Theory of Two Coupled Lasers*

Martin B. Spencer† and Willis E. Lamb, Jr.
Department of Physics, Yale University, New Haven, Connecticut 06520
(Received 10 May 1971)

A transmitting window is used to weakly couple two resonant optical cavities. When active medium is placed in one cavity, the system describes a laser coupled to a resonant cavity, while active medium in both cavities describes two coupled lasers. Equations of motion for the fields in the two cavities are found. These equations are solved in the steady state when the cavities are sufficiently closely tuned that the lasers in the two cavities oscillate with the same frequency. In the locked situation, the second cavity tends to stabilize the frequency of operation against changes in length of the first cavity. When there is no active medium in the second cavity, and that cavity becomes very long, the laser acts as if it were looking into free space.

I. INTRODUCTION

This paper forms the sequel to the preceding paper[1] in which the effects on a laser of one partially transmitting mirror were studied. In that case the output from the laser escaped into free space without returning. In the present situation another mirror on the axis of the system reflects back all the radiation leaking through the window. Thus the entire system may be considered as two cavities coupled by a partially transmitting window. When active medium is put in both cavities, the model describes two lasers weakly coupled and interacting with each other. If only the left-hand cavity encloses active medium, the model describes a single laser coupled to a resonant cavity. In the limit when the resonant cavity becomes very long with some loss, the system approaches the case of a laser oscillator looking through the window into free space described in Ref. 1.

II. TWO CAVITIES

The laser axis is taken along the z direction with the totally reflecting mirrors at $z=-L_1$ and $z=L_2$, and the dielectric window at $z=0$. The complex fields[2] in the two cavities are given by

$$\begin{aligned} E(z,t) &= \mathcal{E}_1(t)\sin k(z+L_1), \qquad -L_1 \le z \le 0 \\ E(z,t) &= \mathcal{E}_2(t)\sin k(z-L_2), \qquad 0 \le z \le L_2 \end{aligned} \tag{1}$$

and

$$\begin{aligned} H(z,t) &= -(i/Z_0)\,\mathcal{E}_1(t)\cos k(z+L_1), \qquad -L_1 \le z \le 0 \\ H(z,t) &= -(i/Z_0)\,\mathcal{E}_2(t)\cos k(z-L_2), \qquad 0 \le z \le L_2 \end{aligned} \tag{2}$$

which satisfy the cavity boundary conditions at the mirrors. $Z_0=(\mu_0/\epsilon_0)^{1/2}$ is the impedance of free space. As before, the electric field is polarized along the x direction, and the magnetic field is along the y direction, while

$$\mathcal{E}_1(t)=E_1(t)\,e^{-i[\nu t+\phi_1(t)]},$$

$$\mathcal{E}_2(t)=E_2(t)\,e^{-i[\nu t+\phi_2(t)]},$$

with the amplitudes and phases varying slowly in the optical period $2\pi/\nu$.

As in the preceding paper the window is described by a dielectric "bump" of the form

$$\epsilon(z)=\epsilon_0[1+(\eta/k)\,\delta(z)] \tag{3}$$

with dimensionless parameter η, which is related to the reflection coefficient of the window. According to Maxwell's equations the boundary conditions at the origin are

$$E(0^+,t)=E(0^-,t), \tag{4}$$

$$H(0^+,t)-H(0^-,t)=i(\eta/Z_0)\,E(0,t). \tag{5}$$

From these conditions we see that

$$\mathcal{E}_1(t)\cos kL_1-\mathcal{E}_2(t)\cos kL_2=\eta\mathcal{E}_1(t)\sin kL_1, \tag{6}$$

$$\mathcal{E}_1(t)\sin kL_1=-\mathcal{E}_2(t)\sin kL_2. \tag{7}$$

On elimination of the field strengths we find the equation

$$\cot kL_1+\cot kL_2=\eta, \tag{8}$$

which determines eigenfrequencies kc for the whole cavity.

The field at the mirror, $E(0,t)$, acts as a source term in the equation for the time dependence of the electric field in either cavity, as shown in Ref. 1:

$$\ddot{\mathcal{E}}_1+(\sigma_1/\epsilon_0)\dot{\mathcal{E}}_1+\Omega_1^2\mathcal{E}_1+(-1)^{m'}(2/L_1)\,\Omega_1 cE(0,t) = -\ddot{\mathcal{P}}_1(t)/\epsilon_0, \tag{9}$$

$$\ddot{\mathcal{E}}_2+(\sigma_2/\epsilon_0)\dot{\mathcal{E}}_2+\Omega_2^2\mathcal{E}_2-(-1)^{m''}(2/L_2)\,\Omega_2 cE(0,t) = -\ddot{\mathcal{P}}_2(t)/\epsilon_0. \tag{10}$$

The extra factors $(-1)^{m'}$ and $(-1)^{m''}$ arise from the different definition of the spatial basis functions in the cavities. The derivations of Eqs. (9) and (10) assume an approximately integral number of halfwavelengths in each cavity, or

$$kL_1 \approx m'\pi \,, \quad (11)$$

$$kL_2 \approx m''\pi \,. \quad (12)$$

The extra minus sign in (10) arises from the fact that the window is at the opposite end of the second cavity. Solving Eq. (5) for $E(0, t)$ in terms of $\mathcal{E}_1(t)$ and $\mathcal{E}_2(t)$, we find

$$E(0, t) = [\mathcal{E}_1(t)\cos kL_1 - \mathcal{E}_2(t)\cos kL_2]/\eta \,. \quad (13)$$

With the same assumption used in the derivation of (9) and (10), Eq. (13) becomes

$$E(0, t) = (-1)^{m'}[\mathcal{E}_1(t) \pm \mathcal{E}_2(t)]/\eta \,. \quad (14)$$

The assumptions (11) and (12) will certainly be valid when the two cavities are approximately the same length. The second cavity can be lengthened in steps of one-half wavelength and the only result will be an alternation in sign in (14). For simplicity we consider the case where the two cavities are the same length to within less than one wavelength. In this case the sign in Eq. (14) is negative.

This expression can be substituted in (9) and (10) to give the equation of motion of the electric fields

$$\ddot{\mathcal{E}}_1 + 2\Gamma_1\dot{\mathcal{E}}_1 + \Omega_1^2\mathcal{E}_1 + (c/L_1)(2\Omega_1/\eta)(\mathcal{E}_1 - \mathcal{E}_2) = -\ddot{\mathcal{P}}_1/\epsilon_0 \,, \quad (15)$$

$$\ddot{\mathcal{E}}_2 + 2\Gamma_2\dot{\mathcal{E}}_2 + \Omega_2^2\mathcal{E}_2 + (c/L_2)(2\Omega_2/\eta)(\mathcal{E}_2 - \mathcal{E}_1) = -\ddot{\mathcal{P}}_2/\epsilon_0 \,. \quad (16)$$

In the slowly-varying-amplitude and -phase approximation these become

$$\dot{\mathcal{E}}_j + [\Gamma_j + i(\Omega_j + M_j)]\mathcal{E}_j - iM_j\mathcal{E}_l = (\tfrac{1}{2}\, i\nu/\epsilon_0)\mathcal{P}_j \,, \quad (17)$$

with $j=1$, $l=2$ or $j=2$, $l=1$. The coupling coefficients are given by

$$M_j = c/\eta L_j \,, \quad (18a)$$

and the internal cavity losses by

$$\Gamma_j = \sigma_j/\epsilon_0 \,. \quad (18b)$$

III. TWO COUPLED LASERS

The complex polarization $\mathcal{P}_j$ in either cavity due to the active medium was shown in Ref. 1 to be given by

$$\mathcal{P}_j = -2i\mathcal{E}_j(\epsilon_0/\nu)(1 - i\xi)f_j(I_j) \,, \quad j = 1, 2 \quad (19)$$

for stationary atoms. We have written the detuning of ν from the atomic frequency in the dimensionless form $\xi = (\omega - \nu)/\gamma$, and

$$f_j(I_j) = [2\alpha_j/I_j\mathfrak{L}(\omega-\nu)]\{1 - [1 + I_j\mathfrak{L}(\omega - \nu)]^{-1/2}\} \,, \quad (20)$$

where

$$\alpha_j = (\nu\wp^2/2\epsilon_0\hbar\gamma)\bar{N}_j\mathfrak{L}(\omega - \nu) \quad (21)$$

is the gain parameter,

$$\mathfrak{L}(\omega - \nu) = 1/(1 + \xi^2) \quad (22)$$

is a dimensionless Lorentzian, and

$$I_j = \wp^2 E_j^2/\hbar^2\gamma_a\gamma_b \quad (23)$$

is the dimensionless intensity.

Using expression (19) for the polarization, Eq. (17) becomes

$$\dot{\mathcal{E}}_j + [\Gamma_j + i(\Omega_j + M_j)]\mathcal{E}_j - iM_j\mathcal{E}_l = (1 - i\xi)f_j(I_j)\mathcal{E}_j \,. \quad (24)$$

For simplicity we assume that the active media in each cavity are identical, so that the parameters ω and γ describing the atoms are equal in each cavity. Assuming also that the rates of excitation of the medium are the same in each cavity, we set $\alpha_1 = \alpha_2$. A different rate of excitation in either cavity would merely multiply the right-hand side of (24) by a constant factor. With these assumptions we can drop the subscript on the f in (24).

Rewriting (24) in terms of the amplitudes and phases of the electric fields, we obtain the four equations of motion

$$\dot{E}_1 = [f(I_1) - \Gamma_1]E_1 + M_1E_2\sin\phi \,, \quad (25)$$

$$\dot{E}_2 = [f(I_2) - \Gamma_2]E_2 - M_2E_1\sin\phi \,, \quad (26)$$

$$\dot{\phi}_1 = \Omega_1 + M_1 - \nu + \xi f(I_1) - (M_1E_2/E_1)\cos\phi \,, \quad (27)$$

$$\dot{\phi}_2 = \Omega_2 + M_2 - \nu + \xi f(I_2) - (M_2E_1/E_2)\cos\phi \,, \quad (28)$$

where $\phi = \phi_2 - \phi_1$ is the phase difference between the fields in the two cavities. These equations can be solved in the stationary state by iteration. Picking a value for ϕ and assuming a value for ξ, we can iterate (25) and (26) to find solutions for I_1, I_2, and E_2/E_1. Using these values, we can solve (27) for ξ

$$\xi = \frac{\omega - \Omega_1 - M_1 + (M_1E_2/E_1)\cos\phi}{\gamma + f(I_1)} \quad (29)$$

and use this in (25) and (26) to find new values for I_1, I_2, and E_2/E_1. In this manner we can iterate (25)–(27) to find self-consistent values for I_1, I_2, E_2/E_1, and ξ for each ϕ. Subtracting (27) from (28) in the stationary state, we find

$$0 = \Omega + \xi[f(I_2) - f(I_1)] + [M_1E_2/E_1 - M_2E_1/E_2]\cos\phi \,, \quad (30)$$

where

$$\Omega = \Omega_2 - \Omega_1 + M_2 - M_1 \quad (31)$$

is the relative cavity frequency. Equation (30) allows us to find Ω using I_1, I_2, E_2/E_1, and ξ from the assumed value of ϕ.

The iteration procedure can be started at $\phi = 0$ or π, when Eqs. (25) and (26) become single-laser equations which can be solved exactly in the stationary state. For the particular choices taken for the M_j and Γ_j, stationary-state solutions exist only for very small values of $\sin\phi$. For each set of parameters two curves are found, one with $\phi \sim 0$ and the other with $\phi \sim \pi$. These correspond to the two very close normal modes of such a double cavity. Thus the choice of one particular sign in (14) is not important, since the opposite choice merely adds π to ϕ.

Figures 1 and 2 show the intensities I_1 and I_2 plotted against the relative cavity frequency Ω. The stationary solutions were investigated for stability (see Appendix) and the unstable portions of the curves are shown dotted. The only stable locked solutions occur when the cavities are closely tuned to each other. When the cavities are sufficiently different in length, the lasers no longer oscillate in phase but tend to act independently of each other.

The second laser cavity has a stabilizing effect on the frequency of the system when the two are locked. For a single laser with a window the frequency of operation is given by[3]

$$\nu = \nu_0 = \frac{(\Omega_1 + \delta\Omega)\gamma + \omega(\Gamma_1 + \Gamma_w)}{\gamma + \Gamma_1 + \Gamma_w} . \tag{32}$$

Variations in length of the resonant cavity result in frequency changes described by

$$\frac{\partial \nu_0}{\partial \Omega_1} = \frac{\gamma}{\gamma + \Gamma_1 + \Gamma_w} . \tag{33}$$

For the coupled lasers, the frequency in the locked state is given by

$$\nu_c = \frac{\{[\Gamma_1 - f(I_1)][\gamma(\Omega_2 + M_2) + \omega f(I_2)] + [\Gamma_2 - f(I_2)][\gamma(\Omega_1 + M_1) + \omega f(I_1)]\}}{\{[\gamma + f(I_2)][\Gamma_1 - f(I_1)] + [\gamma + f(I_1)][\Gamma_2 - f(I_2)]\}} \tag{34}$$

or

$$\nu_c = \frac{\gamma(\Omega_1 + M_1) + \omega f(I_1)}{\gamma + f(I_1)} - \frac{\gamma}{\gamma + f(I_1)} M_1 \frac{E_2}{E_1} \cos\phi . \tag{35}$$

To find the equivalent variation of ν_c with Ω_1, we differentiate Eqs. (25)–(28) with respect to Ω_1 in the steady state. After some algebra this gives

$$1 = \frac{\partial \nu_c}{\partial \Omega_1}\left(2 + \frac{f(I_1) + f(I_2)}{\gamma}\right) + \frac{\partial r}{\partial \Omega_1}\left[\xi\left(M_1 + \frac{M_2}{r^2}\right)\sin\phi + \left(M_1 - \frac{M_2}{r^2}\right)\cos\phi\right]$$

$$+ r\frac{\partial \phi}{\partial \Omega_1}\left[\xi\left(M_1 - \frac{M_2}{r^2}\right)\cos\phi - \left(M_1 + \frac{M_2}{r^2}\right)\sin\phi\right] , \tag{36}$$

where $r = E_2/E_1$.

In the particular case $\Gamma_1 = \Gamma_2$, $r = 1$ when $\sin\phi = 0$ and $\Omega = 0$. Since the cavities are the same length, $M_1 = M_2$, and

$$\frac{\partial \nu_c}{\partial \Omega_1} = \frac{1}{2}\,\frac{\gamma}{\gamma + \Gamma_1} \approx \frac{1}{2}\,\frac{\partial \nu_0}{\partial \Omega_1} . \tag{37}$$

Thus for two identical lasers tuned to equal lengths, movement of one end mirror changes the frequency by only half as much as for a single cavity.

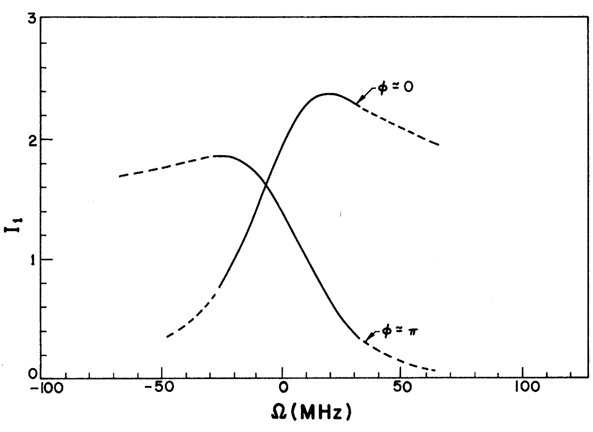

FIG. 1. Two coupled lasers. Plot of the dimensionless intensity in the first cavity vs relative cavity frequency in the stationary state. Unstable solutions are shown dotted. The two solutions correspond to different relative phases between the fields in the two cavities. Losses in the cavities are taken to be equal, $\Gamma_1 = \Gamma_2 = 1$ MHz, $M_1 = M_2 = 20$ MHz, $\alpha_1 = \alpha_2 = 1.6$ MHz, $\gamma = 99$ MHz, and $\omega - \Omega_1 - M_1 = 99$ MHz.

Figure 3 shows the frequency changes brought about by moving the end mirror of cavity 1. The solid curve shows the frequency of the coupled lasers, while the dotted lines show the frequencies of single lasers in cavities equal to cavities 1 and 2. If we neglect Γ_w compared to $\gamma + \Gamma_1$, Eq. (34) shows that the solid curve tends to the dotted lines as $E_1 \to 0$ or $E_2 \to 0$. (Neglect of Γ_w in the comparison is necessary because the extra cavity prevents the loss of radiation represented by Γ_w in a single laser with a window.)

IV. LASER COUPLED TO A RESONANT CAVITY

Removal of the active medium from the right-hand cavity simplifies Eqs. (24) by removing the nonlinearities from one of the two equations:

$$\dot{\mathcal{E}}_1 + [\Gamma_1 + i(\Omega_1 + M_1)]\mathcal{E}_1 - iM_1\mathcal{E}_2 = (1 - i\xi) f(I_1)\mathcal{E}_1 , \tag{38}$$

$$\dot{\mathcal{E}}_2 + [\Gamma_2 + i(\Omega_2 + M_2)]\mathcal{E}_2 - iM_2\mathcal{E}_1 = 0 . \tag{39}$$

In the stationary state (39) gives a relation between the complex fields in the two cavities,

$$\mathcal{E}_2 = \{iM_2/[\Gamma_2 + i(\Omega_2 + M_2 - \nu)]\}\mathcal{E}_1 . \tag{40}$$

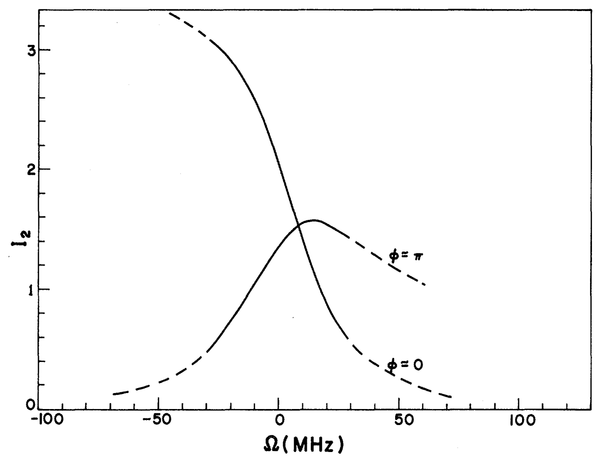

FIG. 2. Two coupled lasers. Plot of the dimensionless intensity in the second cavity corresponding to the two solutions shown in Fig. 1.

Substituting this in (38) gives the steady-state electric field intensity in the laser cavity in the form

$$f(I_1)=\frac{\Gamma_1+i(\Omega_1+M_1-\nu)+M_1M_2/[\Gamma_2+i(\Omega_2+M_2-\nu)]}{1-i\xi}$$
$$=G(\nu)\ . \qquad (41)$$

This can be solved if we note that the right-hand side must be real. Setting $\mathrm{Im}G(\nu)=0$ gives a cubic equation for ν, independent of intensity:

$$[(\Omega_1+M_1)\gamma+\omega\Gamma_1-\nu(\Gamma_1+\gamma)][(\Omega_2+M_2-\nu)^2+\Gamma_2^2]$$
$$+M_1M_2[(\Omega_2+M_2)\gamma-\omega\Gamma_2-\nu(\gamma-\Gamma_2)]=0\ . \qquad (42)$$

Real solutions of (42), ν_i, can be put into (41) to give

$$f(I_1)=\mathrm{Re}G(\nu_i)=G\ . \qquad (43)$$

For all G the nonzero solution of (43) is

$$I_1=\frac{4\alpha-G-[G(G+8\alpha)]^{1/2}}{2G\mathcal{L}(\omega-\nu)}\ , \qquad (44)$$

which reduces to the familiar form

$$I_1=\frac{4}{3}\ \frac{\alpha-G}{G\mathcal{L}(\omega-\nu)}=\frac{\alpha-G}{\beta} \qquad (45)$$

near threshold where $\alpha\sim G$. Thus G as it appears in (43) and (45) is the total effective cavity bandwidth as seen by the active medium in the first cavity. It includes losses in the second cavity as they are reflected into the laser cavity.

The frequency equation has at least one real root. In the limit of a very highly reflecting window when $M_j\to 0$ the only real root $\nu\to\nu_L$, the natural frequency of the laser in a windowless cavity. In this limit $G\to\Gamma_1$ and the intensity approaches its expected value in a windowless cavity.

The intensity in the second cavity is obtained from I_1 using

$$I_2=\{M_2^2/[\Gamma_2^2+(\Omega_2+M_2-\nu)^2]\}I_1\ . \qquad (46)$$

Figures 4 and 5 show the intensities I_1 and I_2 for different Γ_2 as a function of the relative cavity frequency Ω.

Equations for the time dependence of the amplitudes and phases are obtained by removing terms in $f(I_2)$ from (25)-(28). Stability of the stationary states is determined as before.

Where solutions are above threshold ($I_1>0$, $I_2>0$), there is at least one stable state for each Ω. If we change Ω until a solution becomes unstable, the laser will evolve to the other stationary state which is stable.

For sufficiently large loss in the second cavity both solutions are below threshold for small Ω. This occurs because tuning the cavities close in frequency results in more loss being reflected into the laser cavity from the right-hand cavity.

The resonant cavity has a stabilizing effect on the frequency of the laser similar to the case of coupled lasers.

V. LASER RADIATING INTO LINEAR MEDIUM

The case when the laser is radiating into the passive cavity (or any linear medium) can be treated in a more general manner. We consider only the field in the laser cavity, and introduce the effect of the external linear cavity by its "impedance" as seen by the laser just outside the window.

Defining the impedance just outside the window as

$$Z=E(0^+,t)/H(0^+,t), \qquad (47)$$

we can rewrite Eq. (5) as

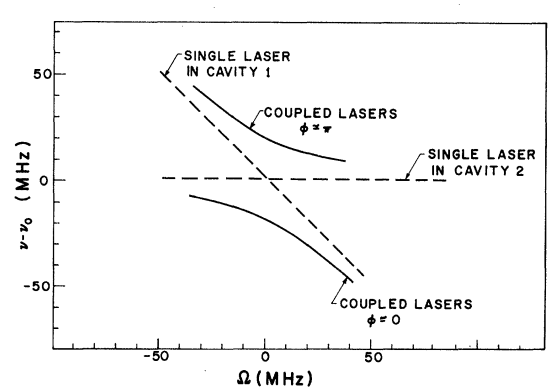

FIG. 3. Curves showing the frequency of operation as the mirror at $z=-L_1$ is moved. The solid line corresponds to the coupled lasers while the dotted lines show the frequency of single lasers in cavities equivalent to cavities 1 and 2. The parameters used are the same as in Figs. 1 and 2.

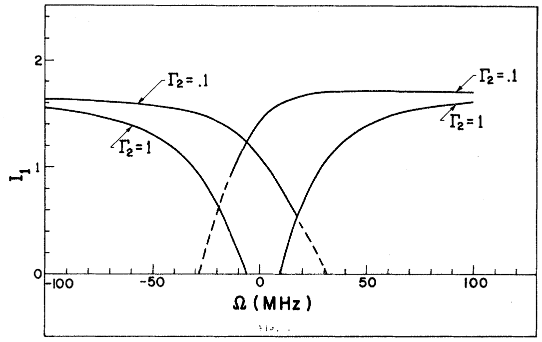

FIG. 4. Steady-state intensities in the laser cavity for a laser coupled to a resonant cavity. The two pairs of curves correspond to different amounts of loss in the resonant cavity. (Unstable solutions are shown dotted.) The parameters used for these solutions are $\Gamma_1=1$ MHz, $M_1=M_2=20$ MHz, $\alpha_1=1.6$, $\gamma=99$ MHz, and $\omega-\Omega_1-M_1=99$ MHz.

$$-i(Z_0/Z)E(0,t)+\mathcal{E}_1(t)\cos kL_1=\eta E(0,t)\ , \tag{48}$$

which can be solved for $E(0, t)$ in terms of $\mathcal{E}_1(t)$:

$$E(0,t)=(-1)^{m'}\mathcal{E}_1(t)/[\eta+i(Z_0/Z)]\ . \tag{49}$$

Substituting this into (9), we find the equation of motion of the electric field in the laser cavity:

$$\ddot{\mathcal{E}}_1+2\Gamma_1\dot{\mathcal{E}}_1+\Omega_1^2\mathcal{E}_1+\frac{2\Omega_1 c/L_1}{\eta+i(Z_0/Z)}\mathcal{E}_1=-\ddot{\mathcal{P}}_1/\epsilon_0\ . \tag{50}$$

In the slowly-varying-amplitude and -phase approximation this becomes

$$\dot{\mathcal{E}}_1+[(\Gamma_1+\Gamma_e)+i(\Omega_1+\Omega_e)]\mathcal{E}_1=(i\nu/2\epsilon_0)\mathcal{P}\ , \tag{51}$$

where

$$\Gamma_e=Y_r\,\frac{c/L_1}{(\eta-Y_i)^2+Y_r^2}\ , \tag{52}$$

$$\Omega_e=(\eta-Y_i)\,\frac{c/L_1}{(\eta-Y_i)^2+Y_r^2}\ , \tag{53}$$

where $Y=Y_r+iY_i=Z_0/Z$ is the normalized admittance of the load seen from the window. Thus the loading of the laser effects the cavity bandwidth and cavity resonance frequency.

In the simple case of a laser oscillator looking into free space $Y_r=1$, $Y_i=0$, and

$$\Gamma_e=\frac{c/L_1}{\eta^2+1}\ , \tag{54}$$

$$\Omega_e=\frac{(c/L_1)\eta}{\eta^2+1}\ , \tag{55}$$

which are just the window loss Γ_w and frequency shift $\delta\Omega$ derived in Ref. 1.

When the laser is looking into the lossless tuned cavity described in Sec. II above, we see that

$$Z=-iZ_0\tan kL_2\ , \tag{56}$$

making

$Y_r=0$ and $Y_i=\cot kL_2$.

In this case $\Gamma_e=0$, and the extra cavity introduces no further losses, but

$$\Omega_e=\frac{c/L_1}{\eta-\cot kL_2}\ , \tag{57}$$

which is highly frequency dependent.

Introduction of loss into the right-hand cavity through a conductivity σ_2 results in an impedence

$$Z=-iZ_0(1+2i\Gamma_2/\nu)^{-1/2}\tan kL_2\,(1+2i\Gamma_2/\nu)^{1/2}\ , \tag{58}$$

where $\Gamma_2=\sigma_2/2\epsilon_0$. If the external cavity is long enough so that $\Gamma_2L_2/c\gg 1$,

$$\tan kL_2\,(1+2i\Gamma_2/\nu)^{1/2}\rightarrow i$$

and $Z\rightarrow Z_0/(1+2i\Gamma_2/\nu)^{1/2}$, which is just the impedance looking into "free space" with loss. In this limit the loss in the right-hand cavity has absorbed the wave before it could be reflected back to influence the laser. When the system is above threshold, and in a steady state, Eq. (51) may be written

$$(\Omega_1+\Omega_e-\nu)/(\Gamma_1+\Gamma_e)=-\mathrm{Re}\mathcal{P}/\mathrm{Im}\mathcal{P}\ . \tag{59}$$

Using the expression (19) for the polarization in the laser cavity, (59) becomes

$$(\omega-\nu)(\Gamma_1+\Gamma_e)=-\gamma(\Omega_1+\Omega_e-\nu)\ . \tag{60}$$

In the above example, looking into free space, Γ_e and Ω_e are independent of ν, and we can solve (60) for the frequency of operation, $\nu=\nu_0$ [see Eq. (32)].

If we remove the loss from the right-hand cavity as in (56), (60) may be written

$$\nu-\nu_L=\frac{\gamma}{\gamma+\Gamma_1}\,\frac{c}{L_1}\,\frac{1}{\eta-\cot(\nu L_2/c)}\ , \tag{61}$$

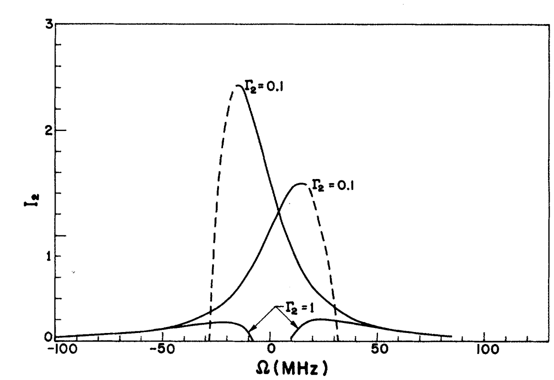

FIG. 5. Intensities in the resonant cavity corresponding to the solutions in Fig. 4.

with

$$\nu_L = (\omega\Gamma_1 + \Omega_1\gamma)/(\Gamma_1+\gamma) . \qquad (62)$$

This transcendental equation, which is valid only when the system is above threshold, has solutions for an unlimited range of frequencies ν, separated by about $\pi c/L_2$. The stability of the different solutions has not been investigated since it is not clear that the system would oscillate with only one frequency, particularly when L_2 becomes large. Introduction of loss in the external cavity reduces the number of such solutions by reducing the excursions of the right-hand side of (61).

APPENDIX: STABILITY OF STATIONARY STATES

Subtracting (27) from (28), we obtain the equation for the time dependence of ϕ:

$$\dot{\phi} = \Omega + \xi[f(I_2) - f(I_1)] - [M_2E_1/E_2 - M_1E_2/E_1]\cos\phi . \qquad (63)$$

The other equation, obtained by adding (27) and (28), gives the time dependence of $\phi_1+\phi_2$. However this equation is effectively decoupled from the other three, since variations in $\phi_1+\phi_2$ have been neglected in factors such as $\mathcal{L}(\omega-\nu)$ and ξ.

Equations (25), (26), and (63) are linearized by expanding to first order about their stationary values, to obtain a set of linear coupled equations of the form

$$\begin{pmatrix} \Delta\dot{E}_1 \\ \Delta\dot{E}_2 \\ E_1\Delta\dot{\phi} \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix} \begin{pmatrix} \Delta E_1 \\ \Delta E_2 \\ E_1\Delta\phi \end{pmatrix} . \qquad (64)$$

The stationary state will be stable if the eigenvalues of this matrix have negative real parts. The criterion for this to occur is

$$r_1 > 0, \quad r_1 r_2 > r_3, \quad \text{and} \quad r_3 > 0 ,$$

where

$$r_1 = -(a+e+j) ,$$

$$r_2 = a(e+j) + ej - hf - bd - cg ,$$

$$r_3 = a(hf - ej) + b(dj - fg) + c(eg - dh) .$$

From (25), (26), and (56) we find

$$a = f(I_1) - \Gamma_1 + E_1 \frac{\partial f}{\partial E_1} ,$$

$$b = M_1 \sin\phi ,$$

$$c = M_1(E_2/E_1)\cos\phi ,$$

$$d = -M_2\sin\phi ,$$

$$e = f(I_2) - \Gamma_2 + E_2\frac{\partial f}{\partial E_2} ,$$

$$f = -M_2\cos\phi ,$$

$$g = -\xi E_1 \frac{\partial f}{\partial E_1} - \left(\frac{M_2E_1}{E_2} + \frac{M_1E_2}{E_1}\right)\cos\phi ,$$

$$h = \xi E_1 \frac{\partial f}{\partial E_2} + \left(\frac{M_2E_1}{E_2} + \frac{M_1E_2}{E_1}\right)\frac{E_1}{E_2}\cos\phi ,$$

$$j = \left(\frac{M_2E_1}{E_2} - \frac{M_1E_2}{E_1}\right)\sin\phi ,$$

with the restrictions

$$[f(I_1) - \Gamma_1]E_1 = -M_1E_2\sin\phi ,$$

$$[f(I_2) - \Gamma_2]E_2 = M_2E_1\sin\phi ,$$

$$\Omega = \xi[f(I_1) - f(I_2)] + (M_2E_1/E_2 - M_1E_2/E_1)\cos\phi ,$$

from which it can be seen that

$$j = f(I_1) + f(I_2) - \Gamma_1 - \Gamma_2 .$$

When the right-hand cavity is empty, the stability criterion is obtained by removing terms containing $f(I_2)$ in the above.

---

*Research sponsored by the Air Force Office of Scientific Research, USAF, under contract F44620-71-C-0042 and in part by NASA.

†Based on material submitted by M. B. Spencer in partial fulfillment of the requirements for the degree of Doctor of Philosophy at Yale University.

[1]M. B. Spencer and W. E. Lamb, Jr., preceding paper, Phys. Rev. A 5, 884 (1972).

[2]To obtain the physical fields we take the real parts of the complex fields.

[3]This is derived as in Sec. V below or as in Ref. 1.