[36]W. L. McMillan, Phys. Rev. 182, 299 (1969).

[37]Our values of the coupling are larger than those quoted by Khalatnikov in Ref. 4 by a factor of 1.414. This factor arises by using the corrected density of states for two rotons; the Khalatnikov result for the density contains an erroneous factor of 2.

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Vertex Correction Contribution to the Decay Rate of Concentration Fluctuations in Binary Liquid Critical Mixtures*

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Recently, a self-consistent scheme for the mode-mode coupling theory of critical fluctuations was developed by Kawasaki in which the decay rate of concentration fluctuations for a binary critical mixture was obtained in the simplest approximation of ignoring all the vertex corrections. In this paper we calculate the contribution of the simplest vertex corrections. We find that the corrections are 2.44% for $q \ll \kappa$, and 0.40% for $q \gg \kappa$, where q and κ are the wave number and the inverse correlation range of concentration fluctuations, respectively.

I. INTRODUCTION

The physical ideas of mode-mode coupling in critical phenomena were perhaps first introduced by Fixman,[1] who considered the critical behavior of shear viscosity in a binary mixture. The idea was reformulated in the language of a time-correlation function by Kawasaki.[2] Later, Kadanoff and Swift[3] developed a formalism for transport coefficients, which can be given a schematic interpretation. In these theories couplings among hydrodynamic fluctuations play a crucial role in determining the macroscopic behavior of the system. Recently this formalism had been further extended[4] with the aid of a generalized Langevin equation due to Mori[5] in which kinetic equations obeyed by critical fluctuations are derived. A main result of the theory is the Dyson-type self-consistent equations for the time correlations of critical fluctuations of the following form:

$$G_{\vec{q}\alpha}(t)=G^0_{\vec{q}\alpha}(t)+\int_0^t dt_1\int_0^{t_1} dt_2 G^0_{\vec{q}\alpha}(t-t_1) \times\Sigma_{\vec{q}\alpha}(t_1-t_2)G_{\vec{q}\alpha}(t_2)\,, \tag{1.1}$$

where $G_{\vec{q}\alpha}(t)\equiv\langle a_{\vec{q}\alpha}(t)a^\dagger_{\vec{q}\alpha}(0)\rangle/\langle a_{\vec{q}\alpha}\, a_{\vec{q}\alpha}{}^\dagger\rangle$ is the renormalized propagator for the gross variable $a_{\vec{q}\alpha}$ with a wave vector $\vec{q}$, $\Sigma_{\vec{q}\alpha}(t_1-t_2)$ is the proper "self-energy," and $G^0_{\vec{q}\alpha}(t)$ is the unperturbed propagator obtained by ignoring coupling among hydrodynamic modes.

By introducing a renormalized vertex represented by a heavy dot •, a corresponding graphical equation for $G_{\vec{q}\alpha}(t)$ is given in Fig. 1, where the renormalized correlation function $U_{\vec{q}\alpha}(t)$ is given by

$$U_{\vec{q}\alpha}(t)\equiv\langle a_{\vec{q}\alpha}(t)a_{\vec{q}\alpha}{}^\dagger(0)\rangle\,. \tag{1.2}$$

The theory has been applied, among others, to the order parameter dynamics of binary liquid critical mixtures, as well as of fluids near the liquid-gas critical point, and excellent agreement with the recent light scattering experiments[6] has been achieved throughout the hydrodynamic and critical regimes. However, this particular calculation ignores all the vertex corrections in the equation shown in Fig. 1. Since the expansion in terms of renormalized propagators[4] contains no obvious small parameter of expansion, there is no *a priori* reason to ignore vertex corrections, and as it stands, a possibility can not be excluded that the excellent agreement with experiments could be fortuitous. Thus it is important to examine the effects of vertex corrections to the order parameter dynamics. A calculation of the contribution of the simplest vertex correction, in the case of a binary fluid mixture, will be presented in Sec. II.

II. CALCULATIONS AND RESULTS

According to the rules given in Ref. 4, the simplest vertex corrections for Fig. 1 are found to be of the type shown in Fig. 2. Note that there all the vertex renormalizations to the vertices at each corner of the "triangles" in the right-hand side of Fig. 2 are ignored. The unperturbed propagator $G^0_{\vec{q}\alpha}(t)$ is given by

$$G^0_{\vec{q}\alpha}(t)=\theta(t)e^{(i\omega_{\vec{q}\alpha}-\gamma_{\vec{q}\alpha})t}\,, \tag{2.1}$$

where

$$\theta(t)=\begin{cases}1, & t\geq 0\\ 0, & t<0\end{cases} \tag{2.2}$$

and $\omega_{\vec{q}\alpha}$ and $\gamma_{\vec{q}\alpha}$ are the frequency and damping constant of the mode $a_{\vec{q}\alpha}$ in the absence of interactions

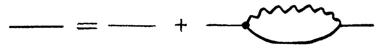

FIG. 1. Graphical equation for $G_{\vec{q}\alpha}(t)$. Thick straight lines represent $G_{\vec{q}\alpha}(t)$. Thin straight lines represent $G^0_{\vec{q}\alpha}(t)$. A wavy line is a renormalized correlation function $U_{\vec{q}\alpha}(t)$ given by (1.2).

between modes. The Dyson equation then becomes

$$\left(\frac{\partial}{\partial t}-i\omega_{\vec{q}\alpha}+\gamma_{\vec{q}\alpha}\right)G_{\vec{q}\alpha}(t)=\int_0^t dt_2\,\Sigma_{\vec{q}\alpha}(t-t_2)G_{\vec{q}\alpha}(t_2)+\delta(t)\,. \tag{2.3}$$

After making a Fourier transformation, we obtain

$$G_{\vec{q}\alpha}(\omega)=1/[-i\omega-i\omega_{\vec{q}\alpha}+\gamma_{\vec{q}\alpha}-\Sigma_{\vec{q}\alpha}(\omega)]\,, \tag{2.4}$$

where

$$G_{\vec{q}\alpha}(\omega)=\int_{-\infty}^{\infty}G_{\vec{q}\alpha}(t)e^{i\omega t}\,dt,\ \text{etc.} \tag{2.5}$$

From this equation one can see that the self-energy $\Sigma_{\vec{q}\alpha}(t)$ plays the role of an additional contribution to the decay rate of critical fluctuations from the nonlinear mode coupling. In this paper we will focus on the investigation of $\Sigma_{\vec{q}\alpha}(t)$.

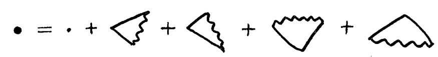

FIG. 2. Simplest vertex corrections.

In the case of a binary fluid mixture where we will only be concerned with slower diffusive motions, the only gross variables[7] associated with them will be considered. They are the relative concentration $c_{\vec{q}}$, the transverse components of the local velocity $\vec{v}_{\vec{q}}$, and the local entropy s. Here, as we are interested in the critical anomaly of the decay rate of concentration fluctuations, we only consider $\Sigma_{\vec{q}c}(\omega)$, and, for simplicity, we drop the local entropy s in view of the absence of any critical anomaly in the thermal conductivity.[8]

The equation for $\Sigma_{\vec{q}c}(\omega)$, which includes the simplest vertex corrections, is given graphically in Fig. 3.

The first two terms on the right-hand side of Fig. 3 have been studied earlier.[4] The third term has the form

$$\Sigma_{\vec{q}B1}(\omega)=\int_{-\infty}^{\infty}e^{i\omega t}dt\int_0^t ds_1\int_0^{s_1}ds_2\sum_{\alpha,\beta,\gamma}\Big[\Big(-2i\sum_{\vec{k}\vec{l}}\mathcal{V}_{\vec{q}c,\vec{k}c,\vec{q}-\vec{k}\alpha}\Big)G_{\vec{q}-\vec{k}v}(t-s_1)\,G_{\vec{k}c}(t-s_2)(-2i\,\mathcal{V}_{\vec{q}-\vec{k}\alpha,\vec{l}c,\vec{q}-\vec{k}-\vec{l}c})$$
$$\times G_{\vec{l}c}(s_1-s_2)\chi_{\vec{l}c}(-2i\,\mathcal{V}_{\vec{k}c,-\vec{l}c,\vec{k}+\vec{l}\beta})\,G_{\vec{q}-\vec{k}-\vec{l}c}(s_1)\chi^{\beta\gamma}_{\vec{k}+\vec{l}v}G_{\vec{k}+\vec{l}v}(s_2)(-2i\mathcal{V}_{\vec{q}-\vec{k}-\vec{l}c,-\vec{k}-\vec{l}\gamma,\vec{q}c})\Big]\,, \tag{2.6}$$

where

$$\chi_{\vec{q}c}=\langle|c_{\vec{q}}|^2\rangle\,,$$
$$\chi^{\alpha\beta}_{\vec{k}v}=\langle v^\alpha_{\vec{k}}v^\beta_{\vec{k}}\rangle=\left(\delta_{\alpha\beta}-\frac{k_\alpha k_\beta}{k^2}\right)\frac{k_BT}{\rho}\,,\quad \alpha,\beta=x,y,z$$
$$\mathcal{V}_{\vec{q}c,\vec{k}c,\vec{q}-\vec{k}\alpha}=\frac{1}{2V^{1/2}}\left(q_\alpha-\frac{\vec{q}\cdot(\vec{q}-\vec{k})}{(\vec{q}-\vec{k})^2}\,(\vec{q}-\vec{k})_\alpha\right), \tag{2.6'}$$
$$\mathcal{V}_{\vec{q}\alpha,\vec{k}c,\vec{q}-\vec{k}c}=\frac{k_BT}{2\rho V^{1/2}}\left(k_\alpha-\frac{\vec{k}\cdot\vec{q}}{q^2}\,q_\alpha\right)\left(\frac{1}{\chi_{\vec{k}c}}-\frac{1}{\chi_{\vec{q}-\vec{k}c}}\right).$$

Near the critical point, we assume that the kinetic shear viscosity η/ρ remains finite as predicted by the theory,[4] and that the diffusion constant behaves as $D\sim\kappa^{d-2}$, where d is the dimensionality.[3,4] Then, denoting all the momentum by q, the scaling property of Σ_{qB1} with $\omega=0$ is found from (2.6) and (2.6′) to be

$$\Sigma_{qB1}\sim q^{2d}q\,\frac{1}{q^2(\eta/\rho)}\,q\,\frac{1}{\chi_{qc}}\,\chi_{qc}q\,\frac{1}{q^2D}\,q\,\frac{1}{q^2(\eta/\rho)}$$
$$\simeq q^{2(d-1)}D^{-1}$$
$$\sim q^d\,. \tag{2.7}$$

Here we have used the fact that a state with only concentration fluctuations has a lifetime of the order of $(q^2D)^{-1}$, whereas a state with at least one transverse velocity has a lifetime of the order $(q^2\eta/\rho)^{-1}$ and also the fact that $\chi_{qc}\sim q^{-2+\eta}$.

The scaling form of (2.7) is then

$$\Sigma_{qB1}=\kappa^d f(q/\kappa)\,. \tag{2.8}$$

where κ is the inverse of the correlation length and f is a suitable scaling function.

Thus in the three-dimensional case, we have

$$\Sigma_{qBi}=\kappa^3 f(q/\kappa)\,. \tag{2.9}$$

Indeed the first six terms on the right-hand side of Fig. 3 can be shown to have the same scaling property, namely, they all have the form $\kappa^3 f(q/\kappa)$. On the other hand, the last four terms scale as

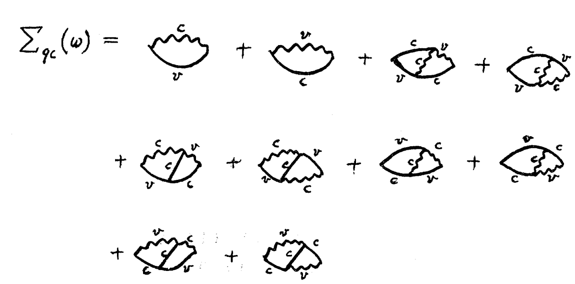

FIG. 3. Self-energy including the simplest vertex corrections.

FIG. 4. Simplest self-energy.

$\kappa^{2(d-1)}f(q/\kappa)$, or $\kappa^4 f(q/\kappa)$ in three-dimension case (here we have used the fact that at T_c, η/ρ is much greater than D). The difference in the behavior of two groups of vertex correction terms arise from the fact that in graphs of the latter group there is no intermediate state in which only concentration fluctuations appear. In investigating the critical anomaly of the decay rate of fluctuation correlations, one considers the case when both the wave number q characterizing spatial inhomogeneity and the inverse correlation length κ are much smaller than the microscopic wave number k_m. Thus the most important contributions to $\Sigma_{\vec{q}c}(\omega)$ are the first six terms of Fig. 3.

Since the contributions from the first two terms to the decay rate $\Sigma_{\vec{q}A}(\omega)$ given graphically by Fig. 4 have already been investigated,[4] we will in the following evaluate the contributions of the vertex corrections $\Sigma_{\vec{q}B}(\omega)$ represented graphically in Fig. 5, in which the propagators and correlation functions are replaced by those obtained earlier.[4]

The complete expression for $\Sigma_{\vec{q}B}(\omega)$ with $\omega \sim \Sigma_{\vec{q}A}$ is thus found to be

$$\Sigma_{\vec{q}B}(\omega) = -\left(\frac{k_B T}{8\pi^3\rho}\right)^2 \int d\vec{k} \int d\vec{l} \left(1 - \frac{\chi_{\vec{q}-\vec{k}-\vec{l}c}}{\chi_{\vec{l}c}}\right)\left(\frac{\chi_{\vec{l}c} - \chi_{\vec{k}c}}{\chi_{\vec{q}c}}\right)\left(\vec{q}\cdot\vec{l} - \frac{[\vec{q}\cdot(\vec{q}-\vec{k})][\vec{l}\cdot(\vec{q}-\vec{k})]}{(\vec{q}-\vec{k})^2}\right)\left(\frac{(\vec{k}+\vec{l})}{(\vec{k}+\vec{l})^2}\cdot[\vec{l}(\vec{q}\cdot\vec{k}) - \vec{k}(\vec{q}\cdot\vec{l})]\right)$$

$$\times[-i\omega + (\eta^*/\rho)(\vec{q}-\vec{k})^2]^{-1}\,[-i\omega + \Sigma^{(0)}_{\vec{k}A} + \Sigma^{(0)}_{\vec{l}A} + \Sigma^{(0)}_{\vec{q}-\vec{k}-\vec{l}A}]^{-1}\,[-i\omega + (\eta^*/\rho)(\vec{k}+\vec{l})^2]^{-1}\,, \qquad (2.10)$$

where η^* is the shear viscosity of the liquid mixture at critical point, and $G_{\vec{k}c}(t) = e^{\Sigma_{\vec{k}A}(0)t}$, etc., with $\Sigma_{\vec{k}A}(0)$ given by Fig. 4 and (2.11) below.

Here we have used the fact that concentration fluctuations decay much more slowly compared to the viscous damping, and the fact that appreciable ω dependence of $\Sigma_{\vec{k}A}(\omega)$, etc., appears only for large ω comparable to viscous damping rate. Since we are interested in the decay rate of concentration fluctuation near the critical point, we take $\omega = 0$ in (2.14). [In contrast to $\Sigma_{\vec{k}A}(\omega)$, here one should expect an appreciable ω dependence already for $\omega \sim \Sigma_{\vec{k}A}$ because of the long lifetime of the intermediate states containing only concentration fluctuations.] We also use the Ornstein-Zernike form for $\chi_{\vec{k}c} \propto (\vec{k}^2 + \kappa^2)^{-1}$, which still gives the correct temperature dependence of the diffusion constant.

The result for $\Sigma_{\vec{q}A}(0)$ was found in Ref. 4 in the closed form[9] as

$$\Sigma_{\vec{q}A}(0) = -(k_B T/6\pi\eta^*)\kappa^3 K(q/\kappa)\,, \qquad (2.11)$$

where

$$K(x) = \tfrac{3}{4}[1 + x^2 + (x^3 - x^{-1})\tan^{-1}x]\,. \qquad (2.12)$$

For the case where $q \ll \kappa$,

$$\Sigma_{\vec{q}A}(0) \simeq -(k_B T/6\pi\eta^*)\kappa q^2\,, \qquad (2.13)$$

and when $q \gg \kappa$,

$$\Sigma_{\vec{q}A}(0) \simeq -(k_B T/16\eta^*)q^3\,. \qquad (2.14)$$

Using this result of $\Sigma_{\vec{q}A}(0)$, one can evaluate $\Sigma_{\vec{q}B}(0)$ numerically. For $q \gg \kappa$ the problem reduces to a three-fold integration, and we obtain

$$\Sigma_{\vec{q}B}(0)/\Sigma_{\vec{q}A}(0) \simeq -2.44\times 10^{-2}\,. \qquad (2.15)$$

For $q \gg \kappa$, we have a five-fold integral, and we find that

$$\Sigma_{\vec{q}B}(0)/\Sigma_{\vec{q}A}(0) \simeq 0.40\times 10^{-2}. \qquad (2.16)$$

Thus, the contributions from the simplest vertex corrections to the decay rate of concentration fluctuation amount to only 2.44% for $q \ll \kappa$ and to 0.40% for $q \gg \kappa$. We did not evaluate $\Sigma_{\vec{q}B}(\omega)$ for more general values of q and ω, but we expect it to be equally small too.

III. CONCLUDING REMARKS

The results obtained in the preceding section give enough confidence in our earlier result [Eq. (2.11)] although we have not yet understood the reason why the corrections turn out to be so small. Note that the situation appears to be not so favorable for magnets where much larger contributions are expected from the vertex corrections.[10] Finally, for detailed comparisons with experiments, the deviations of $\chi_{\vec{k}c}$ from the Ornstein-Zernike form would have to be taken into account[9,11] along with the vertex corrections, and furthermore, a small contribution of vertex correction can in principle be detected by its ω dependence, i.e., a deviation from the Lorentzian line shape already at the frequencies comparable to the Rayleigh linewidth.

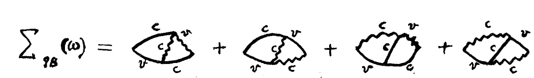

FIG. 5. Simplest vertex correction contribution to the self-energy $\Sigma_{\vec{q}B}(\omega)$.

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[1]M. Fixman, J. Chem. Phys. 36, 310 (1962).

[2]K. Kawasaki, Phys. Rev. 150, 291 (1966).

[3]L. P. Kadanoff and J. Swift, Phys. Rev. 166, 89 (1968); J. Swift, *ibid.* 173, 257 (1968).

[4]K. Kawasaki, Ann. Phys. (N.Y.) 61, 1 (1970); in *Proceedings of the International Summer School of Physics "Enrico Fermi," Varenna, Italy*, 1970 (Academic, New York to be published).

[5]H. Mori, Progr. Theoret. Phys. (Kyoto) 33, 423 (1965).

[6]P. Berge *et al.*, Phys. Rev. Letters 23, 693 (1969); 24, 1223 (1970); D. L. Henry *et al.*, *ibid.* 25, 1170 (1970).

[7]For more general discussion of the gross variables, see M. S. Green, J. Chem. Phys. 20, 1281; (1952) 22, 398 (1954), as well as Ref. 4.

[8]J. V. Sengers, in *Proceedings of the Critical Phenomena Conference, Washington, D.C.*, 1965, edited by M. S. Green and J. V. Sengers, Natl. Bur. Std. (U.S.) Misc. Pub. No. 273 (U. S. GPO, Washington, D. C., 1966); in *Proceedings of the International Summer School of Physics "Enrico Fermi," Varenna, Italy*, 1970 (Academic, New York, to be published).

[9]The same result was also obtained recently by Ferrell using the fluctuation-dissipation theorem. See R. A. Farrell, Phys. Rev. Letters 24, 1169 (1970).

[10]P. Résibois (private communication).

[11]H. L. Swinney (private communication).

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Persistent-Current Measurements of the Superfluid Density and Critical Velocity*†

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Gyroscopic measurements of superfluid persistent currents in liquid helium below and near the λ transition are presented here. Measurements were conducted in a variety of sizes of porous material from 0.05- to 150-μ average size. Near the λ transition, both the superfluid density ρ_s/ρ and the critical velocity ω_c exhibit a $\frac{2}{3}$-power dependence on the temperature difference from the transition. A depression of the λ transition is observed for both ρ_s/ρ and ω_c in the smallest pores. Data for ρ_s/ρ and ω_c away from the transition and down to about 1.25 °K are given and are in agreement with measurements made by other methods.

I. INTRODUCTION

Persistent currents in superfluid helium have been examined extensively since their prediction and discovery.[1–4] This paper will discuss the superfluid critical velocity ω_c and the superfluid density ρ_s/ρ in the region of the superfluid–normal-fluid phase transition.[5,6] Near the λ point both of these quantities exhibit a simple-power-law behavior.

There is a wide class of second-order transitions exhibiting a λ specific-heat anomaly at a critical temperature T_c. One can discuss their thermodynamic functions in terms of an order parameter η that vanishes at T_c.[7] The scaling-law approach[8,9] postulates that $\eta=\eta_0(T-T_c)^\zeta$ near T_c. Although the details of the singularity are not known, it is still possible to determine various relations between their critical exponents. Consider the λ point in liquid helium. Measurements of the specific heat[10] show a logarithmic dependence on the temperature difference from the λ point. The scaling-law approach leads to a connection between the temperature dependence of the specific heat and the superfluid density with a critical exponent of $\frac{2}{3}$ for the superfluid density.[11]

Persistent-current critical velocities discussed in this paper have the same general properties as those obtained by pressure-vs–flow-rate studies.[12] That is, they are temperature independent away from the λ point, and they increase inversely with channel size. Near the λ point the situation is quite different. The critical velocity is independent of channel size and is proportional to $(T_\lambda - T)^{2/3}$. Langer and Fisher[13] have arrived at a critical velocity exponent of $\frac{2}{3}$. They consider the flowing superfluid to be a metastable state which, because of critical fluctuations, breaks up into vortex rings. Their argument leads to a critical velocity proportional to ρ_s/ρ and hence a $\frac{2}{3}$-power-law behavior.

II. EXPERIMENTAL METHOD

One of the most intriguing things about superfluid helium is its similarity to a superconductor. A given ring of superconductor will trap a magnetic field because of a persistent circulation of current in the superconductor. It remains essentially as long as the ring is superconducting and the critical field is not exceeded. Onnes first detected such persistent currents by placing the ring in an external magnetic field and observing the torque produced. The mechanical analog for a superfluid persistent current is shown schematically in Fig. 1. An externally applied rotation of velocity $\vec{\omega}$ is applied to the vector angular momentum $\vec{L}_p$ of the