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Low-Frequency Electric Microfield Distributions in a Plasma Containing Multiply Charged Ions*†

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A theory for calculating low-frequency component electric microfield distributions for a plasma containing more than a single ion species is developed. Calculations at a charged point are made for a plasma containing N^+ singly charged ions and N^{++} doubly charged ions together with a charge neutralizing number of electrons, $N_e (N_e = N^+ + 2N^{++})$. Three different ion ratios ($R \equiv N^{++}/N^+$) are treated: $R = 0.0,\ 1.0,\ \infty$. It is shown that the calculations allow for all ion-ion correlations to a high degree of accuracy. Numerical results are shown both graphically and in tabulated form.

I. INTRODUCTION

In recent years considerable effort has been devoted to the problem of spectral line broadening in plasmas.[1–3] In relation to this problem various theories of the static electric microfield distributions have been formulated.[4,5] However, all of these theories and subsequent calculations have only been concerned with plasmas containing a single positive-ion species. The purpose of this paper is to extend the theory developed by one of us to treat plasmas containing more than a single species of positive ion. Calculations for a plasma containing two positive-ion components have been made; the procedure for extending the calculations to situations with more than two species is indicated.

In this paper, calculations are made for a plasma that contains N^+ singly charged ions and N^{++} doubly charged ions ($N = N^+ + N^{++}$) together with a charge neutralizing number of electrons $N_e (N_e = N^+ + 2N^{++})$. It is assumed that ions interact with each other through an effective potential which includes electron-ion shielding. This model is the two-component analog of the single-component low-frequency model previously developed.[4,5] Since helium plasmas may have both singly and doubly charged species present, the model proposed here is appropriate for discussing the effect of a helium plasma on a radiating He^+ ion ($He^+ \equiv$ He II) or He atom.

As in the papers dealing with singly charged perturbing ions, the calculation of the electric-microfield distribution at a neutral point (e.g., at a He atom) is just a special case of the charged-point development obtained by setting the charge at the origin equal to zero.

To make the mathematical development more general, we make the assumption that it is valid to consider a two-temperature plasma, one temperature for the ions, T_i and one for the electrons T_e. This procedure implies that while the ions may be considered to be in equilibrium with each other, and the electrons with each other, that the ions are not necessarily in equilibrium with the electrons. In the event that a true equilibrium situation prevails, $T_e = T_i$.

All numerical results presented here assume an equilibrium situation. The actual distribution functions are expressed in reduced field units which are a function of electron density only. The calculational programs that we have developed are quite general; they allow for the possibility of a two-temperature plasma, for the possibility that there may be any number of charged-ion perturber species (i.e., singly, doubly, etc.), and for the possibility that the radiator may have any degree of ionization.

Section II of this paper deals with the formal calculations. The asymptotic expressions for the microfield distribution function are presented in

Sec. III. Numerical results and conclusions are given in Sec. IV. An appendix is devoted to the detailed evaluation of various integral expressions by collective coordinate techniques.

II. FORMALISM

Define $Q(\vec{\epsilon})d\vec{\epsilon}$ as the probability of finding an electric field $\vec{\epsilon}$ at the origin of our reference system due to a collection of N ions $(N^{+}+N^{++}+\cdots)$ assumed to interact with each other through shielded Coulomb potentials. The plasma is contained in a volume V and is macroscopically neutral. The origin is assigned a charge q defined by

$$q \equiv \xi e, \quad \xi = 0, 1, 2, \ldots \tag{1}$$

where e is the magnitude of the electronic charge. By choosing the appropriate value of ξ, expressions for microfield distributions at neutral or variously charged radiators are obtained.

The definition of the microfield distribution function, $Q(\vec{\epsilon})$ is given by

$$Q(\vec{\epsilon}) = Z^{-1}\int\cdots\int \exp[-\beta V(\vec{r}_1\cdots\vec{r}_N)] \times \delta(\vec{\epsilon} - \sum_j \vec{\epsilon}_j)\prod_{j=1}^{N} d\vec{r}_j, \quad \beta \equiv \frac{1}{\theta_i} \equiv \frac{1}{kT_i} \tag{2}$$

where Z is the configurational partition function for the two- (or multi-) component system, and $V(\vec{r}_1\cdots\vec{r}_N)$ is the total potential of interaction between ions. $\vec{\epsilon}_i$ is the field at the origin due to the ith ion that is located at $\vec{r}_i$:

$$\vec{\epsilon}_i = q^{-1}\nabla_{\vec{r}_i} V(\vec{r}_1\cdots\vec{r}_N) . \tag{3}$$

Note that because the field point is at the origin and the source at $\vec{r}_i$, the sign of the gradient in this last expression differs from that in the usual relation. The sum $\sum_i \vec{\epsilon}_i$ is given by

$$\sum_{j=1}^{N}\vec{\epsilon}_i = \sum_{j=1}^{N^{+}}\vec{\epsilon}_j + \sum_{m=1}^{N^{++}}\vec{\epsilon}_m . \tag{4}$$

The integrations in Eq. (2) are over the coordinates of all the ions.

By representing the δ function as an integral

$$\delta(\vec{x}) = (2\pi)^{-3}\int\int\int \exp[i\vec{l}\cdot\vec{x}]\,d\vec{l}, \tag{5}$$

we may write

$$Q(\vec{\epsilon}) = Z^{-1}\int\cdots\int (2\pi)^{-3} \times \exp[-\beta V + i\vec{l}\cdot(\vec{\epsilon} - \textstyle\sum_i\vec{\epsilon}_i)]\,d\vec{l}\,\prod_j d\vec{r}_j . \tag{6}$$

Since $\sum_j \vec{l}\cdot\vec{\epsilon}_j$ is independent of the direction of $\vec{l}$, the angular integrations can be done immediately. The result is the commonly occurring expression for $P(\epsilon)$ for an isotropic system[4,5]:

$$P(\epsilon) = 4\pi\epsilon^2 Q(\vec{\epsilon}) = 2\epsilon(\pi)^{-1}\int_0^\infty T(l)\sin(\epsilon l) l\,dl , \tag{7}$$

where $T(l)$ is defined by

$$T(l) = Z^{-1}\int\cdots\int \exp[-\beta V - i\textstyle\sum_j \vec{l}\cdot\vec{\epsilon}_j]\prod_j d\vec{r}_j . \tag{8}$$

The potential energy in the present calculation is

$$V = \sum_{i<j}^{N^{+}} \frac{e^2}{r_{ij}} e^{-r_{ij}/\lambda} + \sum_{m<n} \frac{(2e)^2}{r_{mn}} e^{-r_{mn}/\lambda} + \sum_{j,m} \frac{2e^2}{r_{jm}} e^{-r_{jm}/\lambda}, \tag{9}$$

where

$$\lambda = [(kT_e/4\pi n_e e^2)]^{1/2} .$$

The subscripts i, j are reserved for the N^{+} ions and the subscripts m, n are reserved for the N^{++} ions. It is convenient at this point to define the new quantities w_{j0}, w_{m0}, and V_0:

$$w_{j0} = q(e/r_{j0})\,e^{-\alpha r_{j0}/\lambda}, \tag{10a}$$

$$w_{m0} = q(2e/r_{m0})\,e^{-\alpha r_{m0}/\lambda}, \tag{10b}$$

$$V \equiv V_0 + \sum_{j=1}^{N^{+}} w_{j0} + \sum_{m=1}^{N^{++}} w_{m0} . \tag{11}$$

The w's will take into account the short-range central interactions. V_0 includes all noncentral and long-range interactions. α is an effective range parameter, the choice of which will be considered later. In terms of the form for the potential energy given in Eq. (11), $T(l)$ can be written

$$T(l) = Z^{-1}\int\cdots\int e^{\tilde{V}_0}\prod_j^{N^{+}}\exp[-\beta w_{j0} - iq^{-1}\vec{l}\cdot\vec{\nabla}_0 w_{j0}]\,d\vec{r}_j \times \prod_m^{N^{++}}\exp[-\beta w_{m0} - iq^{-1}\vec{l}\cdot\vec{\nabla}_0 w_{m0}]\,d\vec{r}_m , \tag{12}$$

where

$$\tilde{V}_0 = -\beta V_0 - iq^{-1}\vec{l}\cdot\vec{\nabla}_0 V_0 . \tag{12a}$$

The following definitions can also be made and then substituted into the expression for $T(l)$:

$$\chi^{+}(l,j) \equiv \exp[-\beta w_{j0} - iq^{-1}\vec{l}\cdot\vec{\nabla}_0 w_{j0}] - 1 , \tag{13a}$$

$$\chi^{++}(l,m) \equiv \exp[-\beta w_{m0} - iq^{-1}\vec{l}\cdot\vec{\nabla}_0 w_{m0}] - 1 . \tag{13b}$$

Hence,

$$T(l) = Z^{-1}\int\cdots\int e^{\tilde{V}_0}\prod_j^{N^{+}}[1+\chi^{+}(l,j)]\,d\vec{r}_j \times \prod_m^{N^{++}}[1+\chi^{++}(l,m)]\,d\vec{r}_m . \tag{14}$$

If the factors in $T(l)$ containing the χ's are multiplied out, the following expression is obtained:

$$T(l)=Z^{-1}\int\cdots\int e^{\tilde{V}_0}\left(1+\sum_j^{N^+}\chi^+(l,j)+\sum_m^{N^{++}}\chi^{++}(l,m)+\sum_{i<j}\chi^+(l,j)\chi^+(l,i)+\sum_{m<j}\chi^+(l,m)\chi^+(l,n)\right.$$
$$\left.+\sum_j\sum_m\chi^+(l,j)\chi^+(l,j)\chi^{++}(l,m)+\cdots\right)\prod_j d\vec{r}_j\prod_m d\vec{r}_m \quad (15a)$$

$$=Z^{-1}\int\cdots\int e^{\tilde{V}_0}\prod_j d\vec{r}_j\prod_m d\vec{r}_m+N^+Z^{-1}\int\cdots\int e^{\tilde{V}_0}\chi^+(l,1)\prod_j d\vec{r}_j\prod_m dr_j$$
$$+N^{++}Z^{-1}\int\cdots\int e^{\tilde{V}_0}\chi^+(l,1)\chi^+(l,2)\prod_j d\vec{r}_j\prod_m d\vec{r}_m$$
$$+\tfrac{1}{2}N^+(N^+-1)Z^{-1}\int\cdots\int e^{\tilde{V}_0}\chi^+(l,1)\chi^+(l,2)\prod_j d\vec{r}_j\prod_m d\vec{r}_m$$
$$+\tfrac{1}{2}N^{++}(N^{++}-1)Z^{-1}\int\cdots\int e^{\tilde{V}_0}\chi^{++}(l,1)\chi^{++}(l,2)\prod_j d\vec{r}_j\prod_m d\vec{r}_m$$
$$+N^+N^{++}Z^{-1}\int\cdots\int e^{\tilde{V}_0}\chi^+(l,1)\chi^{++}(l,2)\prod_j d\vec{r}_j\prod_m d\vec{r}_m+\cdots\,. \quad (15b)$$

Now make the following two definitions:

$$T_{jm}(l)\equiv\int\cdots\int e^{\tilde{V}_0}\prod_{i=j+1}^{N^+}d\vec{r}_i\prod_{n=m+1}^{N^{++}}d\vec{r}_n\,,$$
$$Q_{jm}(l)\equiv T_{jm}(l)/T_0(l)\,, \quad (16)$$

where $T_0(l)\equiv T_{00}(l)$. In terms of these new quantities $T(l)$ can be written

$$T(l)=T_0(l)Z^{-1}[1+N^+\int Q_{10}(l)\chi^+(l,1)d\vec{r}_1+N^{++}\int Q_{01}(l)\chi^{++}(l,1)d\vec{r}_1+(1/2!)N^+(N^+-1)$$
$$\times\int\int Q_{20}(l)\chi^+(l,1)\chi^+(l,2)d\vec{r}_1d\vec{r}_2+(1/2!)N^{++}(N^{++}-1)\int\int Q_{02}(l)\chi^{++}(l,1)\chi^{++}(l,2)d\vec{r}_1d\vec{r}_2$$
$$+N^+N^{++}\int\int Q_{11}(l)\chi^+(l,1)\chi^{++}(l,2)d\vec{r}_1d\vec{r}_2+\cdots]\,. \quad (17)$$

Each of the Q's can be expanded in an Ursell expansion according to the following prescription:

$$\mathcal{V}\,Q_{10}(l;\,1)=g_{10}(l;\,1)\,,$$
$$\mathcal{V}\,Q_{01}(l;\,1)=g_{01}(l;\,1)\,,$$
$$\mathcal{V}^2\,Q_{20}(l;\,1,2)=g_{10}(l;\,1)g_{10}(l;\,2)+g_{20}(l;\,1,2),$$
$$\mathcal{V}^2\,Q_{02}(l;\,1,2)=g_{01}(l;\,1)g_{01}(l;\,2)+g_{02}(l;\,1,2)\,,$$
$$\mathcal{V}^2\,Q_{11}(l;\,1,2)=g_{10}(l;\,1)g_{01}(l;\,2)+g_{11}(l;\,1,2)\,. \quad (18)$$

In terms of these g's and in the thermodynamic limit ($N\to\infty$, $\mathcal{V}\to\infty$ so that the density $n=N/\mathcal{V}$ remains constant, $T(l)$ is given by

$$T(l)=T_0(l)Z^{-1}\{1+n^+\int g_{10}(l;\,1)\chi^+(l;\,1)d\vec{r}_1+n^{++}\int g_{01}(l;\,1)\chi^{++}(l;\,1)d\vec{r}_1$$
$$+[(n^+)^2/2!\,][\int\int g_{20}(l;\,1,2)\chi^+(l,2)\chi^+(l;\,2)d\vec{r}_1d\vec{r}_2+(\int g_{10}(l,1)\chi^+(l,1)d\vec{r}_1)^2\,]$$
$$+[(n^+)^2/2!\,][\int\int g_{02}(l;\,1,2)\chi^{++}(l,2)d\vec{r}_1d\vec{r}_2+(\int g_{01}(l,1)\chi^{++}(l,1)d\vec{r}_1)^2\,]$$
$$+n^+n^{++}[\int\int g_{11}(l;\,1,2)\chi^+(l,1)\chi^{++}(l,2)\,d\vec{r}_1d\vec{r}_2+(\int g_{10}(l,1)\chi^+(l,1)\,d\vec{r}_1)\times(\int g_{01}(l,2)\chi^{++}(l,2)d\vec{r}_2)\,]+\cdots\}\,. \quad (19)$$

If h_{jm} is defined by

$$h_{jm}(l) \equiv \int \cdots \int g_{jm}(l)\chi^{+}(l,1)\cdots\chi^{+}(l,j)\chi^{++}(l,1) \times \ldots \chi(l,m) \prod_{i=1}^{j} d\vec{r}_i \prod_{n=1}^{m} d\vec{r}_n , \quad (20)$$

$T(l)$ can be written in terms of these h's as

$$T(l) = T_0(l) Z^{-1}\{1 + n^{+}h_{10}(l) + n^{++}h_{01}(l) + [(n^{+})^2/2!\,][h_{20}(l) + [h_{10}(l)]^2] + [(n^{++})^2/2!\,][h_{02}(l) + [h_{01}(l)]^2] + n^{+}n^{++}[h_{11}(l) + h_{10}(l)h_{01}(l)] + \cdots\} . \quad (21)$$

These terms can be regrouped and written in the form

$$T(l) = T_0(l)Z^{-1}\exp\left(\sum_{j}^{N^+}\sum_{m}^{N^{++}} \frac{(n^+)^j}{j!}\frac{(n^{++})^m}{m!} h_{jm}(l)\right), \quad (22)$$

which is a systematic Ursell cluster expansion.

It can be noted here that the definition of Z would be the same as $T(l)$, if l were set equal to zero. Therefore Z can be written as

$$Z = T_0(0)\exp\left(\sum_{j}^{N^+}\sum_{m}^{N^{++}} \frac{(n^+)^j(n^{++})^m}{j!\;\;m!} h_{jm}(0)\right), \quad (23)$$

This allows $T(l)$ to be written as

$$T(l) = \left(\frac{T_0(l)}{T_0(0)}\right) \times \exp\left(\sum_{j}^{N^+}\sum_{m}^{N^{++}} \frac{(n^+)^j(n^{++})^m}{j!\;\;m!}[h_{jm}(l) - h_{jm}(0)]\right) . \quad (24)$$

If more than two species of perturbing ions are to be considered, Eq. (24) can be generalized:

$$T(l) = \left(\frac{T_0(l)}{T_0(0)}\right) \exp\left(\sum_{j}\cdots\sum_{k}\cdots\sum_{m} \frac{(n)_\cdot^j\ldots(n)_\cdot^k\ldots(n)^m}{j!\ldots k!\ldots m!}[h_{j\ldots k\ldots m}(l) - h_{j\ldots k\ldots m\ldots}(0)]\right) . \quad (25)$$

This allows for any number of perturbing species. The procedure for the evaluation of these terms would be similar to that presented in the Appendix.

Since the calculation of $P(\epsilon)$ considered in this paper always involves a computer calculation, it should be pointed out that the cost of evaluating these terms is considerable.

Consider the individual terms appearing in Eq. (24). By the method developed in the Appendix, the first factor, $T_0(l)/T_0(0)$, becomes

$$T_0(l)/T_0(0) = e^{-\gamma L^2} . \quad (26)$$

In Eq. (26),

$$\gamma = \tfrac{1}{4}a(\theta_i/\theta_e)[\alpha^2 - (1+u)]^2\{\;\} , \quad (26a)$$

$$\{\;\} = \{\alpha^5 u + 2[1 - (1+u)^{3/2}]\alpha^4 + [2u + u^2]\alpha^3 - 4(1+u)[1-(1+u)^{1/2}]\alpha^2 - 3(u+u^2)\alpha + 2[(1+u)^2 - (1+u)^{3/2}]\} ,$$

$$u = \frac{\theta_e}{\theta_i}\left(\frac{1+4R}{1+2R}\right) , \quad R \equiv \frac{n^{++}}{n^+} ,$$

$$\theta_e \equiv kT_e , \quad \theta_i \equiv kT_i , \quad (26b)$$

$$L \equiv \epsilon_0 l , \quad a \equiv r_0/\lambda , \quad \epsilon_0 \equiv e/r_0^2 ,$$

where r_0 is the ion-sphere radius defined by the expression

$$\tfrac{4}{3}\pi r_0^3 n_e = 1 . \quad (26c)$$

Now we consider the factors resulting from the series exponent. There are two first-order terms, one for $j=1$ and $m=0$ and one for $j=0$ and $m=1$. In general, if there are r different species of perturbing ions, there will be r such first-order terms:

$$I_{10} \equiv n^+ [h_{10}(l) - h_{10}(0)]$$
$$= n^+ \mho \int\left(\frac{\int\cdots\int e^{-\beta V_0 - iq^{-1}l\cdot\vec{\nabla}_0 V_0}(e^{-\beta w_{10} - iq^{-1}l\cdot\vec{\nabla}_0 w_{10}} - 1)\prod_{j=2}^{N^+} d\vec{r}_j \prod_{n=1}^{N^{++}} d\vec{r}_n}{\int\cdots\int \exp(-\beta V_0 - iq^{-1}l\cdot\vec{\nabla}_0 V_0)\prod_{1}^{N^+} d\vec{r}_i \prod_{1}^{N^{++}} d\vec{r}_n} - \frac{\int\cdots\int e^{-\beta V_0 - iq^{-1}l\cdot\vec{\nabla}_0 V_0}(e^{-\beta_i w_{10}} - 1)\prod_{2}^{N^+} d\vec{r}_j \prod_{1}^{N^{++}} d\vec{r}_n}{\int\cdots\int e^{-\beta V_0}\prod_{1}^{N^+} d\vec{r}_j \prod_{1}^{N^{++}} d\vec{r}_n}\right) d\vec{r}_1 , \quad (27)$$

and similarly,

$$I_{01} \equiv n^{++}[h_{01}(l) - h_{01}(0)]$$

$$= n^{++}\mathcal{V}\int \left(\frac{\int \cdots \int e^{-\beta V_0 - iq^{-1}\vec{l}\cdot\vec{\nabla}_0 V_0}(e^{-\beta w_{10} - iq^{-1}\vec{l}\cdot\vec{\nabla}_0 w_{10}} - 1) \prod_1^{N^+} d\vec{r}_j \prod_2^{N^{++}} d\vec{r}_m}{\int \cdots \int e^{-\beta V_0} \prod_1^{N^+} d\vec{r}_j \prod_1^{N^{++}} d\vec{r}_m} - \frac{\int \cdots \int e^{-\beta V_0 - iq^{-1}\vec{l}\cdot\vec{\nabla}_0 V_0}(e^{-\beta_i w_{10}} - 1) \prod_1^{N^+} d\vec{r}_j \prod_2^{N^{++}} d\vec{r}_m}{\int \cdots \int e^{-\beta V_0} \prod_1^{N^+} d\vec{r}_j \prod_1^{N^{++}} d\vec{r}_m} \right) d\vec{r}_1 \ . \quad (28)$$

Although these integrals appear formidable, they may be readily reduced through the use of collective coordinates to approximate expressions involving only one-dimensional integrals. The accuracy of this approximation is briefly discussed later and has been extensively discussed elsewhere.[5] Collective coordinates are defined and the nature of the evaluation indicated in the Appendix. The final results are merely stated here:

$$I_{10}(l) = n^+[h_{10}(l) - h_{10}(0)]$$

$$= \frac{3}{1+2R}\int_0^\infty x^2\,dx\,e^{s(x)}\left[e^{-\beta w_{10}}\left(\frac{\sin(LG(x))}{(LG(x))} - 1\right) - \left(\frac{\sin(Lq(x))}{(Lq(x))} - 1\right)\right] \quad (29)$$

and

$$I_{01} = n^{++}[h_{01}(l) - h_{01}(0)]$$

$$= \frac{3R}{1+2R}\int_0^\infty x^2\,dx\,e^{2s(x)}\left[e^{-2\beta w_{10}}\left(\frac{\sin(2LG(x))}{(2LG(x))}\right) - \left(\frac{\sin(2Lq(x))}{(2Lq(x))} - 1\right)\right] , \quad (30)$$

where

$$x = r/r_0 , \quad \beta w_{10} = \xi(\theta_c/\theta_i)(a^2/3x)\,e^{-\alpha a x} ,$$

$$s(x) = \xi\,\frac{ua^2}{3x}\left(\frac{1+2R}{1+4R}\right)\left(\frac{\alpha^2-1}{\alpha^2-(1+u)}\right) \times (e^{-\alpha a x} - e^{(1+u)^{1/2}ax}) , \quad (30a)$$

$$q(x) = -\left(\frac{\alpha^2-1}{\alpha^2-(1+u)}\right)\left(\frac{1}{x^2}\,(e^{-\alpha a x} - e^{-(1+u)^{1/2}ax}) + \frac{a}{x}\,(\alpha e^{-\alpha a x} - (1+u)^{1/2}e^{-(1+u)^{1/2}ax})\right),$$

$$G(x) = q(x) + (e^{-\alpha a x}/x^2)(1+\alpha a x) .$$

The second-order terms appearing in the series exponent are now given explicitly:

$$I_{20}(l) \equiv [(n^+)^2/2!\,][h_{20}(l) - h_{20}(0)]$$

$$= [(n^+)^2/2!\,]\{\mathcal{V}^2\int\int[Q_{20}(l,1,2) - Q_{10}(l,1)Q_{10}(l,2)]\,\chi^+(l,1)\chi^+(l,2)d\vec{r}_1\,d\vec{r}_2 - \mathcal{V}^2\int\int[Q_{20}(0,1,2) - Q_{10}(0,1)Q_{10}(0,2)]\,\chi^+(0,1)\chi^+(0,2)\,d\vec{r}_1\,d\vec{r}_2\} \ . \quad (31)$$

When the $Q_{20}(l)$ and $Q_{20}(0)$ functions are evaluated by the methods presented in the Appendix, they have a very interesting form. Equation (A52) gives for $Q_{20}(l)$

$$Q_{20}(l,1,2) = Q_{10}(l,1)Q_{10}(l,2)\exp\left(-\frac{\theta_e}{\theta_i}\,\frac{a^2}{3x_{12}}\,e^{-(1+u)^{1/2}ax_2}\right) . \quad (32)$$

The expression appearing in large parentheses is the Debye-Hückel pair-correlation function (see Ref. 5) for a discussion of this point). Performing the complete collective-coordinate evaluation, as indicated in the Appendix, we find

$$I_{20}=\frac{\theta_e}{\theta_i}\left(\frac{1}{1+2R}\right)^2 3a^2\sum_k(-1)^k(2k+1)\{20\},$$

$$\{20\}=\left(\int_0^\infty x_2^{3/2}I_{k+1/2}(a'x_2)\,e^{s(x_2)}\left[e^{-\beta_i w(x_2)}j_k(LG(x_2))-j_k(Lq(x_2))\right]\right.$$
$$\times\int_{x_2}^\infty x_1^{3/2}K_{k+1/2}(a'x_1)\,e^{s(x)}\left[e^{-\beta_i w(x_1)}j_k(LG(x_1))-j_k(Lq(x_1))\right]dx_1\,dx_2$$
$$\left.-\delta_{ko}\int_0^\infty x_2^{3/2}I_{1/2}(a'x_2)e^{s(x_2)}\left[e^{-\beta_i w(x_2)}-1\right]\int_{x_2}^\infty x_1^{3/2}K_{1/2}(a'x_1)e^{s(x_1)}\left[e^{-\beta_i w(x_1)}-1\right]dx_1\,dx_2\right). \tag{33}$$

Similarly an expression for I_{02} is given by

$$I_{02}=4\,\frac{\theta_e}{\theta_i}\left(\frac{R}{1+2R}\right)^2 3a^2\sum_k(-1)^k(2k+1)\{02\},$$

$$\{02\}=\int_0^\infty x_2^{3/2}I_{k+1/2}(a'x_2)\,e^{2s(x_2)}\left[e^{-2\beta w(x_2)}j_k(2LG(x_2))-j_k(Lq(x_2))\right]$$
$$\times\int_{x_2}^\infty x_1^{3/2}K_{k+1/2}(a'x_1)e^{2s(x_1)}\left[e^{-2\beta_i w(x_1)}j_k(2LG(x_1))-j_k(2Lq(x_1))\right]dx_1\,dx_2$$
$$-\delta_{ko}\int_0^\infty x_2^{3/2}I_{1/2}(a'x_2)e^{2s(x_2)}\left[e^{-2\beta_i w(x_2)}-1\right]\int_{x_2}^\infty x_1^{3/2}K_{1/2}(a'x_1)e^{2s(x_1)}\left[e^{-2\beta w(x_1)}-1\right]dx_1\,dx_2\,. \tag{34}$$

There is yet another second-order term, viz., I_{11}; it is given by

$$I_{11}=2\,\frac{\theta_e}{\theta_i}\left(\frac{1}{1+2R}\right)\left(\frac{R}{1+2R}\right)3a^2\sum_k(-1)^k(2k+1)\{11\},$$

$$\{11\}=\left(\int_0^\infty x_2^{3/2}I_{k+1/2}(a'x_2)e^{2s(x_2)}\left[e^{-2\beta w(x_2)}j_k(2LG(x_2))-j_k(2Lq(x_2))\right]\right.$$
$$\times\int_{x_2}^\infty x_1^{3/2}K_{k+1/2}(a'x_1)e^{s(x_1)}\left[e^{-\beta w(x_1)}j_k(L\,G(x_1))-j_k(Lq(x_1))\right]dx_1\,dx_2$$
$$+\int_0^\infty x_2^{3/2}K_{k+1/2}(a'x_2)e^{2s(x_2)}\left[e^{-2\beta w(x_2)}j_k(2LG(x_2))-j_k(2Lq(x_2))\right]$$
$$\times\int_0^{x_2} x_1^{3/2}I_{k+1/2}(a'x_1)\,e^{s(x_1)}\left[e^{-s(x_1)}j_k(LG(x_1))-j_k(Lq(x_1))\right]dx_1\,dx_2$$
$$-\delta_{ko}\int_0^\infty x_2^{3/2}K_{1/2}(a'x_2)e^{2s(x_2)}\left[e^{-2\beta w(x_2)}-1\right]\int_{x_2}^\infty x_1^{3/2}K_{1/2}(a'x_1)e^{s(x_1)}\left[e^{-\beta w(x_1)}-1\right]dx_1\,dx_2$$
$$\left.-\delta_{ko}\int_0^\infty x_2^{3/2}K_{1/2}(a'x_2)e^{2s(x_2)}\left[e^{-2\beta w(x_2)}-1\right]\int_0^{x_2} x_1^{3/2}I_{1/2}(a'x_1)e^{s(x_1)}\left[e^{-\beta w(x_1)}-1\right]dx_1\,dx_2\right). \tag{35}$$

I and K refer to modified Bessel functions of the first and third kind, respectively, while $j_k(-)$ specifies a spherical Bessel function of order k.[6] The sums over k in these expressions for the second-order terms converge very rapidly and hence only three terms must be evaluated.

Thus to second order, we may write

$$T(l)=\exp[-\gamma L^2+I_{10}(l)+I_{01}(l)+I_{20}(l)+I_{02}(l)+I_{11}(l)]\,. \tag{36}$$

This result is used in Eq. (7) to calculate $P(\epsilon)$ at a charged point. It may also be shown that this result goes to the Holtsmark limit as $T\to\infty$.[7] If there were r different species of ion, there would be r second-order terms such as I_{20}, and $r!/2!(r-2)!$ second-order cross-term such as I_{11}.

In the event that $P(\epsilon)$ is desired at a neutral point, ξ is set equal to zero which excludes central interactions. Because of the cost of evaluating the terms, it is useful to obtain separate analytic expressions for the neutral case. The expression for $T_0(l)/T_0(0)$ remains unaltered, but the first-and second-order terms in the series exponent are changed owing to the fact that $w(x)$, $s(x)$, and $h_{ij}(0)$ are zero. Hence

$$I_{10}(l)_{\text{neutral}} = n^{+} h_{10}(l)_{\text{neutral}} = \frac{3}{1+2R}\int_0^{\infty} dx\, x^2\left(\frac{\sin(LG(x))}{LG(x)} - \frac{\sin(Lq(x))}{Lq(x)}\right), \tag{37}$$

$$I_{01}(l)_{\text{neutral}} = n^{++} h_{01}(l)_{\text{neutral}} = \frac{3R}{1+2R}\int_0^{\infty} dx\, x^2\left(\frac{\sin(2LG(x))}{2LG(x)} - \frac{\sin(2Lq(x))}{2Lq(x)}\right). \tag{38}$$

The second-order terms for the neutral case are found by again setting $w(x)$, $s(x)$, and $h_{ij}(0)$ equal to zero (or, what is equivalent, by setting $\xi = 0$) and, in addition, the δ_{ko} terms for all k. Hence,

$$I_{20}(l)_{\text{neutral}} = \tfrac{1}{2}(n^{+})^2 h_{20}(l)_{\text{neutral}} = \frac{\theta_e}{\theta_i}\left(\frac{1}{1+2R}\right)^2 3a^2 \sum_k (-1)^k(2k+1)\{20\}_{\text{neutral}},$$

$$\{20\}_{\text{neutral}} = \left\{\int_0^{\infty} x_2^{3/2} I_{k+1/2}(a'x_2)\,[\,j_k(LG(x_2)) - j_k(Lq(x_2))]\int_{x_2}^{\infty} x_1^{3/2} K_{k+1/2}(a'x_1)\,[\,j_k(LG(x_1)) - j_k(Lq(x_1))]\,dx_1\,dx_2\right\}; \tag{39}$$

$$I_{02}(l)_{\text{neutral}} = \tfrac{1}{2}(n^{++})^2 h_{02}(l)_{\text{neutral}} = 4\frac{\theta_e}{\theta_i}\left[\frac{R}{1+2R}\right] 3a^2 \sum_k (-1)^k(2k+1)\{02\}_{\text{neutral}},$$

$$\{02\}_{\text{neutral}} = \int_0^{\infty} x_2^{3/2} I_{k+1/2}(a'x_2)\,[\,j_k(2LG(x_2)) - j_k(2Lq(x_2))]\int_{x_2}^{\infty} x_1^{3/2} K_{k+1/2}(a'x_1)\,[\,j_k(2LG(x_1)) - j_k(2Lq(x_1))]\,dx_1\,dx_2\,; \tag{40}$$

$$I_{11}(l)_{\text{neutral}} = n^{+} n^{++} h_{11}(l)_{\text{neutral}} = 2\frac{\theta_e}{\theta_i}\left(\frac{1}{1+2R}\right)\left(\frac{R}{1+2R}\right) 3a^2 \sum_k (-1)^k(2k+1)\{11\}_{\text{neutral}},$$

$$\begin{aligned}\{11\}_{\text{neutral}} = \Bigg\{ &\int_0^{\infty} x_2^{3/2} I_{k+1/2}(a'x_2)\,[\,j_k(2LG(x_2)) - j_k(2Lq(x_2))]\int_{x_2}^{\infty} x_1^{3/2} K_{k+1/2}(a'x_1)[\,j_k(LG(x_1)) - j_k(Lq(x_1))]\,dx_1\,dx_2\,, \\ &+\int_0^{\infty} x_2^{3/2} K_{k+1/2}(a'x_2)\,[\,j_k(2LG(x_2) - j_k(2Lq(x_2))\,]\int_{x_2}^{\infty} x_2^{3/2} I_{k+1/2}(a'x_1)\,[\,j_k(LG(x_1)) - j_k(Lq(x_1))]\,dx_1\,dx_2 \\ &+\int_0^{\infty} x_2^{3/2} K_{k+1/2}(a'x_2)[\,j_k(2LG(x_2)) - j_k(2Lq(x_2))]\int_{x_2}^{\infty} x_1^{3/2} I_{k+1/2}(a'x_1)\,[\,j_k(LG(x_1)) - j_k(Lq(x_1))]\,dx_1\,dx_2\Bigg\}.\end{aligned} \tag{41}$$

Two approximations have been made thus far. First, we have terminated the series appearing in the exponential with the second-order terms. This may be justified, by consideration of the analytic form of the terms appearing in the series, and by direct numerical calculations.[5] The second approximation concerns the use of collective coordinates in the evaluation of the many-dimensional integrals occurring in this theory.[5,8]

As indicated earlier in this section, the evaluation of the many-dimensional integrals appearing in the expressions for $I_{jm}(l)$ can be transformed into integrals over collective coordinates which have a rather simple form. As is shown in the Appendix, these collective-coordinate integrals may be evaluated as[8]

$$\begin{aligned} I &= \int\cdots\int \exp[-\tfrac{1}{2}\textstyle\sum_k (A_k X_k^2 + 2b_k X_k)]\, J\, \prod_k dX_k \\ &= (\text{const})\exp[\tfrac{1}{2}\textstyle\sum_k b_k^2/(1+A_k)](1 - a_3 + a_4 - \cdots)\,. \end{aligned} \tag{42}$$

The A_k and b_k are specific functions of k, the X_k's represent collective coordinates, and J is the Jacobian of the $\vec{r}\to\vec{X}$ transformation. The series of terms in brackets represents the possible higher-order correction to the first Jacobian approximation. In the calculations made in this paper, a_3 and all other correction terms have been neglected.

Following the procedure previously developed,[5,8] we have shown that the neglect of corrections to the first Jacobian approximation is valid for the temperatures and densities considered in this paper.

III. ASYMPTOTIC MICROFIELD DISTRIBUTION FUNCTION

In this section we deal with the determination of asymptotic microfield distribution functions. Generating the microfield distribution functions for values of $\epsilon > 20$ becomes increasingly expensive because the sine transform routine that is required to evaluate Eq. 7 requires an increasingly finer mesh. For values of $P(\epsilon)$ in this asymptotic region it is more convenient to calculate $P(\epsilon)$ from one of two approximate formulations: the nearest-neighbor approximation (NNA)[8] or the Holtsmark limit.[7]

This problem has been previously considered for the case of singly charged ion perturbers.[5] Hence we are here extending the method to the situation where more than one perturbing ion species is present. Specifically we will examine the situation where both singly and doubly charged perturbers are present. First we will consider the charged-point case, and then the neutral-point problem.

A. Charged-Point Case

For the charged-point case, the additional correlations arising from the presence of a charged particle at the origin make the Holtsmark limit an invalid approximation for the field strengths that we consider. Hence we use the nearest-neighbor approximation which we have shown to be in very close agreement with the near-exact function.

The nearest-neighbor model assumes that for high fields the bulk of the contribution to the total field is due only to the nearest neighbor. This neighbor may be either a singly or a doubly charged ion. The probability of two or more ions producing this high field is very small and is assumed to be zero. If this probability were not small, it would mean that the asymptotic region had not yet been reached and the near-exact microfield calculation would have to be extended. The assumption that the probability of two ions being near the origin is small, has been validated by comparisons between calculated near-exact microfield distributions and the asymptotic results in the region where they join. The asymptotic value is always slightly less than the calculated microfield distribution but the difference decreases as ϵ increases. At the point where the asymptotic form is assumed to be valid, the difference is less than 1%.

The probability of a singly or a doubly charged ion being close to the origin is related to the probability of this same ion producing an electric field, by the following expressions:

$$P_1(\epsilon_1)\,d\epsilon_1 = 4\pi r_1^2 n^+ g^+(r_1)\,dr_1 = \frac{1}{1+2R}\,x_1^2 g^+(x_1)\,dx_1 \;, \tag{43a}$$

$$P_2(\epsilon_2)\,d\epsilon_2 = 4\pi r_2^2 n^{++} g^{++}(r_2)\,dr_2 = \frac{3R}{1+2R}\,x_2^2 g^{++}(x_2)\,dx_2 \;, \tag{43b}$$

where the last expressions on the right are in terms of the reduced quantities already defined. $g^+(r_1)$ and $g^{++}(r_2)$ are pair correlation functions between a singly charged particle and a particle of charge ξ, and between a doubly charged particle and a particle of charge ξ, respectively. They can be found using Eqs. (A48), (A50), and (A52) by allowing for the variable charge ξ:

$$g^+(x_1) = \exp\left(-\xi\,\frac{a^2}{3x_1}\,\frac{\theta_e}{\theta_i}\,e^{-(1+u)^{1/2}ax_1}\right) \;, \tag{44a}$$

$$g^{++}(x_2) = \exp\left[-2\xi\,\frac{a^2}{3x_2}\,\frac{\theta_e}{\theta_i}\,e^{-(1+u)^{1/2}ax_2}\right] \;. \tag{44b}$$

The fields produced by these ions, in units of the reduced field strength ϵ_0 and reduced distance r_0 [defined in Eq. (26b)] are

$$\epsilon_1 = (2/x_1^2)(1+ax_1)\,e^{-ax_1} \;, \tag{45a}$$

$$\epsilon_2 = (2/x_2^2)(1+ax_2)\,e^{-ax_2} \;. \tag{45b}$$

If these field expressions are differentiated, the expressions for $d\epsilon_1$ and $d\epsilon_2$ can be used in Eqs. (43) to obtain $P_1(\epsilon)$ and $P_2(\epsilon)$:

$$P_1(\epsilon_1) = \left(\frac{1}{1+2R}\right)\frac{3x_1^4\exp[-\xi(a^2/x_1)\,(\theta_e/\theta_i)\,e^{-(1+u)^{1/2}}ax_1]}{a\exp(-ax_1)(2+2/ax_1+ax_1)} \;, \tag{46a}$$

$$P_2(\epsilon_2) = \left(\frac{R}{1+2R}\right)\frac{3x_2^4\exp[-2\xi(a^2/3x_2)(\theta_e/\theta_i)\,e^{-(1+u)^{1/2}}ax_2]}{2a\exp(-ax_2)(2+2/ax_2+ax_2)} \;. \tag{46b}$$

The total asymptotic probability is the sum of the probabilities of the two independent events:

$$P(\epsilon)_{\rm asym} = P_1(\epsilon) + P_2(\epsilon) \;. \tag{47}$$

B. Neutral-Point Case

The Holtsmark distribution function is the proper high-temperature limit of the nearly exact microfield distribution functions. As such it represents the situation when the perturbing ions are totally uncorrelated: The ions move independently of one another. For increasingly large values of the field strength ϵ, it can be shown that the neutral-point near-exact distribution function goes over to the Holtsmark result. The Holtsmark result for a two-component plasma is

$$P(\epsilon)_{\rm Holtsmark} = 1.500\,\frac{1+2\sqrt{2}R}{1+2R}\,\epsilon^{-5/2} + 7.680\,\frac{1+2\sqrt{2}R}{1+2R}\,\epsilon^{4} + 21.77\,\frac{1+2\sqrt{2}R}{1+2R}\,\epsilon^{-11/2} + \cdots \;. \tag{48}$$

It can also be shown that for the range of ϵ values considered in this paper, the near-exact distribution function is also well approximated by the NNA which implies equivalence of the two approximation methods in the asymptotic region. The reason for this equivalence can be seen by taking the $T\to\infty$ limit of the NNA and noting that the result is equal to the leading term in the Holtsmark series. For field strengths $\epsilon < 20$, the two approximations may differ from each other and from the near-exact value. However, the differences decrease as the field strength increases and in the high-field region where the shielded potential is numerically almost identical to the Coulomb potential, the NNA reduces to the first term of the Holtsmark series which is overwhelmingly the most important term. This is not too surprising since for large fields one would intuitively expect that nearest-neighbor contributions to the Holtsmark expression would be the largest. For the numerical results presented in this paper, the NNA was used.

IV. NUMERICAL RESULTS AND ANALYSIS

Following the procedure discussed in detail in Ref. 5, we have generated $P(\epsilon)$ values in both tabular and graphical form for a macroscopically neutral plasma containing N^+ singly charged shielded ions and N^{++} doubly charged shielded ions which interact with each other through an effective potential which includes the effect of ion-electron interaction. This model has been and is currently used when dealing with effects of ions on radiating atoms and/or ions immersed in a plasma.[1–3] The form of the effective potential is assumed to be Debye-Hückel.

Values of $P(\epsilon)$ have been calculated for four values of a; for each a value three different ion ratios ($R \equiv n^{++}/n^{+}$) have been considered. Figure 1 shows the three curves for $a = 0.2$: notice that the three curves corresponding to values of $R = 0.0$ (all perturbers singly charged), $R = 1.0$ (50% n^+ and 50% n^{++}), and $R = \infty$ (all perturbers doubly charged) are well separated. Examined in order, Figs. 2, 3, and 4 indicate that as a increases the three R-value curves seem to coalesce. Recall that for a fixed value of temperature an increase in a implies an increase in density. Also note that as a goes from 0.2 to 0.8 the peaks of the three R curves are shifted toward smaller field values and the height of the peaks is raised.

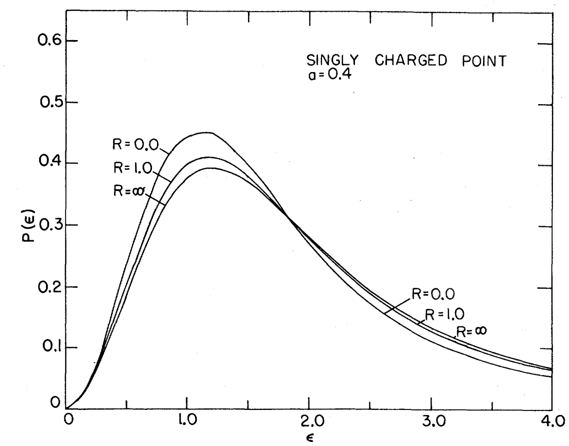

FIG. 2. Electric microfield distribution function $P(\epsilon)$ at a charged point for $a=0.4$, $R=0$, 1, ∞; ϵ is in units of ϵ_0.

The behavior of these R curves as a is varied can be interpreted physically. First, the shift and elevation of the peaks is testimony to the effect of increased interion correlations as a is increased. As these correlations increase, the ions have an increased tendency to "stay away" from one another and hence the probability distribution function is shifted to smaller values of ϵ with increased probability.

Next, consider the fact that as a increases, the relative separation of the curves corresponding to different R values decreases. For all a values discussed in this paper, the curves for $R = \infty$ favor larger field strengths than do those corresponding to $R = 0$. For a given value of N_e, the requirement that $R = \infty$ means that the plasma being treated contains one-half the number of ions than one would for which $R = 0$; but each of these ions is doubly charged.

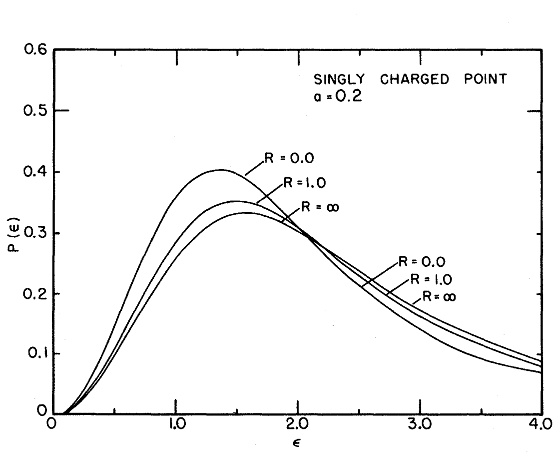

FIG. 1. Electric microfield distribution function $P(\epsilon)$ at a charged point for $a=0.2$, $R=0$, 1, ∞; ϵ is in units of ϵ_0.

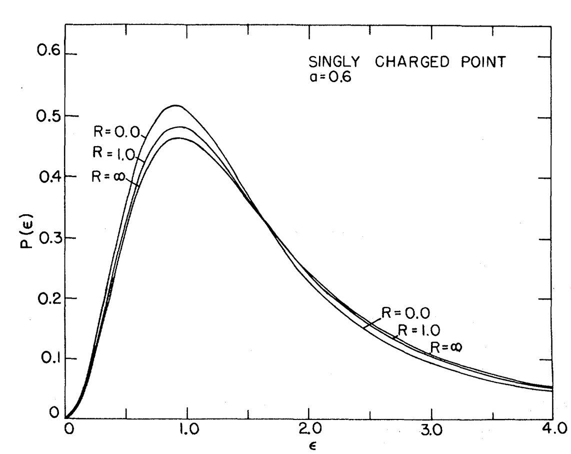

FIG. 3. Electric microfield distribution function $P(\epsilon)$ at a charged point for $a=0.6$, $R=0$, 1, ∞; ϵ is in units of ϵ_0.

TABLE I. Probability distribution $P(\epsilon)$ at a charged point for several values of a, with $R=1.0$. The electric field strength ϵ is in units of ϵ_0. The lines in the columns indicate the point at which asymptotic expressions were used in the calculations.

E	$A=0.2$	$A=0.4$	$A=0.6$	$A=0.8$
0.1	0.422 03E-02	0.100 43E-01	0.203 26E-01	0.422 45E-01
0.2	0.193 53E-01	0.388 71E-01	0.767 21E-01	0.151 15E 00
0.3	0.434 54E-01	0.828 59E-01	0.158 30E 00	0.285 53E 00
0.4	0.744 48E-01	0.136 80E 00	0.243 88E 00	0.407 21E 00
0.5	0.110 20E 00	0.194 90E 00	0.325 69E 00	0.495 29E 00
0.6	0.148 57E 00	0.251 68E 00	0.392 58E 00	0.545 85E 00
0.7	0.187 52E 00	0.302 71E 00	0.440 41E 00	0.564 35E 00
0.8	0.225 13E 00	0.344 96E 00	0.468 90E 00	0.559 09E 00
0.9	0.259 72E 00	0.376 84E 00	0.480 21E 00	0.537 92E 00
1.0	0.289 68E 00	0.398 01E 00	0.477 66E 00	0.507 16E 00
1.1	0.314 16E 00	0.409 09E 00	0.464 73E 00	0.471 42E 00
1.2	0.332 87E 00	0.411 28E 00	0.444 62E 00	0.433 88E 00
1.3	0.345 65E 00	0.406 10E 00	0.420 01E 00	0.396 63E 00
1.4	0.352 71E 00	0.395 15E 00	0.392 97E 00	0.360 94E 00
1.5	0.354 55E 00	0.379 96E 00	0.365 05E 00	0.327 53E 00
1.6	0.351 83E 00	0.361 85E 00	0.337 34E 00	0.296 74E 00
1.7	0.345 30E 00	0.341 96E 00	0.310 56E 00	0.268 68E 00
1.8	0.335 62E 00	0.321 19E 00	0.285 18E 00	0.243 28E 00
1.9	0.323 51E 00	0.300 23E 00	0.261 45E 00	0.220 42E 00
2.0	0.309 72E 00	0.279 61E 00	0.239 48E 00	0.199 90E 00
2.5	0.231 85E 00	0.190 48E 00	0.154 87E 00	0.125 37E 00
3.0	0.163 92E 00	0.129 18E 00	0.102 85E 00	0.821 36E-01
3.5	0.115 07E 00	0.895 56E-01	0.708 02E-01	0.561 61E-01
4.0	0.820 51E-01	0.639 54E-01	0.504 95E-01	0.398 96E-01
4.5	0.599 35E-01	0.470 49E-01	0.371 73E-01	0.292 68E-01
5.0	0.449 16E-01	0.355 64E-01	0.281 28E-01	0.220 70E-01
6.0	0.270 67E-01	0.217 72E-01	0.172 38E-01	0.134 29E-01
7.0	0.176 30E-01	0.143 39E-01	0.113 40E-01	0.875 71E-02
8.0	0.121 92E-01	0.998 25E-02	0.787 38E-02	0.602 49E-02
9.0	0.882 98E-02	0.725 79E-02	0.570 36E-02	0.432 29E-02
10.0	0.662 54E-02	0.545 84E-02	0.426 44E-02	0.322 48E-02
12.0	0.403 84E-02	0.331 74E-02	0.249 31E-02	0.192 68E-02
14.0	0.269 69E-02	0.220 97E-02	0.163 96E-02	0.123 65E-02
16.0	0.188 88E-02	0.153 90E-02	0.113 64E-02	0.836 83E-03
18.0	0.138 75E-02	0.112 16E-02	0.820 06E-03	0.590 08E-03
20.0	0.104 43E-02	0.837 60E-03	0.611 08E-03	0.429 95E-03
22.0	0.817 88E-03	0.633 35E-03	0.464 70E-03	0.321 77E-03
24.0	0.647 75E-03	0.502 92E-03	0.365 34E-03	0.246 25E-03
26.0	0.510 23E-03	0.406 53E-03	0.290 83E-03	0.192 04E-03
28.0	0.422 62E-03	0.333 65E-03	0.235 17E-03	0.152 21E-03
30.0	0.354 60E-03	0.277 45E-03	0.192 75E-03	0.122 36E-03
35.0	0.239 46E-03	0.183 41E-03	0.123 09E-03	0.745 88E-04
40.0	0.170 34E-03	0.127 88E-03	0.830 79E-04	0.481 82E-04
45.0	0.126 08E-03	0.928 70E-04	0.585 04E-04	0.325 49E-04
50.0	0.962 96E-04	0.696 63E-04	0.426 18E-04	0.227 92E-04
60.0	0.603 55E-04	0.422 12E-04	0.244 48E-04	0.121 43E-04
70.0	0.406 20E-04	0.275 36E-04	0.151 61E-04	0.703 35E-05
80.0	0.288 15E-04	0.189 74E-04	0.996 60E-05	0.434 01E-05
90.0	0.212 74E-04	0.136 31E-04	0.685 10E-05	0.281 23E-05
100.0	0.161 95E-04	0.101 08E-04	0.486 80E-05	0.188 68E-05

The implication of Figs. 1, 2, 3, and 4 is that as a increases, the relative importance of correlation between the ions becomes increasingly important so that at $a=0.8$, the curves for the several values of R are not too different. Figure 5 shows the electric microfield distribution, at a charged point, for several values of a, with $R=1$; Fig. 6 shows the same results but for $R=\infty$.

TABLE II. Probability distribution $P(\epsilon)$ at a charged point for several values of a, with $R=\infty$. The electric field strength ϵ is in units of ϵ_0. The lines in the columns indicate the point at which asymptotic expressions were used in the calculations.

E	$A=0.2$	$A=0.4$	$A=0.6$	$A=0.8$
0.1	0.214 70E-02	0.910 50E-02	0.195 29E-01	0.430 09E-01
0.2	0.151 70E-01	0.352 86E-01	0.736 63E-01	0.153 08E 00
0.3	0.366 92E-01	0.753 80E-01	0.149 87F 00	0.286 86E 00
0.4	0.644 92E-01	0.124 83E 00	0.233 58E 00	0.405 39E 00
0.5	0.965 15E-01	0.178 52E 00	0.311 75E 00	0.488 90E 00
0.6	0.130 88E 00	0.231 57E 00	0.375 87E 00	0.535 29E 00
0.7	0.165 88E 00	0.279 97E 00	0.422 20E 00	0.551 19E 00
0.8	0.200 03E 00	0.320 90E 00	0.450 51E 00	0.545 13E 00
0.9	0.232 03E 00	0.352 76E 00	0.462 81E 00	0.524 60E 00
1.0	0.260 79E 00	0.375 06E 00	0.462 09E 00	0.495 37E 00
1.1	0.285 46E 00	0.388 17E 00	0.451 52E 00	0.461 60E 00
1.2	0.305 32E 00	0.393 01E 00	0.434 01E 00	0.426 16E 00
1.3	0.319 83E 00	0.390 85E 00	0.411 98E 00	0.390 91E 00
1.4	0.329 35E 00	0.383 04E 00	0.387 37E 00	0.357 03E 00
1.5	0.334 17E 00	0.370 91E 00	0.361 63E 00	0.325 17E 00
1.6	0.334 74E 00	0.355 67E 00	0.335 80E 00	0.295 69E 00
1.7	0.331 62E 00	0.338 37E 00	0.310 61E 00	0.268 70E 00
1.8	0.325 28E 00	0.319 86E 00	0.286 52E 00	0.244 16E 00
1.9	0.316 31E 00	0.300 84E 00	0.263 82E 00	0.221 96E 00
2.0	0.305 39E 00	0.281 82E 00	0.242 64E 00	0.201 95E 00
2.5	0.237 54E 00	0.196 75E 00	0.159 64E 00	0.128 39E 00
3.0	0.172 83E 00	0.135 76E 00	0.107 31F 00	0.849 15E-01
3.5	0.123 77F 00	0.952 32F-01	0.744 88E-01	0.584 50E-01
4.0	0.894 11E-01	0.685 47E-01	0.534 26E-01	0.417 08E-01
4.5	0.658 55E-01	0.506 96E-01	0.394 77E-01	0.306 80E-01
5.0	0.496 09E-01	0.384 60E-01	0.299 51E-01	0.231 81E-01
6.0	0.300 46E-01	0.236 43E-01	0.184 12E-01	0.141 36E-01
6.6	0.195 99E-01	0.155 98E-01	0.121 22E-01	0.109 11E-01
7.6	0.135 55E-01	0.108 70E-01	0.842 50E-02	0.745 65E-02
9.0	0.981 40E-02	0.790 84E-02	0.610 27E-02	0.468 58E-02
10.0	0.736 17E-02	0.594 77E-02	0.456 29E-02	0.349 16E-02
12.0	0.448 41E-02	0.361 44E-02	0.270 41E-02	0.208 04E-02
14.0	0.298 50E-02	0.240 18E-02	0.177 47E-02	0.133 07E-02
16.0	0.209 05E-02	0.167 15E-02	0.122 47E-02	0.897 40E-03
18.0	0.153 40E-02	0.121 86E-02	0.883 66E-03	0.630 44E-03
20.0	0.115 49E-02	0.908 30E-03	0.656 98E-03	0.457 60E-03
22.0	0.903 05E-03	0.688 99E-03	0.501 36E-03	0.341 13E-03
24.0	0.718 35E-03	0.546 54E-03	0.390 99E-03	0.260 04E-03
26.0	0.561 84E-03	0.441 35E-03	0.310 55E-03	0.202 00E-03
28.0	0.465 26E-03	0.361 86E-03	0.250 55E-03	0.159 48E-03
30.0	0.390 28E-03	0.300 62E-03	0.204 90E-03	0.127 69E-03
35.0	0.263 40E-03	0.198 26E-03	0.130 14E-03	0.770 83E-04
40.0	0.187 27E-03	0.137 91E-03	0.873 72E-04	0.493 13E-04
45.0	0.138 53E-03	0.999 38E-04	0.612 12E-04	0.329 97E-04
50.0	0.105 75E-03	0.748 04E-04	0.443 67E-04	0.228 89E-04
60.0	0.662 21E-04	0.451 48E-04	0.252 07E-04	0.119 73E-04
70.0	0.445 29E-04	0.293 40E-04	0.154 88E-04	0.681 20E-05
80.0	0.315 53E-04	0.210 38E-04	0.100 87E-04	0.412 76E-05
90.0	0.232 69E-04	0.144 11E-04	0.686 90E-05	0.262 51E-05
100.0	0.177 13E-04	0.106 62E-04	0.195 98E-05	0.173 65E-05

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APPENDIX

In order to evaluate the terms used in Eq. (24) for $T(l)$, it is convenient to employ the collective-coordinate techniques of Bohm and Pines as used by

Broyles.[8]

The definitions for V_0, w_{j0}, and w_{m0}, and the potential energy V, are given by Eqs. (9)–(11). V can be written in terms of a Fourier series, excluding the $k=0$ term to allow for the boundary condition of charge neutrality:

$$V=\frac{4\pi\lambda^2}{\mathcal{V}}\sum_k{}'\left(\frac{1}{2}\sum_{i\neq j}\frac{e^2e^{-i\vec{k}\cdot\vec{r}_{ij}}}{(k\lambda)^2+1}+\frac{1}{2}\sum_{m\neq n}\frac{(2e)^2e^{-i\vec{k}\cdot\vec{r}_{mn}}}{(k\lambda)^2+1}+\sum_{j,m}\frac{2e^2e^{-i\vec{k}\cdot\vec{r}_{jm}}}{(k\lambda)^2+1}\right). \quad \text{(A1)}$$

The prime indicates that the $k=0$ term has been omitted. This will be understood in all subsequent expressions and the prime will be omitted:

$$\vec{k}=2\pi(\mathcal{V})^{-1/3}\,[n_x\hat{i}+n_y\hat{j}+n_z\hat{k}]\;, \quad \text{(A2)}$$

i.e., box normalization. The n's are positive or negative integers, not all of which can be simultaneously zero. The first term of Eq. A1 can be written in terms of trigonometric functions:

$$\sum_{\vec{k}}\frac{e^{-i\vec{k}\cdot\vec{r}_{ij}}}{(k\lambda)^2+1}=\sum_{\vec{k}}\left(\frac{1}{(k\lambda)^2+1}\right)[\cos(\vec{k}\cdot\vec{r}_{ij})-i\sin(\vec{k}\cdot\vec{r}_{ij})]\;. \quad \text{(A3)}$$

The imaginary part can be shown to sum to zero since the sine is an odd function. Then applying a familiar cosine identity, Eq. (A3) becomes

$$\sum_{\vec{k}}\frac{e^{-i\vec{k}\cdot\vec{r}_{ij}}}{(k\lambda)^2+1}=\sum_{\vec{k}}\left(\frac{1}{(k\lambda)^2+1}\right)[\cos(\vec{k}\cdot\vec{r}_i)\cos(\vec{k}\cdot\vec{r}_j)+\sin(\vec{k}\cdot\vec{r}_i)\sin(\vec{k}\cdot\vec{r}_j)]$$

$$=2\sum_{k_z\geq 0}\frac{\cos(\vec{k}\cdot\vec{r}_i)\cos(\vec{k}\cdot\vec{r}_j)}{(k\lambda)^2+1}+2\sum_{k_z<0}\frac{\sin(\vec{k}\cdot\vec{r}_i)\sin(\vec{k}\cdot\vec{r}_j)}{(k\lambda)^2+1}. \quad \text{(A4)}$$

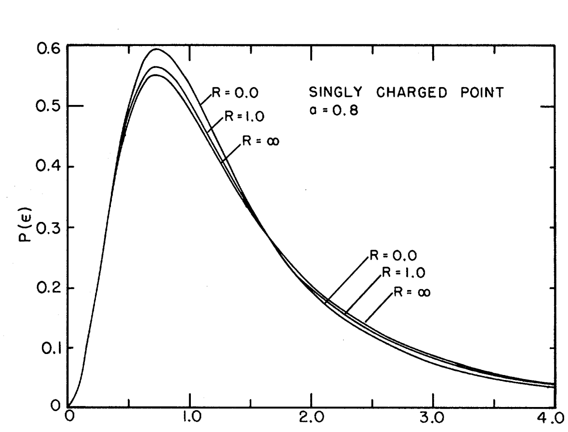

FIG. 4. Electric microfield distribution function $P(\epsilon)$ at a charged point for $a=0.8$, $R=0$, 1, ∞; ϵ is in units of ϵ_0.

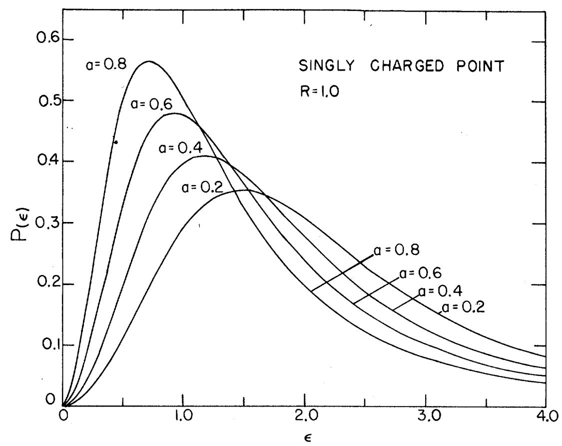

FIG. 5. Electric microfield distribution function $P(\epsilon)$ at a charged point for $R=1$, $a=0.2$, 0.4, 0.6, 0.8; ϵ is in units of ϵ_0.

Next, define

$$S(\vec{k}\cdot\vec{r})\equiv\begin{cases}\cos(\vec{k}\cdot\vec{r}), & k_z\geq 0\\ \sin(\vec{k}\cdot\vec{r}), & k_z<0\;.\end{cases} \quad \text{(A5)}$$

This allows the following equation to be written:

$$\sum_{\vec{k}}\frac{e^{-i\vec{k}\cdot\vec{r}_{ij}}}{(k\lambda)^2+1}=\sum_{\vec{k}}\frac{2S(\vec{k}\cdot\vec{r}_i)S(\vec{k}\cdot\vec{r}_j)}{(k\lambda)^2+1}\;. \quad \text{(A6)}$$

If similar definitions are made for the terms involving $\exp(-i\vec{k}\cdot\vec{r}_{mn})$ and $\exp(-i\vec{k}\cdot\vec{r}_{jm})$, the potential energy can be written as

$$V=\frac{4\pi\lambda^2}{\mathcal{V}}\frac{\sigma}{2}\sum_{\vec{k}}\left(\frac{1}{(k\lambda)^2+1}\right)\left(\sum_{i,j}\frac{2e^2}{\sigma}S_i^+S_j^+ +\sum_{m,n}\frac{2(2e)^2}{\sigma}S_m^{++}S_n^{++}\right.$$

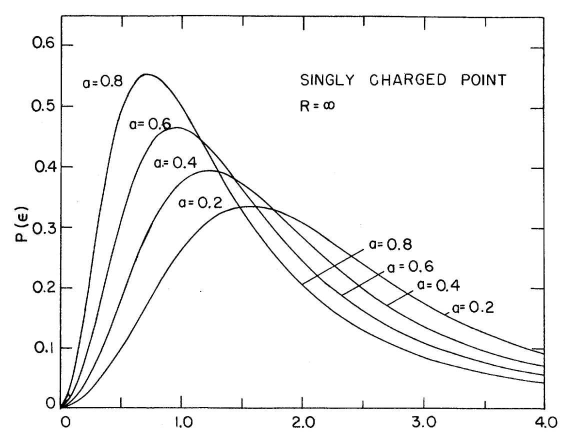

FIG. 6. Electric microfield distribution function $P(\epsilon)$ at a charged point for $R=\infty$, $a=0.2$, 0.4, 0.6, 0.8; ϵ is in units of ϵ_0.

$$+2\sum_{j,m}\frac{2(2e^2)}{\sigma}S_j^+S_m^{++}+2\sum_i\frac{2eq}{\sigma}S_i^+S_0$$
$$+2\sum_m\frac{4eq}{\sigma}S_m^{++}S_0-\frac{2}{\sigma}e^2N^+-\frac{2}{\sigma}(2e)^2N^{++}\bigg),\quad \text{(A7)}$$

where

$$\sigma=N^+e^2+N^{++}(2e)^2 .$$

The last two terms subtract the self-energy that is added by removing the restrictions $i=j$ and $m=n$ from the first two terms. The two terms involving q take into account the charge $q=\xi e$ placed at the origin. Previously, these terms have not been explicitly written, but were understood to be present for $\xi=0$, 1, and 2 in the general interaction potential, Eq. (9). In subsequent expressions it will be necessary to treat them explicitly. It can be shown that Eq. (A7) is correct for any positive value of ξ. By defining the generalized coordinates

$$X_k^+\equiv\sum_j(2e^2/\sigma)^{1/2}S_j^+ \text{ and } X_k^{++}\equiv\sum_m[2(2e)^2/\sigma]^{1/2}S_m^{++}$$

this expression can be written in a more compact form:

$$V=\frac{2\pi\lambda^2\sigma}{\mathcal{V}}\sum_{\vec{k}}\left(\frac{1}{(k\lambda)^2+1}\right)[(X_k^+)^2+(X_k^{++})^2+2X_k^+X_k^{++}$$
$$+2(2q^2/\sigma)^{1/2}S_0(X_k^++X_k^{++})-2].\quad \text{(A8)}$$

According to the definition [Eq. (A5)], $S_0=1$ if $k_z\geq 0$ and equals zero if $k_z<0$. By using the expressions

$$\frac{2\pi\lambda^2\sigma}{\mathcal{V}}=\frac{\theta_e}{2}\left(\frac{1+4R}{1+2R}\right),\quad R=\frac{n^{++}}{n^+},\quad \theta_e=kT_e,$$
$$A_k=\frac{1}{(k\lambda)^2+1},\qquad Y_k\equiv X_k^++X_k^{++},\quad \text{(A9)}$$

V can be written as

$$V=\frac{\theta_e}{2}\left(\frac{1+4R}{1+2R}\right)$$
$$\times\left[\sum_{\vec{k}}A_kY_k^2+2(2q^2/\sigma)^{1/2}\sum_{k_Z\geq 0}A_kY_k-2\sum_{\vec{k}}A_k\right],\quad \text{(A10)}$$

where the Y_k represent a new collective coordinate. Similar transformations can be made on w_{j0} and w_{m0}, which result in

$$\sum_j w_{j0}=2\frac{\theta_e}{2}\left(\frac{1+4R}{1+2R}\right)\left(\frac{2q^2}{\sigma}\right)^{1/2}\sum_{k_Z\geq 0}\frac{X_k^+}{(k\lambda)^2+\alpha^2},$$
$$\sum_m w_{m0}=2\frac{\theta_e}{2}\left(\frac{1+4R}{1+2R}\right)\left(\frac{2q^2}{\sigma}\right)^{1/2}\sum_{k_Z\geq 0}\frac{X_k^{++}}{(k\lambda)^2+\alpha^2},$$

$$\sum_j w_{j0}+\sum_m w_{m0}=\theta_e\left(\frac{1+4R}{1+2R}\right)\left(\frac{2q^2}{\sigma}\right)^{1/2}\sum_{k_Z\geq 0}\frac{Y_k}{(k\lambda)^2+\alpha^2}.\quad \text{(A11)}$$

Now it is possible to write V_0 in terms of the collective coordinate Y_k:

$$V_0=V-\sum_j w_{j0}-\sum_m w_{m0}$$
$$=\frac{\theta_e}{2}\left(\frac{1+4R}{1+2R}\right)\bigg(\sum_{\vec{k}}A_kY_k^2+2(2q^2/\sigma)^{1/2}$$
$$\times\sum_{k_Z\geq 0}f_k(\alpha)A_kY_k-2\sum_{\vec{k}}A_k\bigg),\quad \text{(A12)}$$
$$f_k(\alpha)=\frac{\alpha^2-1}{(k\lambda)^2+\alpha^2},\quad A_k=\frac{1}{(k\lambda)^2+1)}.$$

In order to evaluate $\vec{\nabla}_0 V$, it is necessary to consider the following relation:

$$\vec{\nabla}_0 S(\vec{k}\cdot\vec{r})=\begin{cases}\vec{\nabla}_0\cos(\vec{k}\cdot\vec{r})=-\vec{k}\sin(\vec{k}\cdot\vec{r}), & k_Z\geq 0\\ \vec{\nabla}_0\sin(\vec{k}\cdot\vec{r})=\vec{k}\cos(\vec{k}\cdot\vec{r}), & k_Z<0 .\end{cases}\quad \text{(A13)}$$

Changing $\vec{k}$ to $-\vec{k}$ we find that

$$\vec{\nabla}_0 S(\vec{k}\cdot\vec{r})=\begin{cases}-\vec{k}\cos(\vec{k}\cdot\vec{r}), & k_Z\geq 0\\ -\vec{k}\sin(\vec{k}\cdot\vec{r}), & k_Z<0 .\end{cases}\quad \text{(A14)}$$

Therefore,

$$\vec{\nabla}_0 S(\vec{k}\cdot\vec{r})=-\vec{k}S(\vec{k}\cdot\vec{r}),\quad \text{(A15)}$$

which implies that

$$\vec{\nabla}_0\sum_{k\geq 0}Y_k=-\sum_{k<0}Y_k\vec{k},\quad \text{(A16)}$$

since Y_k is a linear combination of the $S(\vec{k}\cdot\vec{r})$. If the gradients of the expressions given by Eqs. (9), (11), and (12) are evaluated, the terms which are summed over all k_Z go to zero. The only terms that contribute are the ones summed only over negative k_Z. The results are

$$\vec{\nabla}_0 V=-\theta_e\left(\frac{1+4R}{1+2R}\right)\left(\frac{2q^2}{\sigma}\right)^{1/2}\sum_{k_Z\geq 0}A_kY_k\vec{k},$$
$$\vec{\nabla}_0 V_0=-\theta_e\left(\frac{1+4R}{1+2R}\right)\left(\frac{2q^2}{\sigma}\right)^{1/2}\sum_{kZ<0}f_k(\alpha)A_kY_k(\vec{k}),\quad \text{(A17)}$$
$$\vec{\nabla}_0\left(\sum_j w_{j0}+\sum_m w_{m0}\right)=\vec{\nabla}_0 V-\vec{\nabla}_0 V_0$$
$$=\frac{\theta_e}{2}\left(\frac{1+4R}{1+2R}\right)\left(\frac{2q^2}{\sigma}\right)^{1/2}\sum_{kZ<0}\left[\frac{1}{(k\lambda)^2+\alpha^2}\right]Y_k\vec{k}.$$

Sufficient definitions have now been made so that $T_0(l)/T_0(0)$, the first factor in $T(l)$, can be written in terms of collective coordinates

$$\frac{T_0(l)}{T_0(0)}=\frac{\int\cdots\int\exp(-\beta_i V_0-iq^{-1}\vec{l}\cdot\vec{\nabla}_0 V_0)\prod_{j=1}^{N^+}\prod_{m=1}^{N^{++}}d\vec{r}_j\,d\vec{r}_m}{\int\cdot\int\exp(-\beta_i V_0)\prod_{j=1}^{N^+}\prod_{m=1}^{N^{++}}d\vec{r}_j\,d\vec{r}_m}$$

$$=\frac{\int\cdots\int \exp\{-\beta_i\theta_e/2)[(1+4R)/(1+2R)]\sum_{\vec{k}}[A_k Y_k^2+2b_k(l)Y_k]\}J\prod dY_k}{\int\cdots\int\exp\{(-\beta_i\theta_e/2)[(1+4R/(1+2R)]\sum_{\vec{k}}[A_kY_k^2+2b_k(0)Y_k]\}J\prod dY_k},$$

$$b_k(l)=\left(\frac{2q^2}{\sigma}\right)^{1/2} f_k(\alpha)A_k\times\begin{cases}1, & k_z\geq 0\\ -i\theta_i q^{-1}\vec{l}.\vec{k}, & k_z<0\end{cases}$$

$$b_k(0)=\left(\frac{2q^2}{\sigma}\right)^{1/2} f_k(\alpha)A_k\times\begin{cases}1, & k_z\geq 0\\ 0, & k_z<0\,.\end{cases}$$

The self-energy portions of V_0, appearing in both the numerator and the denominator, are independent of the integrations and cancel. J is the Jacobian of the transformation from $\vec{r}_{ij}$ and $\vec{r}_{mn}$ to $\vec{Y}_k$. If we define

$$A_k'\equiv\frac{\theta_e}{\theta_i}\left(\frac{1+4R}{1+2R}\right)A_k\equiv uA_k\,,$$

$$b_k'\equiv\frac{\theta_e}{\theta_i}\left(\frac{1+4R}{1+2R}\right)b_k\equiv ub_k\,, \tag{A19}$$

the expression for $T_0(l)/T_0(0)$ can be put in the form

$$\frac{T_0(l)}{T_0(0)}=\frac{\int\cdots\int\exp[-\frac{1}{2}\sum_{\vec{k}}(A_k'Y_k^2+2b_k'(l)Y_k)]J\prod dY_k}{\int\cdots\int\exp[-\frac{1}{2}\sum_{\vec{k}}(A_k'Y_k^2+2b_k'(0)Y_k]J\prod dY_k}\,. \tag{A20}$$

This allows for evaluation of the integrals according to the formalism used by Broyles. The multiple integrations are thus evaluated:

$$\frac{T_0(l)}{T_0(0)}=\frac{\exp\{\frac{1}{2}\sum_{\vec{k}}[b_k'(l)]^2/(1+A_k')\}[1-a_3(l)+a_4(l)\cdots]}{\exp\{\frac{1}{2}\sum_{\vec{k}}[b_k'(0)]^2/(1+A_k')\}[1-a_3(0)+a_4(0)\cdots]}\,. \tag{A21}$$

a_3 and a_4 can be shown to be small and are neglected. The definitions of $b_k'(l)$ and $b_k'(0)$ further allow us to write

$$\frac{T_0(l)}{T_0(0)}=\exp\left\{\frac{-\theta_e^2}{\sigma}\left(\frac{1+4R}{1+2R}\right)^2\sum_{k_z<0}\left[\frac{f_k^2(\alpha)A_k^2(\vec{l}\cdot\vec{k})^2}{1+A_k'}\right]\right\}. \tag{A22}$$

The sum over k is now replaced by an integral and evaluated. The result of this integration, in terms of conveniently defined reduced variables, is

$$T_0(l)/T_0(0)=e^{-\gamma L^2}\,,$$

$$\gamma=\frac{a}{4}\frac{\theta_i}{\theta_e}\frac{1}{[\alpha^2-(1+u)]^2}\{\alpha^5u+2[1-(1+u)^{3/2}]\alpha^4$$

$$+(2u+u^2)\alpha^3-4(1+u)[(1-u)^{1/2}]\alpha^2$$

$$-3(u+u^2)\alpha+2[(1+u)^2-(1+u)^{3/2}]\}\,,$$

$$u=\frac{\theta_e}{\theta_i}\left(\frac{1+4R}{1+2R}\right),\quad R=\frac{n^{++}}{n^+},\quad \theta_e=kT_e\,,\quad \theta_i=kT\,,$$

$$\tfrac{4}{3}\pi r_0^3 n_e=1\,,\quad a=\frac{r_0}{\lambda}\,,\quad \epsilon_0=\frac{e}{r_0^2}\,,\quad L=\epsilon_0 l\,. \tag{A23}$$

With this choice of reduced quantities, the unit of electric field strength ϵ_0 is a function of electron density only.

In order to evaluate the remaining terms in the exponential of Eq. (24), it is necessary to derive an expression for $Q_{jm}(l)$ in terms of these same reduced quantities. Collective coordinates are used again with the modification indicated below. $Q_{jm}(l)$ is defined by Eq. (16). In terms of collective coordinates

$$Q_{jm}(l)=\frac{\int\cdots\int\exp\{-\frac{1}{2}\sum_{\vec{k}}[A_k'Y_k^2+2b_k'(l)Y_k]\}\prod_{j+1}^{N^+}\prod_{m+1}^{N^{++}}d\vec{r}_i\,d\vec{r}_n}{\int\cdots\int\exp\{-\frac{1}{2}\sum_{\vec{k}}[A_k'Y_k^2+2b'_k(l)Y_k]\}\prod_{i=1}^{N^+}\prod_{n=1}^{N^{++}}d\vec{r}_i\,d\vec{r}_n}\,. \tag{A24}$$

In order to do the $(N^+-j)+(N^{++}-m)$ integrals, it is convenient to introduce another collective coordinate:

$$Y_k'\equiv Y_k-y_k\equiv(X_k^++X_k^{++})-(a_k^++a_k^{++})\,, \tag{A25}$$

where

$$a_k^+\equiv\left(\frac{2e^2}{\sigma}\right)^{1/2}\sum_{i=1}^{j}S^+(\vec{k}\cdot\vec{r}_i)\,,$$

$$a_k^{++}=\left(\frac{2(2e)^2}{\sigma}\right)^{1/2}\sum_{n=1}^{m}S^{++}(\vec{k}\cdot\vec{r}_n)\,.$$

In terms of this new collective coordinate the exponentials in Eq. (A24) are

$$-\tfrac{1}{2}\sum_{\vec{k}}(A_k'Y_k^2+2b_k'Y_k)=-\tfrac{1}{2}\sum_{\vec{k}}[A_k'(Y_k')^2+2(y_kA_k'+b_k')Y_k'+y_k^2A_k'+2y_kb_k']\,. \tag{A26}$$

This allows factors of $Q_{jm}(l)$ to be integrated by the collective-coordinate technique as follows:

$$Q_{jm}(l)=\frac{\exp\{-\frac{1}{2}\sum_{\vec{k}}[A_k'y_k^2+2y_kb_k']\}\int\cdots\int\exp\{-\frac{1}{2}\sum_{\vec{k}}[A_k'(Y_k')^2+2(y_kA_k'+b_k')]\}JdY_k}{\int\cdots\int\exp\{-\frac{1}{2}\sum_{\vec{k}}[A_k'y_k^2+2y_kb_k']\}\int\cdots\int\exp\{-\frac{1}{2}\sum_{\vec{k}}[A_k'(Y_k')^2+2(y_kA_k'+b_k')]\}JdY_k\prod^{j}\prod_{n}^{m}d\vec{r}_i\,d\vec{r}_n}$$

$$=\frac{\exp\{\frac{1}{2}\sum_{\vec{k}}[y_kA_k'+b_k')^2/(1+A_k')-y_k^2A_k'-2y_kb_k']\}}{\int\cdots\int\exp\{\frac{1}{2}\sum_k[(y_kA_k'+b_k')^2/(1+A_k')-y_k^2A_k'-2y_kb_k']\}\prod_{i}^{j}\prod_{n}^{m}d\vec{r}_i\,d\vec{r}_n} \tag{A27a}$$

$$= \frac{(\mathcal{V})^{-j-m}\exp\{-\frac{1}{2}\sum_{\vec{k}}(y_k^2A_k')/(1+A_k')-\sum_{\vec{k}}(y_kb_k')/(1+A_k')\}}{(\mathcal{V})^{-j-m}\int\cdots\int\exp\{-\frac{1}{2}\sum_{\vec{k}}(y_k^2A_k')(1+A_k')-\sum_{\vec{k}}(y_kb_k'(1+A_k')\}\quad \Pi_j\Pi_m d\vec{r}_j d\vec{r}_m} . \tag{A27b}$$

The last equation has been simplified algebraically and the terms in the numerator and denominator that are independent of $y_k(\vec{r})$ have been canceled. For Q_{10} and Q_{01}, the sum $\sum_{\vec{k}}[y_k^2A_k'/(1+A_k')]$ appearing in the numerator and denominator is independent of the coordinates of integration and will also cancel. In the thermodynamic limit, it can be shown that the denominator of Eq. (A37b) becomes

$$\mathcal{V}\to\infty \left\{\frac{1}{\mathcal{V}}\int \exp\left[-\sum_{\vec{k}}\frac{y_kb_k'}{1+A_k'}\right]d\vec{r}\right\}=1 . \tag{A28}$$

It is also necessary to have expressions for $-\beta_i w_{j0}$ and $-\beta_i w_{m0}$ and their respective gradients, in terms of the reduced variables defined in Eqs. (A23) and (A31). These are listed below:

$$-\beta_i w_{j0}\equiv-\frac{1}{\theta_i}\frac{qe}{r_{j0}}e^{-\alpha r_{j0}/\lambda}=-\xi\left(\frac{\theta_e}{\theta_i}\right)\frac{a^2}{3x}e^{-\alpha a x}=-\beta w ,$$

$$-\beta_i w_{m0}=-\frac{1}{\theta_i}q\frac{2e}{r_{m0}}e^{-\alpha r_{m0}/\lambda}$$

$$=-2\xi\left(\frac{\theta_e}{\theta_i}\right)\frac{a^2}{3x}e^{-\alpha a x}=-2\beta w , \tag{A33}$$

$$-iq^{-1}\vec{\ell}\cdot\vec{\nabla}_0 w_{j0}=iL\cos\theta[(e^{-\alpha a x}/x^2)(1+\alpha a x)] ,$$

$$-iq^{-1}\vec{\ell}\cdot\vec{\nabla}_0 w_{m0}=2iL\cos\theta(e^{-\alpha a x}/x^2)(1+\alpha a x) .$$

By comparing the expressions for $T(l)$ given by Eqs. (19) and (24), we can define I_{10} and I_{01}:

$$I_{10}\equiv n^+[h_{10}(l)-h_{10}(0)]$$
$$=n^+[\int g_{10}(l)\chi^+(l,1)d\vec{r}-\int g_{10}(0)\chi^+(0,1)d\vec{r}] , \tag{A34}$$
$$I_{01}\equiv n^{++}[h_{01}(l)-h_{01}(0)]$$
$$=n^{++}[\int g_{01}(l)\chi^{++}(l,1)d\vec{r}-\int g_{01}(0)\chi^+(0,1)d\vec{r}] .$$

The remaining variables can be transformed to the set of dimensionless quantities defined by Eq. (A23):

$$n^+d\vec{r}=n^+r^2dr\,d\Omega=[3/(1+2R)]x^2dx\,d\Omega ,$$
$$n^{++}d\vec{r}=n^{++}r^2dr\,d\Omega=[3R/(1+2R)]x^2dx\,d\Omega . \tag{A35}$$

The relationship between g_{jm} and Q_{jm} [Eq. (18)] together with Eq. (A30) allows I_{10} to be written as

$$I_{10}=\frac{3}{1+2R}\frac{1}{2}\int_0^\infty\left(\int_0^\pi\int_0^{2\pi} \times e^{s(x)+iLq(x)\cos\theta}(e^{-\beta w_{j0}-iL(\nabla_0 w)\cos\theta}-1)d\Omega - \int_0^\pi\int_0^{2\pi}e^{s(x)}(e^{-\beta w_{j0}}-1)d\Omega\right)x^2dx . \tag{A36}$$

The angular integration can be easily performed to give

$$I_{10}=\frac{3}{1+2R}\int_0^\infty x^2dx\,e^{s(x)}\left[e^{-\beta w}\left(\frac{\sin(LG(x))}{(LG(x))}-1\right) - \left(\frac{\sin(Lq(x))}{(Lq(x))}-1\right)\right] , \tag{A37}$$

where

$$G(x)=q(x)+\frac{e^{-\alpha a x}}{x^2}[1+\alpha a x] . \tag{A38}$$

In a similar manner I_{01} can be evaluated as

$$I_{01}=\frac{3R}{1+2R}\int_0^\infty x^2dx\,e^{2s(x)}\left[e^{-2\beta w}\left(\frac{\sin(2LG(x))}{2LG(x)}-1\right) - \left(\frac{\sin(2Lq(x))}{2Lq(x)}\right)\right] . \tag{A39}$$

In order to calculate I_{20}, I_{02}, and I_{11}, it is necessary to have expressions for $\sum_{\vec{k}}[y_kb_k'/(1+A_k')]$ and $\sum_{\vec{k}}[y_k^2A_k'/(1+A_k')]$ in each case. For convenience we label these cases (a), (b), and (c), respectively.

Case (a). As seen in Eq. (A25), y_k is given by

$$y_k=(2e^2/\sigma)^{1/2}[S(\vec{k}\cdot\vec{r}_1)+S(\vec{k}\cdot\vec{r}_2)] . \tag{A40}$$

Therefore

$$-\sum_{\vec{k}}\frac{y_kA_k'}{1+A_k}=[s(x_1)+iLq(x_1)\cos\theta_1] + [s(x_2)+iLq(x_2)\cos\theta_2] , \tag{A41}$$

which is a linear combination of the expressions for the first-order term. Similar considerations give the necessary expressions for cases (b) and (c).

Case (b):

$$-\sum_{\vec{k}}\frac{y_kA_k'}{1+A_k'}=2[s(x_1)+iLq(x_1)\cos\theta_1] + 2[s(x_2)+iLq(x_2)\cos\theta_2] . \tag{A42}$$

Case (c):

$$-\sum_{\vec{k}}\frac{y_kA_k'}{1+A_k'}=[s(x_1)+iLq(x_1)\cos\theta_1] + 2[s(x_2)+iLq(x_2)\cos\theta_2] . \tag{A43}$$

For case (c), particle 2 is assumed to be doubly charged.

In order to evaluate the second term of these expressions, we also need an expression for y_k^2 in each case.

Case (a):

$$y_k^2=(2e^2/\sigma)[S^2(\vec{k}\cdot\vec{r}_1)+2S(\vec{k}\cdot\vec{r}_1)S(\vec{k}\cdot\vec{r}_2)+S^2(\vec{k}\cdot\vec{r}_2)] . \tag{A44}$$

When summed over k, the $S^2(\vec{k}\cdot\vec{r})$ terms in the numerator and denominator of (A44) are independent of r and will cancel in the same manner as that seen when I_{10} was evaluated [see Eq. (A27)]. Hence, the necessary terms are

$$y_k^2 = (4e^2/\sigma)\, S(\vec{k}\cdot\vec{r}_1)\, S(\vec{k}\cdot\vec{r}_2)\ . \tag{A45}$$

Case (*b*):

$$y_k^2 = [4(2e)^2/\sigma]\, S(\vec{k}\cdot\vec{r}_1)\, S(\vec{k}\cdot\vec{r}_2)\ . \tag{A46}$$

Case (*c*):

$$y_k^2 = [4(2e^2)/\sigma]\, S(\vec{k}\cdot\vec{r}_1)\, S(\vec{k}\cdot\vec{r}_2)\ . \tag{A47}$$

If Eqs. (A45)–(A47) are used to evaluate $-\frac{1}{2}\sum_{\vec{k}} \times[y_k^2 A_k'/(1+A_k')]$, the results are as follows.

Case (*a*):

$$-\tfrac{1}{2}\sum_{\vec{k}} \frac{y_k^2 A_k'}{1+A_k'} = -\frac{2e^2}{\sigma}\sum_{\vec{k}} \frac{S(\vec{k}\cdot\vec{r}_1)\, S(\vec{k}\cdot\vec{r}_2)}{1+A_k'}\, A_k'$$

$$= \frac{-2e^2}{\sigma}\sum_{k_Z \geq 0} \frac{u\cos[\vec{k}\cdot(\vec{r}_1-\vec{r}_2)]}{(k\lambda)^2+(1+u)}$$

$$= \frac{-\theta_e}{\theta_i}\, \frac{a^2}{3x_{12}}\, \exp[-(1+u)^{1/2}ax_{12}]. \tag{A48}$$

Case (*b*):

$$-\tfrac{1}{2}\sum_{\vec{k}} \frac{y_k^2 A_k'}{1+A_k'} = -4\,\frac{\theta_e}{\theta_i}\, \frac{a^2}{3x_{12}}\, \exp[-(1+u)^{1/2}ax_{12}]\ . \tag{A49}$$

Case (c):

$$-\frac{1}{2}\sum_{\vec{k}} \frac{y_k^2 A_k'}{1+A_k'} = -2\,\frac{\theta_e}{\theta_i}\, \frac{a^2}{3x_{12}}\, \exp[-(1+u)^{1/2}ax_{12}]. \tag{A50}$$

In Eqs. (A48)–(A50) the summations over k were replaced by integrals which were then evaluated.

The expression for I_{20}, obtained by comparing the expressions for $T(l)$ in Eqs. (19) and (24) is

$$I_{20} \equiv \frac{(n^+)^2}{2!}\,[h_{20}(l)-h_{20}(0)]$$

$$= \frac{(n^+)^2}{2!}\{\mathcal{V}^2 \int\int [Q_{20}(l;1,2) - Q_{10}(l,1)\,Q_{10}(l,2)] \times \chi^+(l,1)\,\chi^+(l,2)\,d\vec{r}_1 d\vec{r}_2$$

$$- \mathcal{V}^2\int\int [Q_{20}(0;1,2) - Q_{10}(0,1)Q_{10}(0,2)] \times \chi^+(0,1)\,\chi^+(0,2)\,d\vec{r}_1 d\vec{r}_2\}\ . \tag{A51}$$

If Eqs. (A48)–(A50) are used to evaluate $Q_{20}(l)$, as given in Eq. (A27), the result is

$$Q_{20}(l;1,2) = Q_{10}(l;1)\, Q_{10}(l;2) \times \exp\left(\frac{-\theta_e}{\theta_i}\, \frac{a^2}{3x_{12}}\, e^{-(1+u)^{1/2}ax_{12}}\right). \tag{A52}$$

This can be substituted into the definition of $h_{20}(l)$:

$$h_{20}(l) = \mathcal{V}^2(a\lambda)^6 \int\int Q_{10}(l;1)\, Q_{10}(l,2)\, \chi^+(l;1)\chi^+(l,2) \times\left[\exp\left(\frac{-\theta_e}{\theta_i}\, \frac{a^2}{3x_{12}}\, e^{-(1+u)^{1/2}ax}\right) - 1\right] d\vec{x}_1 d\vec{x}_2. \tag{A53}$$

As indicated in the body of this paper, the expression appearing in brackets is the Debye-Hückel pair correlation function which can be simplified by employing the linearized approximation:

$$\left[\exp\left(\frac{-\theta_e}{\theta_i}\, \frac{a^2}{3x_{12}}\, e^{-(1+u)^{1/2}ax}\right) - 1\right] \cong -\frac{\theta_e}{\theta_i}\, \frac{a^2}{3x_{12}}\, e^{-(1+u)^{1/2}ax} \equiv \frac{\theta_e}{\theta_i}\, V(x_{12})\ . \tag{A54}$$

The pair potential $V(x_{12})$ can be decoupled by the method of Swiatecki[9]:

$$V(x_{12}) \equiv \frac{-a^2}{3x_{12}}\, \exp[-a'x_{12}] = +\sum_{k=0}^{\infty}(2k+1)\, V_k(x_1,x_2)\, P_k(\cos\theta_{12})\ , \tag{A55}$$

$$V_k(x_1,x_2) = -\tfrac{1}{2}\int_0^{\pi} \frac{a^2}{3x_{12}}\, e^{-a'x_{12}}\, P_k(\cos\theta)\sin\theta\, d\theta\ , \tag{A56}$$

where $a' = (1+u)^{1/2}a$. V_k has been evaluated by Swiatecki as

$$V_k(x_1,x_2) = \frac{-a^2}{3}\, \frac{K_{k+1/2}(a'x_1)}{\sqrt{x_1}}\, \frac{I_{k+1/2}(a'x_2)}{\sqrt{x_2}}\ , \quad x_1 > x_2\ . \tag{A57}$$

I and K are modified Bessel functions.[6] The Legendre polynomial $P_k(\cos\theta_{12})$ can be written as[10]

$$P_k(\cos\theta_{12}) = \sum_{m=0}^{k} \epsilon_m\left(\frac{(k-m)!}{(k+m)!}\right) \times P_k^m(\cos\theta_1)\, P_k^m(\cos\theta_2)\cos(m(\varphi_1-\varphi_2)),$$

$$\epsilon_m = \begin{cases} 1, & m=0 \\ 2, & m\neq 0\ . \end{cases} \tag{A58}$$

Since $\int_0^{2\pi}\int_0^{2\pi} d\varphi_1 d\varphi_2 \cos(m(\varphi_1-\varphi_2))$, used in the evaluation of Eq. (A53), is nonzero only if $m=0$, the only contributing term in Eq. (A58) is $P_k(\cos\theta_1) \times P_k(\cos\theta_2)$. This form for $P_k(\cos\theta_{12})$ and the Swiatecki expression [Eq. (A57)] allow $h_{20}(l)$ and $h_{20}(0)$ to be written as

$$h_{20}(l) = (a\lambda)^6\, \frac{\theta_e}{\theta_i}\left(\frac{-a^2}{3}\right)\sum_k (2k+1) \times\int\int d\vec{x}_1 d\vec{x}_2 f(x_1) g(x_2)\ , \tag{A59}$$

$$f(x_1)=\exp[s(x_1)+iLq(x_1)\cos\theta_1]\times\{\exp[-\beta w(x_1)+iL\nabla w(x_1)\cos\theta_1]-1\}\times P_k(\cos\theta_1)K_{k+1/2}(a'x_1)/\sqrt{x_1}\ ,\quad \text{(A60)}$$

$$g(x_2)=\exp[s(x_2)+iLq(x_2)\cos\theta_2]\times\{\exp[-\beta w(x_2)+iL\nabla w(x_2)\cos\theta_2]-1\}\times P_k(\cos\theta_2)I_{k+1/2}(a'x_2)/\sqrt{x_2}\quad \text{for } x_1>x_2\ ,\quad \text{(A61)}$$

$$h_{20}(0)=(a\lambda)^6\frac{\theta_e}{\theta_i}(4\pi)^2\left(\frac{-a^2}{3}\right)\times 2\delta_{k0}\int_0^\infty dx_2\int_{x_2}^\infty dx_1 f_0(x_1)g_0(x_2)\ ,\quad \text{(A62)}$$

$$f_0(x_1)=x_1^{3/2}K_{1/2}(a'x_1)e^{s(x_1)}(e^{-\beta w(x_1)}-1)\ ,\quad \text{(A63)}$$

$$g_0(x_2)=x_2^{3/2}I_{1/2}(a'x_2)e^{s(x_2)}(e^{-\beta w(x_2)}-1)\ .\quad \text{(A64)}$$

By using the integral definition of the spherical Bessel function

$$j_k(Z)\equiv\tfrac{1}{2}(-i)^k\int_0^\pi e^{iZ\cos\theta}P_k(\cos\theta)\sin\theta\,d\theta\ ,\quad \text{(A65)}$$

the angular portion of Eq. (A59) can be evaluated. If these forms for $h_{20}(l)$ and $h_{20}(0)$ are used in Eq. (A51), the final form for I_{20} becomes

$$I_{20}=\frac{\theta_e}{\theta_i}\left[\frac{1}{1+2R}\right]^2 3a^2\sum_k(-1)^k(2k+1)\{20\}\ ,$$
$$\{20\}=\left\{\int_0^\infty x_2^{3/2}I_{k+1/2}(a'x_2)e^{s(x_2)}[e^{-\beta w(x_2)}j_k(LG(x_2))-j_k(Lq(x_2))]-\delta_{k0}\int_0^\infty x_1^{3/2}K_{k+1/2}(a'x_1)e^{s(x_1)}\right.$$
$$\times[e^{-\beta w(x_1)}j_k(LG(x_1))-j_k(Lq(x_1))]dx_1\,dx_2\ x_2^{3/2}I_{1/2}(a'x_2)e^{s(x_2)}$$
$$\left.\times(e^{-\beta w(x_2)}-1)\int_{x_2}^\infty x_1^{3/2}K_{1/2}(a'x_1)e^{s(x_1)}[e^{-\beta w(x_1)}-1)dx_1\,dx_2\right\}\ .\quad \text{(A66)}$$

I_{02} is relatively simple and can be treated similarly with only appropriate factors of two needing to be considered. The final form is

$$I_{02}=4\frac{\theta_e}{\theta_i}\left(\frac{R}{1+2R}\right)^2 3a^2\sum_k(-1)^k(2k+1)\{02\}\ ,$$
$$\{02\}=\left\{\int_0^\infty x_2^{3/2}I_{k+1/2}(a'x_2)e^{2s(x_2)}[e^{-2\beta w(x_2)}j_k(2LG(x_2))-j_k(2Lq(x_2))]\right.$$
$$\times\int_{x_2}^\infty x_1^{3/2}K_{k+1/2}(a'x_1)e^{2s(x_1)}[e^{-2\beta w(x_1)}j_k(2LG(x_1))-j_k(2Lq(x_1))]dx_1dx_2$$
$$\left.-\delta_{k0}\int_0^\infty x_2^{3/2}I_{1/2}(a'x_2)e^{2s(x_2)}(e^{-2\beta w(x_2)}-1)\int_{x_2}^\infty x_1^{3/2}K_{1/2}(a'x_1)e^{2s(x_1)}(e^{-2\beta w(x_1)}-1)\,dx_1\,dx_2\right\}\ .\quad \text{(A67)}$$

I_{11} has more subtle differences. First, there is a factor of 2 multiplying $V(r_{12})$ [compare Eqs. (A48) and (A50)]. Second, as can be seen from the definition of I_{11}, obtained by comparing Eqs. (22) and (36), there is no 2! dividing the densities. Third, because the integrand is not symmetric in x_1 and x_2, the cases $x_1>x_2$ and $x_1<x_2$ cannot be replaced by one integration which results in another factor of 2. With these differences accounted for, the final expression is

$$I_{11}=\frac{2\theta_e}{\theta_i}\left(\frac{1}{1+2R}\right)\left(\frac{R}{1+2R}\right)3a^2\sum_k(-1)^k(2k+1)\{11\}\ ,$$
$$\{11\}=\left\{\int_0^\infty x_2^{3/2}I_{k+1/2}(a'x_2)e^{2s(x_2)}[e^{-2\beta w(x_2)}j_k(2LG(x_2))-j_k(2Lq(x_2))]\right.$$
$$\times\int_{x_2}^\infty x_1^{3/2}K_{k+1/2}(a'x_1)e^{s(x_1)}[e^{-\beta w(x_1)}j_k(LG(x_1))-j_k(Lq(x_1))]dx_1\,dx_2$$
$$+\int_0^\infty x_2^{3/2}K_{k+1/2}(a'x_2)e^{2s(x_2)}[e^{-2\beta w(x_2)}j_k(2LG(x_2))-j_k(2Lq(x_2))]\quad \text{(A68)}$$

$$\times\int_0^{x_2} x_1^{3/2} I_{k+1/2}(a'x_1)\, e^{s(x_1)}\left[e^{-\beta w(x_1)} j_k(LG(x_1)) - j_k(Lq(x_1))\right] dx_1\, dx_2$$

$$-\delta_{k0}\int_0^{\infty} x_2^{3/2} I_{1/2}(a'x_2)\, e^{2s(x_2)}(e^{-2\beta\omega(x_2)}-1)\int_{x_2}^{\infty} x_1^{3/2} K_{1/2}(a'x_1) e^{s(x_1)}(e^{-\beta w(x_1)}-1)\, dx_1\, dx_2$$

$$-\delta_{k0}\int_0^{\infty} x_2^{3/2} K_{1/2}(a'x_2)\, e^{2s(x_2)}(e^{-2\beta w(x_2)}-1)\int_0^{x_2} x_1^{3/2} I_{1/2}(a'x_1)\, e^{s(x_1)}(e^{-\beta w(x_1)}-1)\, dx_1\, dx_2 \Big\} .$$

With these expressions for $I_{jm}(l)$ up to second order, we can evalute $T(l)$:

$$T(l) = \exp[-\gamma L^2 + I_{10}(l) + I_{01}(l) + I_{20}(l) + I_{02}(l) + I_{11}(l)] . \qquad \text{(A69)}$$

If we omit the last three terms in the exponent we get

$$T(l) = \exp[-\gamma L^2 + I_{10}(l) + I_{01}(l)] , \qquad \text{(A70)}$$

which is referred to as the first approximation to $T(l)$. Then in this sense the second approximation to $T(l)$ is given by Eq. (A69).

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Laser with a Transmitting Window*

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The effect of transmission of radiation through one mirror of a laser is investigated. For a laser oscillator the result is to change the effective resonance frequency and Q of the cavity. Using the same model for the cavity, a signal is injected into the active medium through the transmitting window, and its effect on the system studied. When the external signal is strong enough and sufficiently close to the natural frequency of the laser oscillator, the laser locks its frequency to the input signal. The equations describing the system are solved over the range of input frequencies where the laser is locked, and the resulting gain found. In the high-intensity limit the medium saturates, and the gain tends to that of a lossy cavity. As the input intensity vanishes, the gain approaches infinity and the system tends to a laser oscillator.

I. INTRODUCTION

It is the purpose of this paper to investigate the effects on the operation of a laser arising from the fact that to some extent it is in communication with the rest of space outside the resonant cavity; i.e., some of its internal energy is escaping through the windows. As a result the effective cavity Q is lowered, and there is a slight change in operating frequency. Furthermore, using the same techniques it is possible to consider the case of an external signal applied to the laser through one of its