$$\times \int_0^{x_2} x_1^{3/2} I_{k+1/2}(a'x_1) e^{s(x_1)} [e^{-\beta w(x_1)} j_k(LG(x_1)) - j_k(Lq(x_1))] dx_1 dx_2$$

$$- \delta_{k0} \int_0^{\infty} x_2^{3/2} I_{1/2}(a'x_2) e^{2s(x_2)} (e^{-2\beta\omega(x_2)} - 1) \int_{x_2}^{\infty} x_1^{3/2} K_{1/2}(a'x_1) e^{s(x_1)} (e^{-\beta w(x_1)} - 1) dx_1 dx_2$$

$$- \delta_{k0} \int_0^{\infty} x_2^{3/2} K_{1/2}(a'x_2) e^{2s(x_2)} (e^{-2\beta w(x_2)} - 1) \int_0^{x_2} x_1^{3/2} I_{1/2}(a'x_1) e^{s(x_1)} (e^{-\beta w(x_1)} - 1) dx_1 dx_2 \Bigg\} .$$

With these expressions for $I_{jm}(l)$ up to second order, we can evalute $T(l)$:

$$T(l) = \exp[-\gamma L^2 + I_{10}(l) + I_{01}(l) + I_{20}(l) + I_{02}(l) + I_{11}(l)] . \tag{A69}$$

If we omit the last three terms in the exponent we get

$$T(l) = \exp[-\gamma L^2 + I_{10}(l) + I_{01}(l)] , \tag{A70}$$

which is referred to as the first approximation to $T(l)$. Then in this sense the second approximation to $T(l)$ is given by Eq. (A69).

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Laser with a Transmitting Window*

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The effect of transmission of radiation through one mirror of a laser is investigated. For a laser oscillator the result is to change the effective resonance frequency and Q of the cavity. Using the same model for the cavity, a signal is injected into the active medium through the transmitting window, and its effect on the system studied. When the external signal is strong enough and sufficiently close to the natural frequency of the laser oscillator, the laser locks its frequency to the input signal. The equations describing the system are solved over the range of input frequencies where the laser is locked, and the resulting gain found. In the high-intensity limit the medium saturates, and the gain tends to that of a lossy cavity. As the input intensity vanishes, the gain approaches infinity and the system tends to a laser oscillator.

I. INTRODUCTION

It is the purpose of this paper to investigate the effects on the operation of a laser arising from the fact that to some extent it is in communication with the rest of space outside the resonant cavity; i.e., some of its internal energy is escaping through the windows. As a result the effective cavity Q is lowered, and there is a slight change in operating frequency. Furthermore, using the same techniques it is possible to consider the case of an external signal applied to the laser through one of its windows. The reflected wave may be amplified or otherwise modified by the active device. Under certain circumstances the applied signal may cause the laser frequency to lock to the applied frequency, giving a single-frequency amplified output.

A one-dimensional model for the laser cavity is taken to consist of one totally reflecting mirror, and one partially transmitting window consisting of a "bump" in the dielectric constant. The window is taken to have a reflection coefficient slightly less than unity to simulate the leakage through, and diffraction around, a real laser mirror.

The calculation of the field in the laser cavity is similar to that used in earlier laser theory[1] with the difference that the "leaky" mirror introduces an extra term into Maxwell's equations. This extra term results in the field being "driven" not only by the polarization of the inverted medium, but also by the field at the window. The polarization is calculated quantum mechanically using an assumed field strength and frequency, and the usual self-consistency requirements determine the actual field strengths and frequencies.

II. MAXWELL'S EQUATIONS APPLIED TO ONE-DIMENSIONAL CAVITY WITH A PARTIALLY TRANSMITTING WINDOW

Since the window will be taken to be only slightly transmitting, it will be convenient to consider the leakage through the mirror as a small perturbation, and that the field inside the cavity can be well represented by its expansion in normal modes of a closed cavity of similar dimensions.

The cavity of length L may be taken to lie along the z axis between $-L$ and 0, with the electric field polarized in the x direction and the magnetic field polarized in the y direction.

As discussed by Slater,[2] basis modes suitable for expansion of the electric and magnetic fields are

$$u_n(z) = \sin k_n z \text{ and } v_n(z) = \cos k_n z \ , \tag{1}$$

with

$$k_n = n\pi/L \ . \tag{1a}$$

They satisfy the relationships

$$\frac{\partial u_n(z)}{\partial z} = k_n v_n(z) \ , \tag{2a}$$

$$\frac{\partial v_n(z)}{\partial z} = -k_n u_n(z) \ , \tag{2b}$$

but are unnormalized. Fields in the cavity are expanded as

$$E(z, t) = \textstyle\sum_n A_n(t) u_n(z) \ , \tag{3}$$

$$H(z, t) = \textstyle\sum_n H_n(t) v_n(z) \ , \tag{4}$$

while the expansion of their derivatives follows from the above definitions:

$$\frac{\partial E(z,t)}{\partial z} = \frac{2}{L} \sum_n v_n(z) \left(\int_{-L}^{0} dz\, v_n(z) \frac{\partial E(z,t)}{\partial z} \right)$$
$$= \sum_n v_n(z) \left(\frac{2}{L} E(0,t) + k_n A_n(t) \right) \ , \tag{5}$$

$$-\frac{\partial H(z,t)}{\partial z} = \frac{-2}{L} \sum_n u_n(z) \left(\int_{-L}^{0} dz\, u_n(z) \frac{\partial H(z,t)}{\partial z} \right)$$
$$= \textstyle\sum_n u_n(z) k_n H_n(t) \ . \tag{6}$$

The polarization of the medium in the cavity can also be expanded in terms of the u_n:

$$P(z, t) = \textstyle\sum_n P_n(t) u_n(z) \ , \tag{7}$$

where $P_n(t)$ is the projection of the polarization on the nth cavity mode,

$$P_n(t) = (2/L) \textstyle\int_{-L}^{0} u_n(z) P(z,t)\, dz \ . \tag{8}$$

Maxwell's equations in conjunction with (5) and (6) become

$$k_n A_n(t) + \mu_0 \dot{H}_n(t) + (2/L) E(0,t) = 0 \ , \tag{9}$$

$$k_n H_n(t) - \epsilon_0 \dot{A}_n(t) = J_n + \dot{P}_n(t) \ , \tag{10}$$

where, as usual, we set $J_n = \sigma A_n$ to provide some loss in the cavity. We then obtain the wave equation

$$\epsilon_0 \mu_0 \ddot{A}_n(t) + \sigma \mu_0 \dot{A}_n(t) + k_n^2 A_n(t)$$
$$= -(2/L) k_n E(0,t) - \mu_0 \ddot{P}_n(t) \ . \tag{11}$$

The first term on the right-hand side expresses the fact that although the main electric field in the laser will have the space dependence of the basis modes $u_n(z)$ which vanish at the mirrors, a small amount of radiation leaking through the window means that the electric field is not quite zero at the origin.

Since the window will be taken to be strongly reflecting, the resulting field at the origin will be small, thus minimizing the convergence problems of (3). With this in mind we can consider a single mode and from now on drop the subscript n, writing

$$\ddot{A}(t) + (\sigma/\epsilon_0) \dot{A}(t) + \Omega^2 A(t)$$
$$= -(2/L) \Omega c E(0,t) - (1/\epsilon_0) \ddot{P}(t) \ , \tag{12}$$

where $\Omega = k(\epsilon_0 \mu_0)^{-1/2}$ is the cavity resonance frequency. The field at the window provides a "driving force" in the equation for the field in the cavity in addition to the usual term provided by the laser medium.

III. MODEL FOR TRANSMITTING WINDOW

The window at $z = 0$ is taken to be a "bump" in the dielectric permittivity given by

$$\epsilon(z)=\epsilon_0[1+\Lambda\delta(z)] . \quad (13)$$

Λ is a parameter, with dimensions of length, that determines the reflectivity of the window.

In order to find how the window modifies the field, we integrate Maxwell's equation across the bump. From

$$\frac{\partial E(z,t)}{\partial z}=-\mu_0\dot{H}(z,t) , \quad (14)$$

we find that

$$E(0^+,t)=E(0^-,t) , \quad (15)$$

while from

$$\frac{-\partial H(z,t)}{\partial z}=\epsilon(z)\dot{E}(z,t) , \quad (16)$$

we see that there is a discontinuity of

$$H(0^+,t)-H(0^-,t)=-\Lambda\epsilon_0\dot{E}(0,t) \quad (17)$$

in the magnetic field related to the "height" of the dielectric bump and the time rate of change of the electric field strength at the mirror.

IV. POLARIZATION OF MEDIUM

The active atoms are taken to have two levels a and b, separated by energy $\hbar\omega$ and to be represented by a density matrix ρ rather than a wave function. In order to avoid unnecessary complications, the atoms are considered stationary.

The equation of motion of the density matrix is

$$\dot{\rho}=-i[H,\rho]-\tfrac{1}{2}(\Gamma\rho+\rho\Gamma)+\lambda ,$$

where

$$\rho=\begin{pmatrix}\rho_{aa} & \rho_{ab}\\ \rho_{ba} & \rho_{bb}\end{pmatrix} , \quad H=\begin{pmatrix}W_a & V\\ V & W_b\end{pmatrix} ,$$
$$\Gamma=\begin{pmatrix}\gamma_a & 0\\ 0 & \gamma_b\end{pmatrix} , \quad \lambda=\begin{pmatrix}\lambda_a & 0\\ 0 & \lambda_b\end{pmatrix} . \quad (18)$$

The perturbation Hamiltonian is $\hbar V$, while $\hbar W_a$ and $\hbar W_b$ are the unperturbed energies of the levels a and b. The two levels decay naturally with damping constants γ_a and γ_b, and are populated by pumping at rates λ_a and λ_b.

Using the expansion (3) of the electric field, the electric dipole approximation for the perturbation becomes

$$V(t)=-A(t)\wp\, u(z)/\hbar . \quad (19)$$

The time dependence of the electric field $A(t)$ is written in terms of an amplitude $E(t)$ and phase $\phi(t)$, which vary slowly in an optical period $2\pi/\nu$,

$$A(t)=E(t)\cos[\nu t+\phi(t)] . \quad (20)$$

The quantity

$$\mathcal{E}(t)=E(t)e^{-i[\nu t+\phi(t)]} \quad (21)$$

is the positive frequency part of the electric field, and also satisfies Eq. (12) with $E(0,t)$ and $P(t)$ suitably defined.

Thus the off-diagonal term in the density-matrix equation may be written

$$\dot{\rho}_{ab}=-(i\omega+\gamma)\rho_{ab}-i(\rho_{aa}-\rho_{bb})A(t)\wp\, u(z)/\hbar ,$$

where (22)

$$\gamma=\tfrac{1}{2}(\gamma_a+\gamma_b) .$$

The steady-state solution of this equation gives

$$\rho_{ab}=-\tfrac{1}{2}i(\rho_{aa}-\rho_{bb})\mathcal{E}(t)\wp\, u(z)/\{\hbar[\gamma+i(\omega-\nu)]\} , \quad (23)$$

where it is assumed that the population inversion varies slowly compared with ρ_{ab}, and we have used the rotating-wave approximation.

Combining (19) and (23), we find

$$iV(t)(\rho_{ab}-\rho_{ba})=-\frac{1}{2}\left(\frac{\wp}{\hbar}\right)^2(\rho_{aa}-\rho_{bb})A(t)\times\left(\frac{\mathcal{E}(t)}{\gamma+i(\omega-\nu)}+\frac{\mathcal{E}^*(t)}{\gamma-i(\omega-\nu)}\right)u^2(z) . \quad (24)$$

If only the slowly varying terms are kept in this expression, the diagonal terms in the equation of motion (18) become rate equations,

$$\dot{\rho}_{aa}=-\gamma_a\rho_{aa}+\lambda_a+R(\rho_{bb}-\rho_{aa}) , \quad (25a)$$

$$\dot{\rho}_{bb}=-\gamma_b\rho_{bb}+\lambda_b-R(\rho_{bb}-\rho_{aa}) , \quad (25b)$$

with a rate R of transitions $a \leftrightarrow b$

$$R=[\wp^2E(t)^2/2\gamma\hbar^2]\,\mathfrak{L}(\omega-\nu)u^2(z)$$
$$=IR_s\mathfrak{L}(\omega-\nu)u^2(z) , \quad (26)$$

where

$$R_s=\tfrac{1}{2}\gamma_a\gamma_b/\gamma , \quad (27a)$$

$$I=\wp^2E(t)^2/\hbar^2\gamma_a\gamma_b , \quad (27b)$$

$$\mathfrak{L}(\omega-\nu)=\gamma^2/[\gamma^2+(\omega-\nu)^2] . \quad (27c)$$

I is the dimensionless intensity of the field in the cavity, while the Lorentzian factor $\mathfrak{L}(\omega-\nu)$ describes the effect of detuning the laser frequency from the atomic frequency.

The solution of (25) for the steady-state population inversion is

$$\rho_{aa}-\rho_{bb}=\frac{\lambda_a/\gamma_a-\lambda_b/\gamma_b}{1+R/R_s} , \quad (28)$$

which can be substituted in Eq. (23) for ρ_{ab}. The polarization of the medium is given by

$$P(z,t)=\wp(\rho_{ab}+\rho_{ba})$$
$$=-\left(\frac{\tfrac{1}{2}i(\wp^2/\hbar)\mathcal{E}(t)N(z)u(z)}{(1+R/R_s)[\gamma+i(\omega-\nu)]}+\text{c. c.}\right) . \quad (29)$$

$N(z)$ is the steady-state population inversion

$(\lambda_a/\gamma_a - \lambda_b/\gamma_b)$ that would exist in the absence of induced transitions. The denominator factor $(1+R/R_s)$ describes the saturation of the medium due to the presence of optical radiation in the cavity.

Projecting $P(z,t)$ onto the cavity mode[3] to find $P(t)$ for use in (12) gives

$$P(t) = -\left(\frac{\frac{1}{2}i\wp^2\bar{N}/\hbar}{\gamma+i(\omega-\nu)}\,\frac{(2\mathcal{E})}{I\mathcal{L}(\omega-\nu)}\times\{1-[1+I\mathcal{L}(\omega-\nu)]^{-1/2}\}+\text{c.c.}\right), \quad (30)$$

where

$$\bar{N} = (1/L)\int_{-L}^{0} N(z)\,dz\ .$$

If we define

$$\alpha = (\tfrac{1}{2}\nu\wp^2/\epsilon_0\hbar\gamma)\,\bar{N}\mathcal{L}(\omega-\nu) \quad (31)$$

and

$$f(I) = [2\alpha/I\mathcal{L}(\omega-\nu)]\{1-[1+I\mathcal{L}(\omega-\nu)]^{-1/2}\}, \quad (32)$$

the saturation gain parameter, then the polarization can be written

$$P(t) = -2i\mathcal{E}\,\frac{\epsilon_0}{\nu}\,\frac{\gamma-i(\omega-\nu)}{\gamma}\,f(I)+\text{c.c.} \quad (33)$$

For low intensities $f(I)\sim\alpha-\beta I$, where

$$\beta = \tfrac{3}{4}\,\alpha\mathcal{L}(\omega-\nu)\ , \quad (34)$$

and (33) has a form reminiscent of the small signal single-mode theory of Ref. 1.

V. APPLICATION TO A LASER RADIATING INTO FREE SPACE

We can calculate the magnetic field just inside the laser cavity in terms of the amplitude of the electric field in the cavity. Similarly the magnetic field just outside the cavity is related to the electric field outside the cavity. From these expressions, the electric field at the mirror can be found using (17). Throughout the following we shall find it convenient to work with complex quantities whose real parts are the corresponding physical quantities. In the same way as $\mathcal{E}(t)$ was introduced, we define $\mathcal{P}(t)$ as the positive frequency part of the polarization $P(t)$, and introduce a complex magnetic field $\mathcal{H}(t)$. Inside the laser, Maxwell's equations give

$$(k/\mu_0)\,\mathcal{E}(t) = -\dot{\mathcal{H}}(t) = ikc\mathcal{H}(t)\ . \quad (35)$$

Therefore (4) can be rewritten as

$$H(z,t) = -(i/Z_0)\mathcal{E}(t)v(z)\ , \quad (36)$$

where $Z_0 = (\mu_0/\epsilon_0)^{1/2}$ is the impedance of free space.

Outside the cavity the fields take the form of a plane wave travelling away from the laser in the $+z$ direction:

$$E(z,t) = E_+(t)e^{ikz}\ , \quad z\geq 0 \quad (37)$$

$$H(z,t) = Z_0^{-1}E_+(t)e^{ikz}\ , \quad z\geq 0\ . \quad (38)$$

From (15) and (37) we see

$$E_+(t) = E(0,t)\ , \quad (39)$$

so that substitution of (36), (38), and (39) in (17) leads to a relationship between $E(0,t)$ and the electric field amplitude $\mathcal{E}(t)$ in the laser:

$$E(0,t) = \mathcal{E}(t)/(\eta+i)\ , \quad (40)$$

where the dimensionless quantity $\eta = k\Lambda$ measures the strength of the dielectric "bump."

Replacing $E(0,t)$ in (12) by (40) gives the complex electric field amplitude and phase equation for a laser radiation into free space:

$$\ddot{\mathcal{E}}(t) + \frac{\sigma}{\epsilon_0}\,\dot{\mathcal{E}}(t) + \Omega^2\mathcal{E}(t) + \frac{2\Omega c}{L(\eta+i)}\,\mathcal{E}(t) = \frac{\ddot{\mathcal{P}}(t)}{\epsilon_0}$$

or

$$\ddot{A}(t) + \left(\frac{\sigma}{\epsilon_0} + \frac{2c}{L(\eta^2+1)}\right)\dot{A}(t) + \Omega^2\left(1+\frac{2c\eta}{L\Omega(\eta^2+1)}\right)A(t) = \frac{\ddot{P}(t)}{\epsilon_0}\ , \quad (41)$$

where the imaginary part of the new term has been moved to the term in $\dot{A}(t)$. An equation of this form was solved in Ref. 1. It is seen that the partially transmitting window introduces some extra loss and a small change in operating frequency.

The increase in cavity bandwidth introduced by the window is

$$\Omega/Q_w = 2c/L(\eta^2+1)\ .$$

When a plane wave is incident on an isolated dielectric discontinuity (13), the intensity transmission coefficient is

$$T' = 4/(\eta^2+4)\ . \quad (42)$$

Thus, in the case at hand where η is large, the increase in bandwidth introduced by the transmitting window may be written

$$\Omega/Q_w = (c/2L)T'\ . \quad (43)$$

The right-hand side can be interpreted as the fractional rate of loss of energy through the mirror, since $2L/c$ is the average time taken by a photon to return to the window, and T' is the probability of transmission.

The total Q of the cavity is given by

$$Q^{-1} = Q_w^{-1} + Q_0^{-1}\ , \quad (44)$$

where Q_0 is the unloaded Q of the cavity, and Q_w is the external Q due to the window.

The (time-averaged) power emitted by the laser may be found by evaluating the Poynting vector for the external radiation:

$$S = |\vec{E}\times\vec{H}| = E^2/2Z_0(\eta^2+1)\ . \tag{45}$$

Using (43) this can be written

$$S = (\Omega/Q_w)(\tfrac{1}{4}\epsilon_0 E^2 L)$$
$$= (\Omega/Q_w)W\ , \tag{46}$$

where W is the electromagnetic energy in the cavity.

The frequency shift $\delta\Omega \approx [(c/L)(\Omega/2Q_w)]^{1/2}$, although a small fraction of Ω, can be significant compared to the atomic width γ.

VI. LASER SUBJECTED TO EXTERNAL SIGNAL

As in Sec. V we consider a laser cavity with one partially reflecting mirror. Incident on this mirror is a plane wave with frequency close to the frequency at which the undisturbed laser would oscillate. Most of the incident wave is reflected from the mirror, but a fraction is transmitted and acts as a source for the internal field.

If the laser is below threshold, it can amplify or attenuate the incident signal depending on the gain parameter. Above threshold the laser will oscillate at its natural frequency ν_0 in the absence of an input signal. The incident signal will beat with the laser output unless it is close to ν_0 and sufficiently strong to lock the laser to its frequency.

In the same way as for the laser oscillator of Sec. V, we take the fields in the laser cavity to be

$$E(z,t) = \mathcal{E}(t)u(z)\ , \tag{47}$$

$$H(z,t) = -(i/Z_0)\mathcal{E}(t)v(z)\ . \tag{48}$$

The external fields consist of an incident wave $\mathcal{E}_I(t)$, a reflected wave $\mathcal{E}_R(t)$, and a field from the laser $\mathcal{E}_L(t)$. We write

$$E(z,t) = \mathcal{E}_I(t)e^{-ik'z} + \mathcal{E}_R(t)e^{ik'z} + \mathcal{E}_L(t)e^{ikz}\ , \tag{49}$$

$$H(z,t) = [-\mathcal{E}_I(t)e^{-ik'z} + \mathcal{E}_R(t)e^{ik'z} + \mathcal{E}_L(t)e^{ikz}]/Z_0\ , \tag{50}$$

where the incident wave has frequency ν' and wave number k'.

Substituting Eqs. (47)–(50) in (17), we find

$$-\mathcal{E}_I(t) + \mathcal{E}_R(t) + \mathcal{E}_L(t) + i\mathcal{E}(t) = i\eta\,[\mathcal{E}_L(t) + \mathcal{E}_R(t) + \mathcal{E}_I(t)]\ , \tag{51}$$

with

$$\eta = k\Lambda \gg 1\ . \tag{52}$$

We can solve (51) for the (small) electric field at the window:

$$E(0,t) = \mathcal{E}_L(t) + \mathcal{E}_R(t) + \mathcal{E}_I(t)$$
$$= [\mathcal{E}(t) + 2i\mathcal{E}_I(t)]/(\eta + i)\ . \tag{53}$$

Using (53), Eq. (12) becomes

$$\ddot{\mathcal{E}}(t) + \frac{\sigma}{\epsilon_0}\dot{\mathcal{E}}(t) + \Omega^2\mathcal{E}(t) + \frac{2\Omega c}{L(\eta+i)}\ \mathcal{E}(t)$$
$$= \frac{-\ddot{\mathcal{P}}(t)}{\epsilon_0} - \frac{4\Omega c/L}{(1-i\eta)}\mathcal{E}_I(t)\ . \tag{54}$$

Writing $\mathcal{E}_I(t)$ and $\mathcal{E}(t)$ in terms of amplitude and phase factors we have

$$\mathcal{E}_I(t) = Fe^{-i\nu' t}\ , \quad \mathcal{E}(t) = E(t)e^{-i\nu t} = E(t)e^{-i[\nu' t + \phi(t)]}\ , \tag{55}$$

where the real amplitude $E(t)$ and relative phase angle $\phi(t)$ are slowly varying compared to $e^{-i\nu t}$. We have written the relation between the input frequency and the laser frequency

$$\nu = \nu' + \dot{\phi}\ .$$

Defining

$$1/(1-i\eta) = iTe^{-i\psi}\ , \tag{56}$$

where

$$T = (1+\eta^2)^{-1/2} \text{ and } \psi = \tan^{-1}(1/\eta)\ , \tag{57}$$

and using (33), Eq. (54) can be written

$$\dot{E}(t) = [f(I) - \Gamma]E(t) + (2c/L)TF\cos(\psi - \phi)\ , \tag{58}$$

$$\dot{\phi}(t) = \xi f(I) - \Delta + (2c/L)T[F/E(t)]\sin(\psi - \phi)\ , \tag{59}$$

where

$$\Gamma = \sigma/2\epsilon_0 + c/L(\eta^2+1) = \Gamma_0 + \Gamma_w \tag{60}$$

is the effective cavity bandwidth of the loaded laser. The phase equation (59) depends on the detuning parameter

$$\xi = (\omega - \nu)/\gamma\ , \tag{61}$$

and the separation

$$\Delta = \nu' - \Omega' = \nu' - \Omega - \frac{c}{L}\ \frac{\eta}{\eta^2+1} \tag{62}$$

of the input frequency ν' from the effective cavity resonance frequency Ω'.

Under certain conditions the phase difference $\phi(t)$ will tend to a constant, meaning that the laser frequency has "locked" to the frequency of the input signal. From the form of (59) this can only occur if the amplitude $E(t)$ is constant. Therefore the frequency-locked solutions of (58) and (59) are the stationary states of the system

$$\dot{E}(t) = 0 \text{ and } \dot{\phi}(t) = 0\ . \tag{63}$$

If we define a dimensionless input intensity

$$\mathcal{I} = (\wp F)^2/\hbar^2\gamma_a\gamma_b\ , \tag{64}$$

the stationary laser intensity I [defined as in (27b)] can be found as the solution of the transcendental equation

$$(2c/L)^2T^2\mathcal{I} = I\{[\Gamma - f(I)]^2 + [\Delta - \xi f(I)]^2\}\ . \tag{65}$$

For low intensities this equation takes the form of a cubic in I, while at high intensities, when

$f(I)\to 0$, it becomes linear, with solution

$$I=\frac{(2c/L)^2T^2}{\Gamma^2+\Delta^2}\,\mathcal{I}\,. \tag{66}$$

Graphs of (65) above threshold for various values of the parameters are shown in Figs. 1 and 2. Below threshold, when $\alpha<\Gamma$, only one solution for $\mathcal{I}$ occurs for each input intensity I and that solution is stable.

Above threshold, the cubic shape of (65) makes it possible to have three solutions for I corresponding to small values of $\mathcal{I}$. It is shown in the Appendix that at most two, and usually only one of these solutions is stable. When $\mathcal{I}=0$, Eq. (65) has a nonzero root only when $\alpha>\Gamma$ and

$$\Delta=\xi\Gamma\,, \tag{67}$$

implying that the device will oscillate with no input signal only if condition (67) is satisfied. This condition can be reexpressed in terms of the effective cavity resonance frequency and the atomic resonance frequency to give the well-known[4] expression for the frequency ν_0 at which the laser would oscillate without an input signal:

$$\nu_0=(\gamma\Omega'+\omega\Gamma)/(\gamma+\Gamma)\,. \tag{68}$$

If the input frequency ν' is detuned slightly from ν_0, the curve (65) will no longer touch the $\mathcal{I}$ axis, so that a certain minimum input intensity $\mathcal{I}(I_0)$ will be required before the laser will lock to the input signal. Figures 3 and 4 show the transient behavior of the amplitude and frequency difference when $\mathcal{I}$ is less than and greater than $\mathcal{I}(I_0)$. In the former case the laser field cannot settle down to a

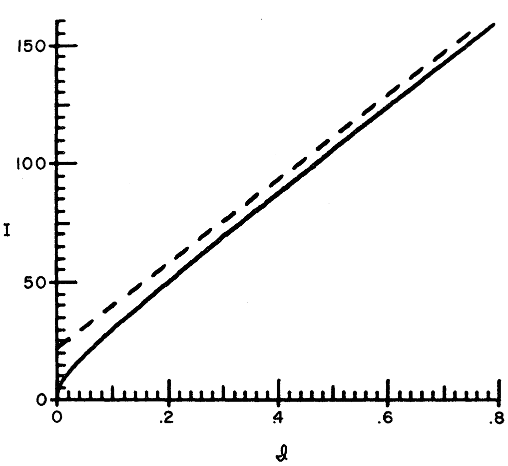

FIG. 1. Plot of Eq. (65) for high input intensities showing the variation of the steady-state intensity I in the laser cavity with input intensity $\mathcal{I}$. The input frequency ν' is taken to be tuned to the natural laser frequency ν_0, and $\Gamma=1$MHz, $\Gamma_w=\frac{3}{4}$ MHz, $\gamma=99$ MHz, $\omega-\Omega'=200$ MHz, and $\alpha=1.1$ MHz. The dashed line is a plot of (66) showing the asymptotic high-intensity behavior.

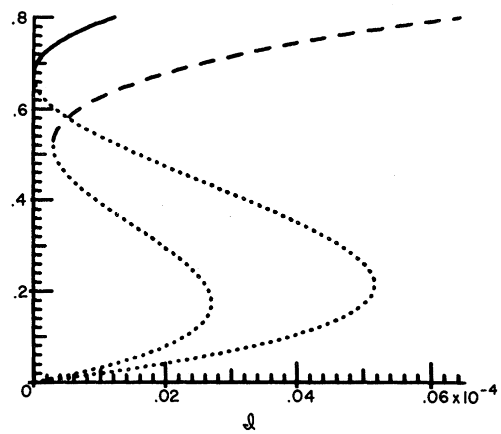

FIG. 2. Plot of Eq. (65) for very low input intensities showing the cubic shape. The unstable parts of the curves are shown dotted. The continuous line shows the behavior when $\nu'=\nu_0$, while the dashed line shows (65) with $\nu'=\nu_0=0.05$ MHz. The other parameters are the same as in Fig. 1.

steady state, but has amplitude and phase modulation.

The factor $(2c/L)T$ in (65) can be much larger than Γ, so that even in the case of a linear passive cavity described by Eq. (66) we may find $I\sim 400\mathcal{I}$. Despite the reflection of the incident signal by the mirror, the same property of high reflectance traps the field in the cavity and allows it to become larger than the incident field.

VII. GAIN OF LASER AMPLIFIER

From (49) the complex voltage gain of the amplifier is

$$\mathcal{G}=(\mathcal{E}_L+\mathcal{E}_R)/\mathcal{E}_I\,. \tag{69}$$

Rewriting (51) as

$$[\mathcal{E}_L(t)+\mathcal{E}_R(t)](1-i\eta)=\mathcal{E}_I(t)(1+i\eta)-i\mathcal{E}(t)\,, \tag{70}$$

(69) becomes

$$\mathcal{G}=-\frac{\eta-i}{\eta+i}\left(1-\frac{\mathcal{E}(t)}{\mathcal{E}_I(t)(\eta-i)}\right) \tag{71}$$

or

$$\mathcal{G}=-\frac{\eta-i}{\eta+i}\left(1-\frac{E(t)e^{-i\phi}}{(\eta-i)F}\right)\,. \tag{72}$$

In the steady state (58) and (59) imply

$$F[\cos(\psi-\phi)-i\sin(\psi-\phi)]$$
$$=-(L/2c)[E(t)/T]\{[f(I)-\Gamma]-i[\xi f(I)-\Delta]\} \tag{73}$$

or

$$E(t)e^{-i\phi}/F=-\frac{e^{-i\psi}(2c/L)T}{[f(I)-\Gamma]-i[\xi f(I)-\Delta]} \tag{74}$$

$$=-\frac{(2c/L)(\eta-i)/(\eta^2+1)}{[f(I)-\Gamma]-i[\xi f(I)-\Delta]}\,. \tag{75}$$

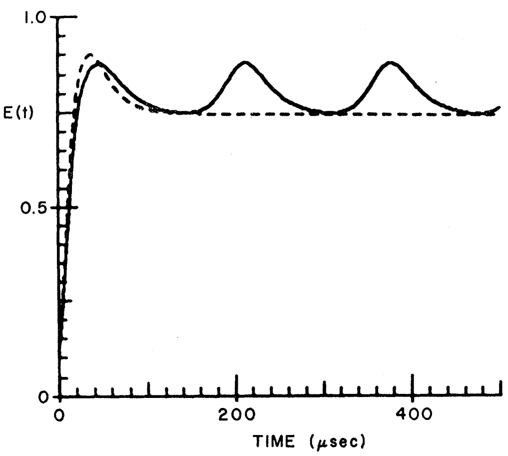

FIG. 3. Examples of the transient behavior of the electric field amplitude $E(t)$ in the cavity, as described by Eq. (58), where $\nu' - \nu_0 = 0.05$ MHz as in Fig. 2. The continuous curve shows the case when the input intensity is insufficient to lock the laser ($\mathcal{I} = 1.6\times10^{-7}$), while the dashed curve shows the laser approaching a stable locked state ($\mathcal{I} = 3.6\times10^{-7}$). The remaining parameters are the same as in the previous figures.

Therefore

$$\mathcal{G} = -\frac{\eta - i}{\eta + i}\left(1 + \frac{2\Gamma_w}{[f(I) - \Gamma] - i[\xi f(I) - \Delta]}\right), \quad (76)$$

where

$$\Gamma_w = c/L(\eta^2 + 1) = \tfrac{1}{2}\,\Omega/Q_w \quad (77)$$

is the cavity bandwidth due to the window [see Eq. (60)]. The voltage gain in (72) tends to -1 in the limit of a perfectly reflecting mirror, indicating a phase shift of π but no change in amplitude. The term proportional to $E(t)$ in (72) describes the contribution from the field in the cavity. This term becomes infinite when the laser is oscillating with no input signal.

If we remove the nonlinear laser medium from the cavity but leave the linear losses, the voltage gain (76) becomes

$$\mathcal{G} = \frac{\eta - i}{\eta + i}\;\frac{\Gamma_w - \Gamma_0 + i\Delta}{\Gamma_w + \Gamma_0 - i\Delta}\;. \quad (78)$$

The cavity will be matched to the input if $\Gamma_w = \Gamma_0$, making the window losses equal to the linear losses in the cavity. If the input frequency is tuned to the effective cavity resonance frequency Ω', so that $\Delta = 0$, the entire input signal will be absorbed in the cavity, and the voltage gain $\mathcal{G} = 0$.

Removing both the laser medium and the linear losses from the cavity results in

$$\mathcal{G} = \frac{\eta - i}{\eta + i}\;\frac{\Gamma_w + i\Delta}{\Gamma_w - i\Delta}\;, \quad (79)$$

which has unit amplitude. In this case when the input frequency equals the natural cavity resonance frequency Ω,

$$\Delta = -\eta\Gamma_w \quad (80)$$

and

$$\mathcal{G} = -1\,, \quad (81)$$

so that the laser cavity acts like a short circuit at $z = 0$.

The intensity gain G is obtained from (76) as

$$G = |\,\mathcal{G}\,|^2 = 1 + \frac{4\Gamma_w[f(I) - \Gamma_0]}{[f(I) - \Gamma]^2 + [\xi f(I) - \Delta]^2}\;, \quad (82)$$

where I is the stable solution of (65) for given input intensity $\mathcal{I}$. If the laser is below threshold, it will still amplify as long as the gain from the inverted medium exceeds the linear losses [$f(I) > \Gamma_0$]. The critical condition $f(I) = \Gamma_0$ when $G = 1$ is just the threshold condition for the same medium in a cavity without window. An expression similar to (82) has been obtained previously,[5] in the special case $\Delta = 0$, $\xi = 0$, and without the expression (32) for $f(I)$. Figure 5 shows the intensity gain plotted against input frequency for various values of $\mathcal{I}$, over the range of frequencies about ν_0 where the laser is frequency locked. This range of locking varies with input intensity, being smaller for lower intensity. For the lowest intensity shown in Fig. 5, the range of locking is about 4.5 MHz. The curves with smaller $\mathcal{I}$ show larger gain at $\nu' = \nu_0$. As $\mathcal{I}$ is increased G decreases, and the range of locked frequencies increases. In the limit of very large $\mathcal{I}$, $f(I) \to 0$ and the medium becomes saturated. This is equivalent to removing the active medium, leaving only the cavity with its linear loss. In this limit

$$G = [(\Gamma_w - \Gamma_0)^2 + \Delta^2]/[(\Gamma_w + \Gamma_0)^2 + \Delta^2]\,, \quad (83)$$

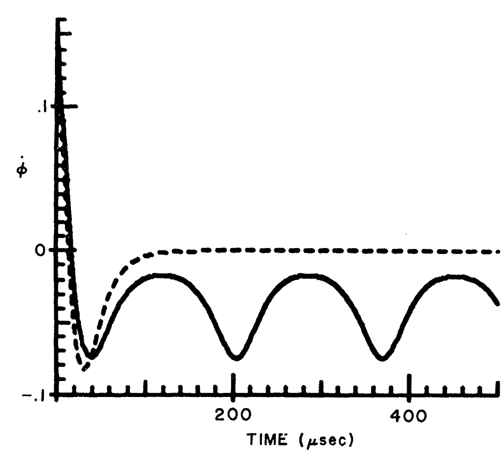

FIG. 4. Examples of the transient behavior of the frequency difference $\dot{\phi} = \nu' - \nu$ corresponding to the two cases in Fig. 3. In the locked case $\nu' - \nu \to 0$ after its initial excursion.

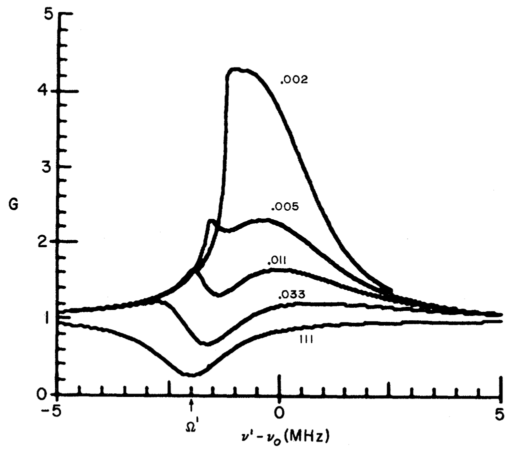

FIG. 5. Intensity gain G given by (82) plotted against $\nu' - \nu_0$, the difference between the incident frequency and the natural frequency of the laser, for various dimensionless input intensities. High intensities saturate the medium giving a characteristic dip in gain near the effective cavity resonance frequency Ω'. Values of the parameters used are $\Gamma = 1$ MHz, $\Gamma_w = \frac{3}{4}$ MHz, $\gamma = 99$ MHz, $\omega - \Omega' = 200$ MHz, and $\alpha = 1.1$ MHz, making $\Omega' - \nu_0 = -2$ MHz.

which shows a characteristic minimum centered on the effective cavity resonance frequency Ω'. For frequencies many cavity bandwidths away from Ω', $G \sim 1$. Figure 5 shows that for large input intensity G does in fact take this form. The dip persists for smaller input intensities because the laser intensity I becomes locally large and saturates the medium. The detuning of the effective cavity resonance frequency Ω' from the atomic resonance ω is purposely taken large in the calculations in order to separate Ω' from the natural laser frequency ν_0.

VIII. COMPARISON WITH PREVIOUS TREATMENT

A qualitative treatment of the laser amplifier was given by Wittke in 1957,[6] in which he took a simple model for the power emitted by the atoms. He then equated the sum of the powers from the atoms and the signal generator to the total power absorbed by loss mechanisms, and obtained an equation similar to (58) with $\dot{E} = 0$.

Our treatment differs from Wittke's in three respects. We use field amplitudes rather than intensities and can treat the phenomenon of phase locking. This also allows us to examine the solution when the input frequency does not equal the effective cavity resonance frequency. Secondly we are able to allow for the fact that the active medium is confined in a finite cavity and so find expressions for quantities that must otherwise be described by "filling factors." Thirdly, our treatment does not assume that the Q_0 of the cavity is matched to the Q attributed to the window, Q_w.

If we incorporate Wittke's assumptions ($\Gamma_0 = \Gamma_w$, $\Delta = 0$, and $\xi = 0$) in (82), the steady-state equation for the intensity amplification, we find

$$G = f^2/(f - \Gamma)^2 . \tag{84}$$

An expression similar to (84) may be obtained using Wittke's method. Calling the field in the cavity

$$E = E_i + E_0 , \tag{85}$$

where E_i is the amplitude of the incident radiation and E_0 is that of the output (taken to be in phase), the loss of power from the cavity is assumed to be

$$P_L = \Gamma'(E^2 + E_0^2) . \tag{86}$$

The first term is due to linear losses in the cavity, while the second is the loss out the window. Sources of power are the input field, and the inverted medium,

$$P_G = \Gamma' E_i^2 + 2f'(I)E^2 . \tag{87}$$

Γ' is proportional to the cavity bandwidth, and $f'(I)$ to the saturated gain parameter for the medium. Equating P_L to P_G and solving for $(E_0/E_i)^2$ gives an equation

$$G = f'^2/(\Gamma' - f')^2 , \tag{88}$$

similar to (84). The form taken for f' by Wittke was

$$f'(I) = \frac{\alpha'}{1 + (\beta'/\alpha')I} , \tag{89}$$

which is the saturated gain parameter for a travelling wave in free space.[7]

In the low-intensity limit (89) and (32) have the same I dependence, but the presence of the cavity modifies the parameters β' and α' by numerical factors when compared with (34). The intensity of oscillation with no input signal may be obtained from Wittke's work as

$$I = (\alpha' - \Gamma')/\beta' \tag{90}$$

in the low-intensity approximation, and again this differs from our result by numerical factors that can be absorbed into filling factors.

APPENDIX

Equations (58) and (59) have only one stable stationary state below threshold, but when $\alpha > \Gamma$ there can sometimes be three solutions of $\dot{E} = 0$ and $\dot{\phi} = 0$ for a given input field. We wish to investigate the stability of these stationary states and so expand the amplitude and phase about their stationary values E_0 and ϕ_0, giving in matrix form

$$\begin{pmatrix} \Delta\dot{E} \\ \Delta\dot{\phi} \end{pmatrix} = \begin{pmatrix} f-\Gamma+E_0\dfrac{\partial f}{\partial E} & \dfrac{2c}{L}TF\sin(\psi-\phi_0) \\ \xi\dfrac{\partial f}{\partial E}-\dfrac{2c}{L}T\dfrac{F}{E_0^2}\sin(\psi-\phi_0) & -\dfrac{2c}{L}T\dfrac{F}{E_0}\cos(\psi-\phi_0) \end{pmatrix}\begin{pmatrix} \Delta E \\ \Delta\phi \end{pmatrix}. \tag{91}$$

This can be simplified using the conditions for a steady state, $\dot{E}=0$, $\dot{\phi}=0$:

$$(2c/L)TF\cos(\psi-\phi_0)=-(f-\Gamma)E_0 \tag{92}$$

and

$$(2c/L)TF\sin(\psi-\phi_0)=-(\xi f-\Delta)E_0 \tag{93}$$

to give

$$\begin{pmatrix} \Delta\dot{E} \\ \Delta\dot{\phi} \end{pmatrix} = \begin{pmatrix} f-\Gamma+E_0\dfrac{\partial f}{\partial E} & -(\xi f-\Delta)E_0 \\ \dfrac{\xi f-\Delta}{E_0}+\xi\dfrac{\partial f}{\partial E} & f-\Gamma \end{pmatrix}\begin{pmatrix} \Delta E \\ \Delta\phi \end{pmatrix} \tag{94}$$

$$= \begin{pmatrix} g-\Gamma & -(\xi f-\Delta)E_0 \\ (\xi g-\Delta)/E_0 & f-\Gamma \end{pmatrix}\begin{pmatrix} \Delta E \\ \Delta\phi \end{pmatrix}, \tag{95}$$

where

$$g=E_0\frac{\partial f}{\partial E}+f=\{2\alpha[1+I\mathfrak{L}(\omega-\nu)]^{-3/2}-f(I)\}_0 . \tag{96}$$

Stationary solutions of (58) and (59) will be stable if the real part of the eigenvalues of this characteristic matrix are negative. Using a modified Routh's criterion[8] (or by simply observing conditions for the real parts of the roots of a quadratic equation to be negative), we find the conditions for stability are

$$(\Gamma-f)+(\Gamma-g)\geq 0 \tag{97}$$

and

$$(\Gamma-f)(\Gamma-g)+(\xi f-\Delta)(\xi g-\Delta)\geq 0 . \tag{98}$$

With the help of (96), the condition (97) implies

$$2\Gamma\geq 2\alpha[1+I\mathfrak{L}(\omega-\nu)]^{-3/2} \tag{99}$$

or

$$I\geq\frac{(\alpha/\Gamma)^{2/3}-1}{\mathfrak{L}(\omega-\nu)} .$$

This places a lower bound on the intensity in the cavity, independent of input intensity. The condition (98) is interpreted by noting that

$$E\frac{\partial f}{\partial E}=2E^2\frac{\partial f}{\partial E^2}=2I\frac{\partial f}{\partial I} ,$$

so that with the help of (96), Eq. (98) becomes

$$(\Gamma-f)\left(\Gamma-f-2I\frac{\partial f}{\partial \Gamma}\right)+(\Delta-\xi f)\left(\Delta-\xi f-2\xi I\frac{\partial f}{\partial I}\right)\geq 0$$

or using (65) and some algebra,

$$\left(\frac{2c}{L}\right)^2 T^2\frac{\partial \mathcal{I}}{\partial I}\geq 0 . \tag{100}$$

The limiting condition of equality will occur at the turning points of the curve of $\mathcal{I}$ vs I occurring for positive I and $\mathcal{I}$. Condition (100) limits the number of stable solutions for given $\mathcal{I}$ to two at most. Usually one of these is eliminated by condition (99), but it can be shown that under certain circumstances both stable states may exist for a particular $\mathcal{I}$. The range of parameters for which two stable states exist is extremely limited, and it would seem unlikely that the system would ever reach the state with smaller I. When $\nu'=\nu_0$, the upper turning point occurs with $\mathcal{I}=0$, and corresponds to the state of oscillation of a laser with no input. This is known to be stable, whereas the other solution for $\mathcal{I}=0$, (viz., $I=0$) is unstable.

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[1]W. E. Lamb, Jr., Phys. Rev. 134, A1429 (1964).

[2]J. C. Slater, *Microwave Electronics* (Van Nostrand, Princeton, N. J., 1950), p. 64. In Chap. 9, Slater gives a general theory of microwave oscillators and later applies it to klystron and magnetron oscillators. The nonlinear electronic medium is treated by approximate methods appropriate for relatively small signals. Masers and lasers are systems which are governed by the same general theory. The present paper and its sequel apply this general theory to a simple but realistic model of the nonlinear laser medium for which the calculations can be carried out for arbitrarily strong signals.

[3]See Ref. 1, Sec. 16.

[4]This is the "pulling relation" found in the usual small signal theory. That it is true in general here, independent of intensity, is a consequence of taking stationary atoms.

[5]J. Weber, Rev. Mod. Phys. 31, 681 (1959), Eq. (28).

[6]J. P. Wittke, Proc. IRE 45, 291 (1957).

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