Photon production by the dynamical Casimir effect

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In this paper we present some calculations regarding the average number of photons produced in the dynamical Casimir effect for the ideal case of two perfectly conducting uncharged parallel plates, using the zero-point energy summation method. We show that it is possible to create intense photon radiation when the two plates are modulated periodically.

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I. INTRODUCTION

The idea that the "shape" of the vacuum can be changed, for example, by inserting two perfectly conducting plates, to produce observable effects, led Casimir in 1948 to formulate what is commonly referred to as the Casimir effect [1,2]. Since then, the Casimir effect has been extensively studied and exploited in a variety of situations [3–7]. Examples include the Casimir effect for plane dielectric surfaces [8,9], in a given cavity [10], between two polarizable particles [11], etc. The Casimir effect has been generally considered as a manifestation of the change in the zero-point electromagnetic energy due to the presence of matter, even if it can also be derived very elegantly by source theory with no reference to the zero-point field [12,13].

Regardless of the way Casimir forces are calculated, they are obtained either by holding the geometry fixed or by varying the geometry only quasistatically. The dynamical Casimir effect occurs when the geometry of the system varies more quickly in time so that the vacuum is perturbed and emission of photons becomes possible. Work by other authors regarding photon production in the presence of moving boundaries can be found in Refs. [14–17].

Recently Schwinger [18–20] has proposed a more general procedure to evaluate the Casimir effect within source theory: the real part of the action gives the Casimir energy, which is the only contribution we have in the static effect; instead the imaginary part describes the photon production, which characterizes the dynamical situation. In this paper we present a calculation of the average number of photons produced in the dynamical Casimir effect for the ideal case of two perfectly conducting plates, using the zero-point energy summation method. More explicitly, we consider two examples of plate modulation: a pulse in time and a square-well periodic potential. We demonstrate that for the case of a periodic potential intense photon radiation may occur, mainly as visible light. In all cases, the calculations are expected to be adequate when the time scales of the frequency modulation of the photons are long on the time scale of the photon frequencies themselves, i.e., in the adiabatic approximation.

II. FORWARD AND BACKWARD MOTION OF AN ELECTROMAGNETIC MODE

The static Casimir energy for two perfectly conducting plates separated by a distance d ($L_x = L_y = L$ and $L_z = d$, where $L >> d$) is given by the difference between the zero-point energy when the plates are placed at a distance d and when the distance between them is infinite

$$U(d) = E_0(d) - E_0(\infty) = -\frac{\pi^2 \hbar c L^2}{720 d^3} . \tag{1}$$

Now suppose that the distance d varies as a function of time $d(t)$, as in the example shown in Fig. 1. We assume a constant initial value d_i and after a time ΔT the modulation stops and d has the value d_f.

In this case the frequency ω_k is a function of the time $\omega_k(t)$

$$\omega_k(t) = c\left[\left[\frac{\pi l}{L}\right]^2 + \left[\frac{\pi m}{L}\right]^2 + \left[\frac{\pi n}{d(t)}\right]^2\right]^{1/2} , \tag{2}$$

where $l, m, n = 0, 1, 2, \ldots$, with the restriction that only one integer at a time can be zero.

The Hamiltonian for one single mode is (see the Appendix for more details)

$$H(t) = \tfrac{1}{2}[P^2 + \omega_k^2(t) Q^2] . \tag{3}$$

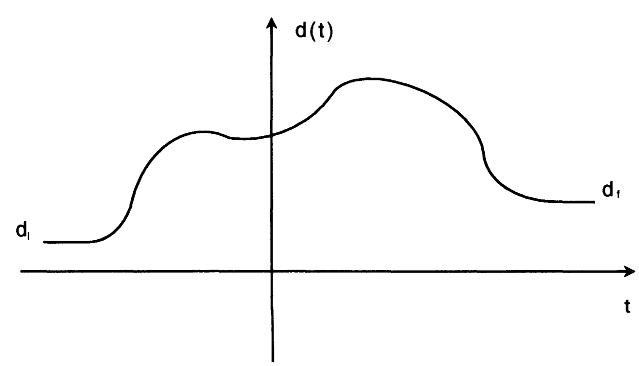

FIG. 1. The distance between the two perfectly conducting plates varies as a function of time. Before the modulation the distance is d_i and after the modulation stops, the distance is d_f.

If we work in the Heisenberg picture and introduce the creation and annihilation operators

$$a^{\dagger}(t)=\left[\frac{\omega_k(t)}{2\hbar}\right]^{1/2}Q(t)-i\left[\frac{1}{2\omega_k(t)\hbar}\right]^{1/2}P(t)\ , \tag{4a}$$

$$a(t)=\left[\frac{\omega_k(t)}{2\hbar}\right]^{1/2}Q(t)+i\left[\frac{1}{2\omega_k(t)\hbar}\right]^{1/2}P(t)\ , \tag{4b}$$

with

$$[a,a^{\dagger}]=1\ , \tag{4c}$$

Eq. (3) becomes

$$H(t)=\hbar\omega_k(t)[a^{\dagger}(t)a(t)+\tfrac{1}{2}]\ . \tag{5}$$

Before and after the modulation, the Hamiltonian $H(t)$ is the usual Hamiltonian for a simple harmonic oscillator. Suppose that the modulation is on between $-T<t<T$. For $t_1<-T$

$$H_{\alpha}(t_1)=\hbar\omega_k(t_1)[a_{\alpha}^{\dagger}(t_1)a_{\alpha}(t_1)+\tfrac{1}{2}]\ , \tag{6a}$$

with

$$\omega_k(t_1)=\omega_i=c\left[\left[\frac{\pi l}{L}\right]^2+\left[\frac{\pi m}{L}\right]^2+\left[\frac{\pi n}{d_i}\right]^2\right]^{1/2}\ , \tag{6b}$$

and for $t_2>T$

$$H(t_2)=\hbar\omega_k(t_2)[a_{\alpha}^{\dagger}(t_2)a_{\alpha}(t_2)+\tfrac{1}{2}]\ , \tag{6c}$$

with

$$\omega_k(t_2)=\omega_f=c\left[\left[\frac{\pi l}{L}\right]^2+\left[\frac{\pi m}{L}\right]^2+\left[\frac{\pi n}{d_f}\right]^2\right]^{1/2}\ . \tag{6d}$$

The mean number of Casimir photons produced for a given electromagnetic mode is

$$N(\omega_f)=\langle 0,t_1|a^{\dagger}(t_2)a(t_2)|0,t_1\rangle\ . \tag{7}$$

The total number is obtained by taking the sum over all the modes

$$N=\sum_{\mathbf{k}\lambda}N_{\mathbf{k}\lambda}(\omega_f)\ , \tag{8}$$

where $\lambda=1,2$ is the polarization index. The corresponding energy is

$$\Delta E=\sum_{\mathbf{k}\lambda}\hbar\omega_f N_{\mathbf{k}\lambda}(\omega_f)\ . \tag{9}$$

We want to find an expression for $N_{\mathbf{k}\lambda}$. If we introduce the complex quantity $q_{\mathbf{k}\lambda}$, such that

$$Q_{\mathbf{k}\lambda}(t)=q_{\mathbf{k}\lambda}+q_{\mathbf{k}\lambda}^{*}(t)\ , \tag{10a}$$

it is possible to show that $q_{\mathbf{k}\lambda}(t)$ satisfies the equation (see, for example, Refs. [21,22])

$$\frac{d^2}{dt^2}q_{\mathbf{k}\lambda}(t)+\omega_k^2(t)q_{\mathbf{k}\lambda}(t)=0\ . \tag{10b}$$

Equation (10b) is formally identical to a one-dimensional Schrödinger equation, where t replaces the spatial coordinate x.

Using a formalism originally due to Brown and Carson [23], we can determine $N_{\mathbf{k}\lambda}(\omega_f)$ in terms of the scattering solutions of Eq. (10b). For $t<-T$ and $t>T$, Eq. (10b) becomes

$$\frac{d^2}{dt^2}q_{\mathbf{k}\lambda}(t)+\omega_{i,f}^2 q_{\mathbf{k}\lambda}(t)=0\ . \tag{11}$$

Suppose for simplicity $\omega_i=\omega_f=\omega$. Equation (11) admits solutions of the type

$$\phi(t)=e^{-i\omega t}\ , \tag{12a}$$

which represent a signal of frequency ω moving forward in time, or

$$\phi^{*}(t)=e^{+i\omega t}\ , \tag{12b}$$

which represents a signal moving backward in time. Equations (10b) and (11) have the same asymptotic behavior. We can write two asymptotic solutions to Eq. (10b).

One solution $q_f(t)$ (omitting the subscript $\mathbf{k}\lambda$ for simplicity) gives a photon moving forward in time, which has some probability to be scattered backward in time (see Fig. 2)

$$q_f(t)=\begin{cases}e^{-i\omega t}+R_b(\omega)e^{+i\omega t}\ , & t\to-\infty\\ T_f(\omega)e^{-i\omega t}\ , & t\to+\infty\ .\end{cases} \tag{13}$$

The other solution $q_b(t)$ gives a signal moving backward in time, which has some probability to be scattered forward in time due to the modulation (see Fig. 3)

$$q_b(t)=\begin{cases}T_b(\omega)e^{+i\omega t}\ , & t\to-\infty\\ e^{+i\omega t}+R_f(\omega)e^{-i\omega t}\ , & t\to+\infty\ .\end{cases} \tag{14}$$

Therefore a scattering matrix can be defined

$$S=\begin{bmatrix}R_b & T_f\\ T_b & R_f\end{bmatrix}\ . \tag{15}$$

The elements of the matrix are not independent because the Wronskian function of any pair of solutions to Eq. (10b) is a constant

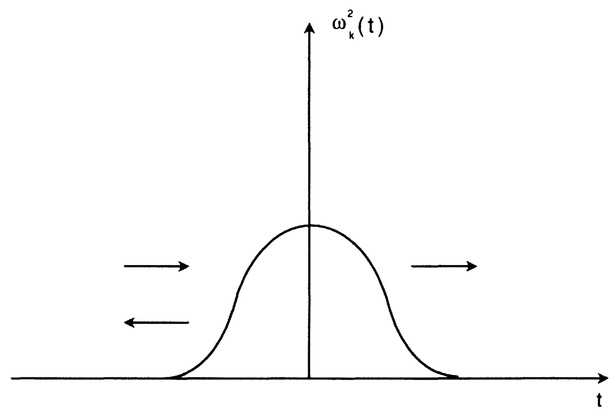

FIG. 2. Photon moving forward in time incident over the potential $\omega_k^2(t)$.

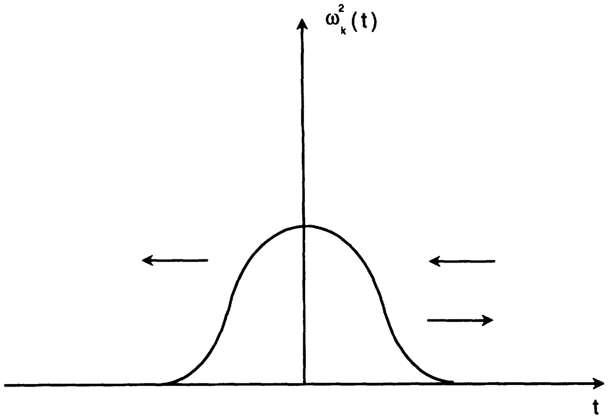

FIG. 3. Photon moving backward in time incident over the potential $\omega_k^2(t)$.

$$W[\psi_1,\psi_2]=\psi_1\frac{d\psi_2}{dt}-\frac{d\psi_1}{dt}\psi_2=\text{const} . \tag{16}$$

Computing the following Wronskian functions at $t\to-\infty$ and $+\infty$ and equating the two asymptotic values leads to the conditions

$$W[q_f^*,q_f]1-|R_b|^2=|T_f|^2 , \tag{17a}$$

$$W[q_b^*,q_b]|T_b|^2=1-|R_f|^2 . \tag{17b}$$

S is a symmetrical unitary matrix and if $\omega_k^2(t)$ is an even function of time, then $R_b=R_f$. In general $|R_b|=|R_f|$.

Let us consider a signal moving backward in time, which means that we consider the boundary condition given by Eq. (14). We can express the mean number of photons produced, in terms of the probability that a signal moving backward in time is scattered forward in time, as follows. If we calculate the constant Wronskian function $[Q,q_b]$, where Q is given by Eqs. (4), for $t_1<-T$ and $t_2>T$ and equate the results, we obtain an expression for $a(t_2)$ and $a^\dagger(t_2)$ as a function of the annihilation and creation operators before the modulation

$$a(t_2)=\frac{T_b e^{i\omega t_1}a(t_1)+T_b^* R_f e^{-i\omega t_1}a^\dagger(t_1)}{e^{i\omega t_2}(1-|R_f|^2)} . \tag{18}$$

$a^\dagger(t_2)$ is obtained in a similar manner. Therefore the mean number of photons produced after the modulation for a given mode

$$N_{\mathbf{k}\lambda}=\langle 0,t_1|a_{\mathbf{k}\lambda}^\dagger(t_2)a_{\mathbf{k}\lambda}(t_2)|0,t_1\rangle$$
$$=\frac{|R_{f\mathbf{k}\lambda}(\omega)|^2}{[1-|R_{f\mathbf{k}\lambda}(\omega)|^2]}=\frac{|R_{b\mathbf{k}\lambda}(\omega)|^2}{[1-|R_{b\mathbf{k}\lambda}(\omega)|^2]} . \tag{19}$$

The same result is obtained considering the boundary condition given by Eq. (13).

Equation (19) is formally identical to the result that has been obtained using a Bogolubov transformation and imposing the Feynman-Stückelberg boundary condition [24,25] for the free Green's function [26–28].

III. PHOTON PRODUCTION FOR A PULSE IN TIME

As a first example we consider the case in which the distance $d(t)$ between the two plates varies as (see Fig. 4)

$$\frac{1}{d^2(t)}=\frac{1}{a^2}\left[1+\frac{b^2}{\cosh^2(\pi t/\tau)}\right] , \tag{20}$$

where τ describes the pulse duration and b determines the modulation strength. Therefore

$$\omega_k^2(t)=c^2\left[\left[\frac{\pi l}{L}\right]^2+\left[\frac{\pi m}{L}\right]^2+\left[\frac{\pi n}{a}\right]^2+\left[\frac{\pi n b}{a\cosh(\pi t/\tau)}\right]^2\right] \tag{21a}$$
$$=\omega_k^2+\frac{\Omega_n^2}{\cosh^2(\pi t/\tau)} ,$$

where

$$\omega_k^2=c^2[(\pi l/L)^2+(\pi m/L)^2+(\pi n/a)^2] \tag{21b}$$

and

$$\Omega_n^2=c^2(\pi n b/a)^2 . \tag{21c}$$

Equation (10b) then becomes

$$\left[\frac{d^2}{dt^2}+\omega_k^2+\frac{\Omega_n^2}{\cosh^2(\pi t/\tau)}\right]q_a(t)=0 . \tag{22}$$

For $n=0$, Eq. (22) reduces to a free equation for the corresponding oscillator mode and the reflection probability coefficient is zero. For $n\neq0$ ($\Omega_n\neq0$), the reflection probability is given by [29]

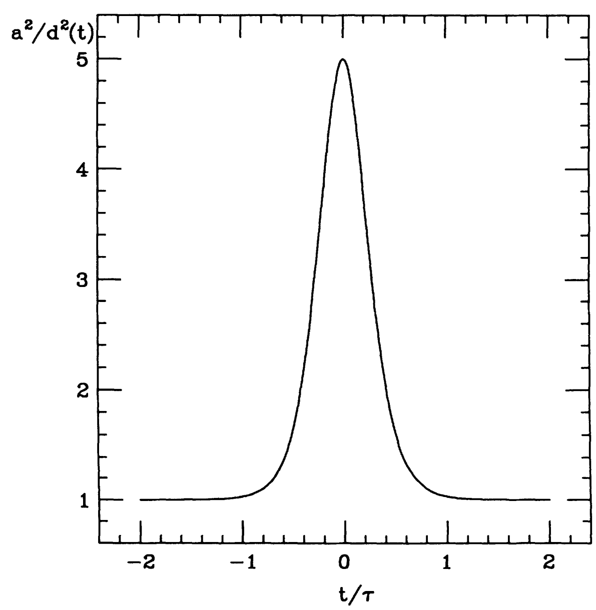

FIG. 4. d as a function of time for the modulation pulse chosen for illustration in Sec. III.

$$|R_{fn}(\omega_k)|^2=\frac{\cos^2\theta_n}{\sinh^2(\omega_k\tau)+\cos^2\theta_n}\ , \tag{23a}$$

with

$$\theta_n=\sqrt{(\pi/2)^2+\Omega_n^2\tau^2}\ . \tag{23b}$$

The mean number of photons produced for a given mode is

$$N=\frac{|R_{fn}(\omega_k)|^2}{[1-|R_{fn}(\omega_k)|^2]}=\frac{\cos^2(\theta_n)}{\sinh^2(\omega_k\tau)}\ . \tag{24a}$$

We notice that in the limit $\omega_k\to 0$ ($V\to\infty$), $N\to\infty$; when $\omega_k\to\infty$, $N\to 0$.

The dynamical Casimir energy is

$$\Delta E=\frac{L^2}{(2\pi)^2}2\hbar c\sum_{n=1}^{\infty}\int d^2k_{\parallel}\sqrt{k_{\parallel}^2+(\pi n/a)^2}\frac{\cos^2\theta_n}{\sinh^2(\omega_k\tau)}\ , \tag{24b}$$

with $k_{\parallel}=\sqrt{(\pi l/L)^2+(\pi m/L)^2}$. If we make the substitution $x^2=(k_{\parallel}a/\pi)^2+n^2$, the Casimir energy is given by

$$\Delta E=\pi^2\frac{L^2\hbar c}{a^3}\left[\sum_{n=1}^{\infty}\cos^2\theta_n\int_{n^2}^{\infty}dx\frac{x^2}{\sinh^2(yx)}\right]\ , \tag{25}$$

where $y=(\pi c\tau/a)$. In experimental situations, y is much larger than one; therefore the integral in Eq. (25) can be written as

$$\begin{aligned}f(n,y)&=4\int_{n^2}^{\infty}dx\,x^2\exp(-2xy)\\&=\exp(-2yn^2)\frac{2yn^2+1+2y^{2n^4}}{y^3}\end{aligned} \tag{26}$$

and we can consider just the first term in the series

$$\Delta E\sim\pi^2\frac{L^2\hbar c}{a^3}\cos^2\theta_1\exp(-2y)\frac{2y^2+2y+1}{y^3}\ , \tag{27}$$

which is of order zero for $y\gg 1$. Therefore the pair production is practically zero in any real situation.

IV. PHOTON PRODUCTION IN A TIME-PERIODIC POTENTIAL

As a second example we consider the case in which the distance between the plates varies periodically as shown in Fig. 5, where we have a succession of period barriers with period $T=2T_1+2T_2$. In a given period $T,d(t)$ varies as follows: for $-T_2-T_1<t<-T_2$ and $T_2<t<T_1+T_2$

$$\frac{1}{d^2(t)}=\frac{1}{a^2}\ , \tag{28a}$$

and for $-T_2<t<T_2$

$$\frac{1}{d^2(t)}=\frac{1}{a^2}(1+b^2)\ , \tag{28b}$$

where b is the modulation strength. Therefore in the valleys ($T_2-T<t-rT<-T_2$)

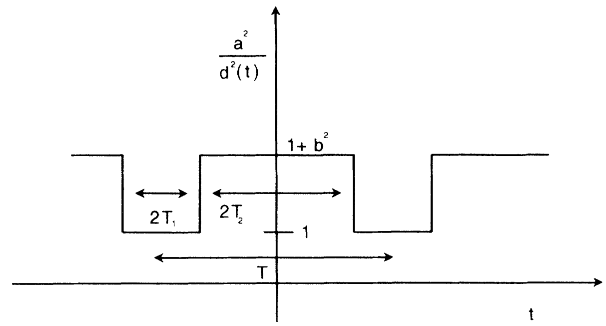

FIG. 5. d as a function of time for the periodic potential chosen for illustration in Sec. IV. The period is $T=2T_1+2T_2$.

$$\left[\frac{d^2}{dt^2}+\omega_k^2\right]q(t)=0\ , \tag{29a}$$

where ω_k is given by Eq. (21b), and in the hills

$$\left[\frac{d^2}{dt^2}+\omega_{k2}^2\right]q(t)=0\ , \tag{29b}$$

with $\omega_{k2}=\sqrt{\omega_k^2+\Omega_n^2}$ and Ω_n given by Eq. (21c).

We wish to determine the mean number of photons produced by the modulation in a period $\Delta t=rT$, where r is a large integer. In the valleys [30]

$$q_r=A_re^{i\omega_k(t-rT)}+B_re^{-i\omega_k(t-rT)}\ . \tag{30}$$

The coefficients A_r,B_r, belonging to successive values of r, can be related by a matrix P, obtained by imposing the continuity for q and its derivative at $t=(-T_2+rT)$ and (T_2+rT). Noting that the centers of the peaks have coordinates $t=rT$,

$$\begin{bmatrix}A_{r+1}\\B_{r+1}\end{bmatrix}=P\begin{bmatrix}A_r\\B_r\end{bmatrix}\ , \tag{31a}$$

with

$$P=\begin{bmatrix}(\alpha_1-i\beta_1)e^{i\omega T} & -i\beta_2e^{i\omega T}\\ i\beta_2e^{-i\omega T} & (\alpha_1+i\beta_1)e^{-i\omega T}\end{bmatrix}\ , \tag{31b}$$

where

$$\begin{aligned}\alpha_1(k,a,b)&=\cos(2\omega_2T_2)\cos(2\omega T_2)\\&\quad+\frac{\epsilon}{2}\sin(2\omega_2T_2)\sin(2\omega T_2)\ ,\end{aligned} \tag{31c}$$

$$\begin{aligned}\beta_1(k,a,b)&=\sin(2\omega T_2)\cos(2\omega_2T_2)\\&\quad-\frac{\epsilon}{2}\sin(2\omega_2T_2)\cos(2\omega T_2)\ ,\end{aligned} \tag{31d}$$

$$\beta_2(k,a,b)=\frac{\eta}{2}\sin(2\omega_2T_2)\ , \tag{31e}$$

and

$$\epsilon=(\omega_2/\omega)+(\omega/\omega_2)\ , \tag{31f}$$

$$\eta=(\omega/\omega_2)-(\omega_2/\omega)\ , \tag{31g}$$

where α_1,β_1,β_2 satisfy the condition

$$\alpha_1^2+\beta_1^2-\beta_2^2=1 . \tag{31h}$$

By iteration we have

$$\begin{bmatrix} A_r \\ B_r \end{bmatrix} = P^r \begin{bmatrix} A_0 \\ B_0 \end{bmatrix} \tag{32}$$

with the condition that for $r \to \pm\infty$ the limit of P^r should exist. If we consider the eigenvalue problem for P, we have

$$P = UP_d U^{-1} , \quad P^r = UP_d^r U^{-1} , \tag{33}$$

where U is the unitary matrix that diagonalizes P and P_d is the corresponding diagonal matrix. The eigenvalues of P are the roots of the characteristic equation

$$p^2 - p \operatorname{Tr}(P) + \det(P) = 0 , \tag{34a}$$

$$p_\pm = \tfrac{1}{2}\{\operatorname{Tr}(P) \pm \sqrt{[\operatorname{Tr}(P)]^2-4}\} . \tag{34b}$$

Acceptable solutions are obtained if and only if $p_\pm$ are complex conjugates

$$|\operatorname{Tr}(P)| \le 2 . \tag{35}$$

If we define a real parameter γ, such that

$$\cos(\gamma) = \tfrac{1}{2}\operatorname{Tr}(P) , \tag{36a}$$

with

$$p_+ = e^{+i\gamma} , \quad p_- = e^{-i\gamma} , \tag{36b}$$

a frequency ω is allowed if the expression

$$\cos(\gamma) = \tfrac{1}{2}\operatorname{Tr}(P) = \alpha_1 \cos(\omega T) + \beta_1 \sin(\omega T) \tag{36c}$$

is satisfied.

Consider the matrix U

$$U = \begin{bmatrix} (p_+ e^{i\omega T} - A) & (p_- e^{i\omega T} - A) \\ i\beta_2 & i\beta_2 \end{bmatrix} , \tag{37}$$

where $A = \alpha_1 + \beta_1$. If we impose the boundary condition given by Eqs. (14)–(32), after substituting Eq. (33) in Eq. (32), we get

$$U^{-1} \begin{bmatrix} 1 \\ R_f \end{bmatrix} = P_d^r U^{-1} \begin{bmatrix} T_b \\ 0 \end{bmatrix} . \tag{38}$$

From Eq. (38) we can derive an expression for R_f and thus $|R_f|^2$:

$$|R_f|^2 = \frac{\beta_2^2}{\alpha_1^2+\beta_1^2-2B\cos(\gamma)+B^2} , \tag{39a}$$

where

$$B = \frac{\sin[(r+1)\gamma]}{\sin(r\gamma)} . \tag{39b}$$

We have photon production for a given mode only if the corresponding frequency ω_k satisfies the relation (assuming $T_1 = T_2 = T/4$)

$$\cos(\gamma k) = \cos\left[\frac{T}{2}\omega_{2k}\right]\cos\left[\frac{T}{2}\omega_k\right] - \frac{1}{2}\left[\frac{\omega_{2k}}{\omega_k} + \frac{\omega_k}{\omega_{2k}}\right]\sin\left[\frac{T}{2}\omega_{2k}\right]\sin\left[\frac{T}{2}\omega_k\right] . \tag{40}$$

The corresponding number of photons produced is

$$N_a = \frac{1}{4}\left[\frac{\omega_k}{\omega_{2k}} - \frac{\omega_{2k}}{\omega_k}\right]^2 \sin^2\left[\frac{T}{2}\omega_{2k}\right]\frac{\sin^2(r\gamma_k)}{\sin^2(\gamma_k)} . \tag{41}$$

If we indicate with $\{\omega_k^*\}$ the set of frequencies such that

$$\cos(\gamma_k^*) = \pm 1 \tag{42}$$

and $\omega_k^* < \omega_c$, where ω_c is the cutoff frequency, we notice that in the limit $r \to \infty$ we have

$$\frac{\sin^2 r\gamma_k}{\sin^2\gamma_k} \sim \begin{cases} r^2 & \text{if } \omega_k \sim \omega_k^* \\ 0 & \text{if } \omega_k \neq \omega_k^* . \end{cases} \tag{43}$$

Therefore the total number of photons produced is on the order of

$$N \sim \frac{r^2}{4}\sum_k \left[\frac{\omega_k^*}{\omega_{2k}^*} - \frac{\omega_{2k}^*}{\omega_k^*}\right]^2 \sin^2\left[\frac{T}{2}\omega_{2k}^*\right] . \tag{44}$$

Let us do a more detailed calculation in the limit $L \gg a$. If we introduce the variables

$$y = \frac{T}{2}c\frac{\pi}{a} , \tag{45a}$$

$$x = \frac{ak_\parallel}{\pi} , \tag{45b}$$

ω_k and ω_{2k} can be written, respectively, as

$$\omega_k = \frac{c\pi}{a}\sqrt{x^2+n^2} , \tag{46a}$$

$$\omega_{k2} = \frac{c\pi}{a}\sqrt{x^2+n^2(1+b^2)} , \tag{46b}$$

and Eq. (40) becomes

$$\cos(\gamma_k) = h(x,y,n,b) = \cos[y\sqrt{x^2+n^2(1+b^2)}]\cos(y\sqrt{x^2+n^2}) - \frac{1}{2}\left[\frac{\sqrt{x^2+n^2(1+b^2)}}{\sqrt{x^2+n^2}} + \frac{\sqrt{x^2+n^2}}{\sqrt{x^2+n^2(1+b^2)}}\right]\sin[y\sqrt{x^2+n^2(1+b^2)}]\sin(y\sqrt{x^2+n^2}) . \tag{47}$$

We notice that in the limit $x \to \infty$, $h \sim \cos(2xy)$.

The total number of photons is obtained by summing Eq. (41) over the values of ω_k satisfying Eq. (40) or (47). For the moment we keep fixed the values of y, n, b and we vary x. Let us call x_i, $i = 1, 2, \ldots, m$, the solutions of Eq. (42) up to a cutoff value x_m. We assume for simplicity that the cutoff x_c is a solution of Eq. (42) and $x_c = x_m$. As we shall see below, this will not change the final result.

The number of photons produced for a given n can be expressed as ($A = L^2$)

$$N(y,n,b,r,A,a) = \frac{\pi A}{4a^2} \sum_{i=1}^{m} \int_{x_i}^{x_{i+1}} f(x,y,n,b) g(x,y,n,b,r) dx \;, \tag{48a}$$

where $f(x,y,n,b)$ is

$$f(x,y,n,b) = x \left[\left[\frac{x^2+n^2}{x^2+n^2(1+b^2)} \right]^{1/2} - \left[\frac{x^2+n^2(1+b^2)}{x^2+n^2} \right]^{1/2} \right]^2 \times \sin^2[y\sqrt{x^2+n^2(1+b^2)}] \;, \tag{48b}$$

$g(x,y,n,b,r)$ is the composite function given by

$$g(x,y,n,b,r) = \frac{\sin^2[r\gamma_k(x,y,n,b)]}{\sin^2[\gamma_k(x,y,n,b)]} \tag{48c}$$

and x_i and x_{i+1} are the end points of a given band. The total number of photons produced by the time periodic potential can be expressed as

$$N = \sum_{n=1}^{n_c} N(y,n,b,r,A,a) \;. \tag{48d}$$

To evaluate the integral given by Eq. (48a) we notice that $g(x,y,n,b,r)$ has singularities at the points $x = x_i$, which are removable because the limit for x going to x_i exists and is equal to r^2. We can therefore define a continuous function $G(x,y,n,b,r)$ as

$$G(x,y,n,b,r) = \begin{cases} g(x,y,n,b,r) & \text{if } x_i < x < x_{i+1} \\ r^2 & \text{if } x = x_i, x_{i+1} \;. \end{cases} \tag{49}$$

The integral

$$J(y,n,b,r,A,a) = \frac{\pi A}{4a^2} \int_{x_i}^{x_{i+1}} f(x,y,n,b) G(x,y,n,b,r) dx \tag{50}$$

has the same value of $N(y,n,b,r,A,a)$ because the two integrals differ only for a finite number of singularities. Moreover $J > 0$ because the integrand function is nonnegative and $J = 0$ only if the integrand function is zero for every value of $x \in [x_i, x_{i+1}]$. Applying the mean value theorem we get

$$J = \frac{\pi A}{4a^2} f(\bar{x}_i, y, n, b) \int_{x_i}^{x_{i+1}} G(x,y,n,b,r) dx \;, \tag{51}$$

where $\bar{x}_i$ is an appropriate value in the interval $[x_i, x_{i+1}]$.

We want now to evaluate the integral in Eq. (51) given by

$$I(y,n,b,r) = \int_{x_i}^{x_{i+1}} G(x,y,n,b,r) dx \;, \tag{52}$$

in the limit $r \to \infty$. The existence of this integral is ensured by the continuity of G in the closed interval $[x_i, x_{i+1}]$. Therefore we can choose the following partition to evaluate it:

$$x_0 = x_i \;, \quad x_1 = x_0 + \Delta x \;, \quad x_2 = x_0 + 2\Delta x, \ldots, \\ x_j = x_0 + j\Delta x, \ldots, x_r = x_{i+1} = x_0 + r\Delta x \;, \tag{53a}$$

where

$$\Delta x = \frac{x_{i+1} - x_i}{r} \;. \tag{53b}$$

Applying the definition of the Riemann integral we can write [31]

$$I(y,n,b,r) = \lim_{r \to \infty} \left[\sum_{j=1}^{r} G(x_0 + j\Delta x, y, n, b, r) \Delta x \right] . \tag{54}$$

For r large, $G(x,y,n,b,r)$ is different from zero only for

$$x \sim x_0, x_r \;. \tag{55}$$

Therefore only two elements of the sum in Eq. (54) are different from zero. Hence $I(y,n,b,r)$ can be approximated as

$$I(y,n,b,r) \sim \lim_{r \to \infty} [G(x_0 + \Delta x, y, n, b, r) + G(x_0 + r\Delta x, y, n, b, r)] \Delta x \cong 2r^2 \left[\frac{x_{i+1} - x_i}{r} \right] . \tag{56}$$

The last expression in Eq. (56) has been obtained by substituting in it the values of Δx and G at $x = x_i, x_{i+1}$ in the limit $r \to \infty$. hence $N(y,n,b,r,A,a)$ is

$$N(y,n,b,r,A,a) = J = \frac{\pi A}{4a^2} f(\bar{x}_i, y, n, b, r) 2r(x_{i+1} - x_i) \;. \tag{57}$$

The total number of photons produced per pulse, per unit of area, is obtain by substituting Eq. (57) in Eq. (48d)

$$(N/Ar) = \frac{\pi}{2a^2} \sum_{n=1}^{n_c} \sum_{i=1}^{m} f(\bar{x}_i, y, n, b)(x_{i+1} - x_i) \;. \tag{58}$$

In experimental situations, y is a very large number; therefore the left-hand side of Eq. (47) oscillates very rapidly. Let us make some numerical estimates. If we choose

$$a \sim 10^{-4} \text{ cm} \;, \quad T \sim 10^{-3} \text{ s} \;, \quad b \sim 5 \;, \tag{59}$$

Eq. (46c) can be written

$$\nu \sim 1.5 \times 10^{14} \sqrt{x^2+n^2} \text{ Hz} \;. \tag{60a}$$

If the cutoff frequency is $\nu_c \sim 10^{15}$ Hz, $\sqrt{x^2+n^2}$ cannot exceed a number of the order of 7. Moreover

$$y \sim 4.7 \times 10^{11} \;. \tag{60b}$$

If we choose a given value of x, and a small interval around x, for example, of the order of 10^{-5}, we can always find at least one value of x such that the left-hand side goes to ± 1. Therefore the sum over i in Eq. (58) can be approximated with an integral as

$$\sum_{i=1}^{m}(x_{i+1}-x_i)\sim\int_0^{x_c(n)}dx\ . \tag{61}$$

The number of photons produced per unit area per pulse is

$$\frac{N}{rA}\sim\frac{1}{4}\frac{\pi}{a^2}\sum_{n=1}^{n_c}\int_0^{x_c(n)}x\left[\frac{\sqrt{x^2+n^2}}{\sqrt{x^2+n^2(1+b^2)}}-\frac{\sqrt{x^2+n^2(1+b^2)}}{\sqrt{x^2+n^2}}\right]^2dx\ , \tag{62}$$

where $n_c=7$ and $x_c(n)$ is the cutoff value of x for a given n. In Eq. (62) we have substituted $\sin^2$ with its mean value. Evaluating the integral

$$\frac{N}{rA}\sim\frac{1}{8}\frac{\pi b^2}{a^2}\sum_{n=1}^{n_c}n^2\left[\ln\left[1+\frac{x_c^2(n)b^2}{x_c^2(n)+n^2(1+b^2)}\right]\right]\ . \tag{63}$$

The above expression grows as $\ln(n_c)$ for large n_c and it is estimated to give, for the number of produced photons,

$$\sim 10^{11}/(\text{area pulse})\ . \tag{64}$$

Therefore we have a large production of photons, mainly in the visible region.

V. CONCLUSIONS

We have shown that it is possible to create intense light by modulating the vacuum between two perfectly conducting plates when the distance between them is sufficiently small. It is possible to apply the method discussed above to different geometries or the presence of dielectrics. As suggested by Schwinger, the intense blue light produced in the sonoluminescence phenomenon [32,33] can be interpreted as being due to the dynamical Casimir effect. We believe that the work detailed in this paper is a confirmation of Schwinger's idea that the dynamical Casimir effect can produce intense radiation.

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APPENDIX

In the original work the notion was discussed that moving mirrors were in contradiction to the conventional Hamiltonian description of quantum mechanics [14]. In more recent work this has been shown not to be essential. Moving mirrors can indeed be described in the conventional Hamiltonian formulation of quantum mechanics, modulo the nonseparability of the Hibert spaces which is present in all relativistic quantum field theories. For completeness of presentation we outline the method which can be employed and refer the reader to the work of Calucci [17] for an exhaustive treatment of the details.

Suppose that we have a cavity in the spatial region Ω. The walls are considered to be perfectly reflecting. The Lagrangian of the electromagnetic modes inside the cavity is given by

$$L=\frac{1}{8\pi}\int_\Omega d^3\mathbf{r}(|\mathbf{E}|^2-|\mathbf{B}|^2)\ . \tag{A1}$$

In the Coulomb gauge $\nabla\cdot\mathbf{A}=0$,

$$\mathbf{E}(\mathbf{r},t)=-\frac{1}{c}\frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t}\ , \tag{A2a}$$

$$\mathbf{B}(\mathbf{r},t)=\nabla\times\mathbf{A}(\mathbf{r},t)\ , \tag{A2b}$$

one has the normal mode expansion

$$\mathbf{A}(\mathbf{r},t)=\sum_\alpha Q_\alpha(t)\,\mathbf{A}_\alpha(\mathbf{r};\Omega(t))\ , \tag{A3}$$

where $\Omega(t)$ indicates that the volume of the cavity varies with time. If the geometry of the cavities is sufficiently simple (for example, the spherical geometry or rectangular "box" geometry of this work), the quantum numbers $\{\alpha\}$ retain their definition and the wave function of the modes in the cavity have at all times the form $\Psi(\ldots,Q_\alpha,\ldots,t)$ no matter what the size of the cavity. Inner products are defined as

$$\langle\Psi_f|\Psi_i\rangle=\int\Psi_f^*\Psi_i\prod_\alpha dQ_\alpha\ . \tag{A4}$$

There are no further complications introduced from the fact that one has moving mirror walls beyond the nonseparable nature of the Hilbert space present even for nonmoving mirror walls.

However, the Lagrangian implied by Eqs. (A1)–(A3) is of the general form

$$L=\tfrac{1}{2}\sum_{\alpha\beta}[K_{\alpha\beta}\dot{Q}_\alpha\dot{Q}_\beta+M_{\alpha\beta}Q_\alpha\dot{Q}_\beta-N_{\alpha\beta}Q_\alpha Q_\beta]\ , \tag{A5}$$

from which the Hamiltonian may be constructed. The structure of the Lagrangian with moving mirrors has additional items from the fact that the walls are moving; i.e., the cavity electric fields are found from Eqs. (A2a) and (A3) to be

$$\mathbf{E}(\mathbf{r},t)=-\frac{1}{c}\sum_\alpha[\dot{Q}_\alpha\,\mathbf{A}_\alpha(\mathbf{r};\Omega(t))+Q_\alpha\,\dot{\mathbf{A}}_\alpha(\mathbf{r};\Omega(t))\ . \tag{A6}$$

When the walls are moving on a time scale slow on the scale of the mode frequencies, then the second term on the right-hand side of Eq. (A6) can be neglected, as discussed by Calucci. The resulting Lagrangian then has the well-known adiabatic form

$$L_{\text{adiabatic}}=\tfrac{1}{2}\sum_\alpha[\dot{Q}_\beta^2-\omega_\alpha(\Omega(t))^2Q_\alpha^2]\ , \tag{A7}$$

used in this work.

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