}\right]^{1/2}\left[\frac{M\dot\gamma}{\pi\hbar}\right]^{1/4}e^{-(1/2+n)\gamma}\exp\left[-\frac{M\dot\gamma}{2\hbar}\left[1-i\frac{\dot\rho}{\rho\dot\gamma}\right]q^2\right]H_n\left[\left[\frac{M\dot\gamma}{\hbar}q\right]^{1/2}\right] . \tag{26}$$

The propagator of the system, with the help of Mehler's formula [22],

$$\sqrt{1-z^2}\exp\left[\frac{2XYz-X^2-Y^2}{1-z^2}\right]=e^{-X^2-Y^2}\sum_{n=0}^{\infty}\frac{z^n}{2^n n!}H_n(X)H_n(Y) \ , \tag{27}$$

is expressed as

$$K(q,t;q',t')=\left[\frac{M\sqrt{\dot\gamma\dot\gamma'}}{2\pi i\hbar\sin(\gamma-\gamma')}\right]^{1/2} \times\exp\left\{\frac{iM}{2\hbar}\left[\left[\frac{\dot\rho}{\rho}q^2-\frac{\dot\rho'}{\rho'}q'^2\right]+\frac{1}{\sin(\gamma-\gamma')}[(\dot\gamma q^2+\dot\gamma' q'^2)\cos(\gamma-\gamma')-2\sqrt{\dot\gamma\dot\gamma'}qq']\right]\right\} , \tag{28}$$

where $\rho'=\rho(t')$ and $\gamma'=\gamma(t')$. Equations (2) and (28) are the same as our previous results using path-integral methods [8].

IV. SQUEEZING OPERATORS

In order to obtain the uncertainty relations, we express q, p, q^2, and p^2 in terms a and $a^\dagger$ [Eqs. (10) and (11)]. Using the definition of the uncertainty product given as

$$(\Delta q\Delta p)_{m,n}=[|\langle m|q^2|n\rangle-|\langle m|q|n\rangle^2| \times|\langle m|p^2|n\rangle-|\langle m|q|n\rangle^2|]^{1/2} \ , \tag{29}$$

one can easily get the uncertainty relation for various states [23]. We note that the diagonal elements in the uncertainty relation are given by

$$(\Delta q\Delta p)_{n,n}=\frac{\hbar}{2}(2n+1)\left[1+\frac{\dot\rho^2}{\dot\gamma\rho^2}\right]^{1/2} . \tag{30}$$

Since the minimum uncertainty of Eq. (29) is larger than $\hbar/2$, the coherent states of the system are not minimum uncertainty states. To find the minimum uncertainty state, we may express the Hamiltonian of Eq. (1) in terms of a and $a^\dagger$ as

$$H=\frac{\hbar}{4\Omega}[\alpha^2+\alpha^* a^{\dagger 2}+\beta\{a,a^\dagger\}] \ , \tag{31}$$

where

$$\{a,a^\dagger\}=aa^\dagger+a^\dagger a \ , \tag{32}$$

$$\alpha=M[\dot\rho^2+\omega(t)^2\rho^2-\rho^2\dot\gamma^2]-2iM\rho\dot\rho\dot\gamma \ , \tag{33}$$

and

$$\beta=M[\dot\rho^2+\omega(t)^2\rho^2+\rho^2\dot\gamma^2] \ . \tag{34}$$

To diagonalize the Hamiltonian, we introduce new creation and annihilation operators [11,23], i.e., the squeezing operators

$$b=\mu a+\nu a^\dagger \ , \quad b^\dagger=\mu^* a^\dagger+\nu^* a \ , \tag{35}$$

where

$$|\mu|^2-|\nu|^2=1 \ . \tag{36}$$

These have the same properties of usual creation and annihilation operators. If b and $b^\dagger$ obey the relation

$$[H,b]=-kb \ , \tag{37}$$

the Hamiltonian of Eq. (31) can be diagonalized in some space. The transformation constants μ and ν satisfying Eqs. (36) and (37) are

$$\mu=\frac{\alpha}{\sqrt{2k(\beta-k)}} \ , \tag{38}$$

$$\nu=\frac{\beta-k}{\sqrt{2k(\beta-k)}} \ , \tag{39}$$

$$k=2\omega(t)\Omega \ . \tag{40}$$

In this case, the Hamiltonian of Eq. (31) is

$$H=\hbar\omega(t)(b^\dagger b+\tfrac{1}{2}) \ . \tag{41}$$

The eigenvalues of the Hamiltonian are

$$\lambda_n = \hbar\omega(t)(n+\tfrac{1}{2}) , \quad n=0,1,2,\ldots , \tag{42}$$

whereas the ground-state wave function is obtained by

$$b\phi_0 = 0 . \tag{43}$$

Using Eqs. (35), (38), and (39), we can write Eq. (43) as

$$\frac{1}{\sqrt{2\hbar\omega(t)M}} \exp\left[i\tan^{-1}\frac{-\omega(t)\rho+\dfrac{\Omega}{M\rho}}{\dot{\rho}} \right] \times [M\omega(t)q+ip]\phi_0 = 0 , \tag{44}$$

whose normalized solution is

$$\phi_0 = \left[\frac{M\dot{\gamma}}{\pi\hbar}\right]^{1/4} e^{-[M\omega(t)/2\hbar]q^2} . \tag{45}$$

The normalized excited eigenstates of the Hamiltonian are

$$\begin{aligned}\phi_n &= \frac{1}{\sqrt{n!}}(b^\dagger)^n u_0 \\ &= \left[\frac{1}{2^n n!}\right]^{1/2} \left[\frac{M\omega(t)}{\pi\hbar}\right]^{1/4} e^{-[M\omega(t)/2\hbar]q^2} \\ &\quad \times H_n\left[\left[\frac{M\omega(t)}{\hbar}\right]^{1/2} q\right] .\end{aligned} \tag{46}$$

This ϕ_n is not a solution of the Schrödinger equation given by Eq. (23), but an important function to describe the system. Equation (46) along with (29) yields the uncertainty product as

$$(\Delta q \Delta p)_{n,n} = \hbar(n+\tfrac{1}{2}) , \tag{47}$$

and since the minimum uncertainty of Eq. (47) is $\hbar/2$, the minimum uncertainty state is an eigenstate of the Hamiltonian of the system. That is, the eigen-coherent-state of the Hamiltonian of the system is the squeezed state of the system.

All of our above results are for a bound system. If $\omega^2(t)$ in the Hamiltonian of Eq. (1) were negative, then the corresponding system would be unbound. In this case, since the creation and annihilation operators do not transform to the different set of operators, the system gives no uncertainty minimum.

V. SUMMARY

Using the invariant operators, we have obtained the quantum-mechanical solution of the time-dependent harmonic oscillator that is always a bound system. In these calculations we have found that the operator method is much simpler for deriving the quantum-mechanical quantities than other methods. We have also found that the eigenfunctions of the invariant operators are not eigenfunctions of the Hamiltonian of the system unless they are in the invariant representation. The minimum uncertainty is obtained through this solution of the Schrödinger equation. A task for future work will be to extend this theory to the investigation of other properties of the time-dependent harmonic oscillator for both bound and unbound systems.

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