we may extend this result at any even d to find

$$\Delta_l'\xi(\omega) \simeq -\frac{A_l(2l)}{(l-1)!}\,(-i\omega)^l \ln|\omega|\ ,\ \omega\to 0,\ d=2l \qquad \text{(C8)}$$

where

$$\Delta_l'\xi(\omega) = \xi(\omega) - \xi - \omega \frac{\partial\xi}{\partial\omega}\bigg|_{\omega=0} - \cdots - \frac{\omega^{l-2}}{(l-2)!}\,\frac{\partial^{l-2}\xi(\omega)}{(\partial\omega)^{l-2}}\bigg|_{\omega=0}.$$

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Onset of Turbulence for Counterflow in He II

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Two mutual friction numbers are derived for superfluid helium which can describe the onset of turbulence in counterflowing helium. The numbers are $M_n = A\rho_s\rho_n\langle\vec{V}_r\rangle^3 d^2/\eta_n\langle\vec{V}_n\rangle$ and $M_s = A\rho_s\rho_n\langle\vec{V}_r\rangle^3 d^2/\eta_n\langle\vec{V}_s\rangle$, where M_n predicts the onset of turbulence in the normal fluid and M_s predicts the onset of turbulence in the superfluid. Staas, Taconis, and van Alphen's Reynolds number is found to apply only to flow conditions for which $\vec{V}_n \simeq \vec{V}_s$.

INTRODUCTION

For many years, now, considerable attention has been given to the heat transport properties of superfluid helium. For this liquid, total isothermal fluid flow in the presence of small heat currents can be described in terms of a two-fluid model in which a counterflow of two-fluid components, normal fluid, and superfluid can be envisioned. As a consequence, the heat transport properties of He II are intimately related to its hydrodynamic flow properties. The same might also be true for larger heat currents where the two-fluid model breaks down and nonlinear relationships develop between the heat current and temperature gradient[1] as well as the heat current and pressure gradient.[2,3] The nonlinearities have been described in terms of an empirical mutual friction force F_{sn} originally proposed by Gorter and Mellink[4] and are known to accompany a developed tangled mass of vorticity and/or turbulence within the fluid.[5] Although several experimental investigations have been made into the nature of the tangled mass of vorticity,[6–9] very little is known about the onset of turbulence in superfluid helium. By deriving a set of dimensionless numbers similar to the Reynolds number[10] of classical hydrodynamics, a possible explanation for this phenomenon will follow.

CRITICAL-HEAT PROBLEM

The generally accepted equations of motion for the steady-state flow of liquid helium are

$$\rho_s\vec{V}_s\cdot\vec{\nabla}\vec{V}_s=-\vec{\nabla}P_s-\vec{F}_{sn} \tag{1}$$

for the superfluid and

$$\rho_n\vec{V}_n\cdot\vec{\nabla}\vec{V}_n=-\vec{\nabla}P_n+\eta_n\vec{\nabla}^2\vec{V}_n+\vec{F}_{sn} \tag{2}$$

for the normal fluid. By definition

$$\vec{\nabla}P_s\equiv(\rho_s/\rho)\vec{\nabla}P-\rho_s S\vec{\nabla}T, \tag{3}$$

$$\vec{\nabla}P_n\equiv(\rho_n/\rho)\vec{\nabla}P+\rho_s S\vec{\nabla}T, \tag{4}$$

where ρ_s, ρ_n, and ρ are the superfluid, normal fluid, and total fluid densities, $\vec{V}_s$ and $\vec{V}_n$ are the superfluid and normal fluid velocities, $\vec{\nabla}P$ and $\vec{\nabla}T$ are the pressure and temperature gradients, η_n is the normal fluid viscosity, and S is the entropy per unit mass of the total fluid. By neglecting the inertial terms of Eqs. (1) and (2), substituting in Eqs. (3) and (4), eliminating $\vec{\nabla}P$, and solving for $\vec{\nabla}T$, it can be shown that

$$\vec{\nabla}T=\frac{\vec{F}_{sn}}{\rho_s S}+\frac{\eta_n\vec{\nabla}^2\vec{V}_n}{\rho S}\ . \tag{5}$$

To this can be added the equations for one-dimensional counterflow,

$$\rho_n\vec{V}_n+\rho_s\vec{V}_s=0, \tag{6}$$

and the net heat flux $\dot{\vec{q}}$ away from a heat source,

$$\dot{\vec{q}}=\rho ST\vec{V}_n=-\rho_s ST(\vec{V}_s-\vec{V}_n). \tag{7}$$

From these equations temperature gradients for limiting cases of heat current can be derived.

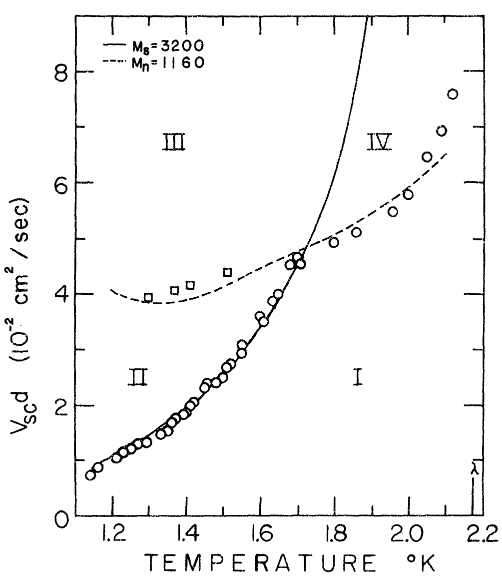

FIG. 1. Temperature dependence of the critical superfluid velocity (Chase). Circles: points obtained directly or by the Vinen delay time technique ($V_{sc1}d$); squares: locus of the break in the $\vec{\nabla}T$-vs-$\dot{\vec{q}}$ curves ($V_{sc2}d$). Lines are a best fit of the mutual friction numbers to the data.

For small heat currents, the mutual friction term of Eq. (5) is negligible and by substituting Eq. (7) into Eq. (5), it can be shown that for a right cylindrical pipe of diameter d, a temperature gradient of

$$\vec{\nabla}T=-(32\eta_n/\rho^2S^2Td^2)\,\dot{\vec{q}} \tag{8}$$

will develop across the length of the pipe as it carries a heat flux of $\dot{\vec{q}}$. Measurements[3,11] of η_n based on this equation have yielded values agreeing well with other experiments.[12] Equation (8) therefore describes a flow regime of superfluid helium which is analogous to the laminar-flow regime of classical hydrodynamics.

For larger heat currents, the viscous term of Eq. (5) becomes negligible and by substituting Eq. (7) into Eq. (5), the temperature gradient becomes

$$|\vec{\nabla}T|=\frac{|\vec{F}_{sn}|}{\rho_s S}=\frac{A\rho_n}{S}\left(\frac{\dot{q}}{\rho_s ST}\right)^m\ , \tag{9}$$

where $m=3$[1,4] and

$$\vec{F}_{sn}=A\rho_s\rho_n\,|\vec{V}_s-\vec{V}_n|^2\,(\vec{V}_s-\vec{V}_n). \tag{10}$$

Although many investigators[9,13-18] have recently questioned the value of 3 for m, in the work which follows the value is retained since only then are the units of the mutual friction constant fixed for all bath temperatures and are suitable fits provided to all the data. Since vorticity and/or turbulence accompanies the mutual friction force,[5] it must be assumed that Eq. (9) describes a flow regime of superfluid helium which is roughly analogous to the turbulent flow regime of classical hydrodynamics.

For still larger heat currents, a third flow regime develops as m in Eq. (9) slightly increases and the temperature gradient continues to rise as $\dot{\vec{q}}^{\,m}$. This phenomenon is unlike anything in classical hydrodynamics and suggests that the three flow regimes are related to a development of turbulence in first one and then both components of the superfluid[19,20] or the development of turbulence in first one and then instability in both of the superfluid components.[9] If $\dot{Q}_{c1}$ and $\dot{Q}_{c2}$ ($\dot{Q}_{c1}<\dot{Q}_{c2}$ by definition) are the "critical" heat currents separating the three flow regimes, then $\dot{Q}_{c1}$ and $\dot{Q}_{c2}$ are both fairly sharply defined for some bath temperatures while they tend to merge or smear out to one for other bath temperatures[13,18,19] (see Figs. 1 and 2). Under different experimental conditions, the onset of the smearing effect can be observed for different bath temperatures after which, for increasing bath temperature, the effect is displayed on up to a temperature near the λ point. The data close to the λ temperature is somewhat more obscured.[21] To date, there has not been a totally successful attempt in predicting $\dot{Q}_{c1}$ or $\dot{Q}_{c2}$, but

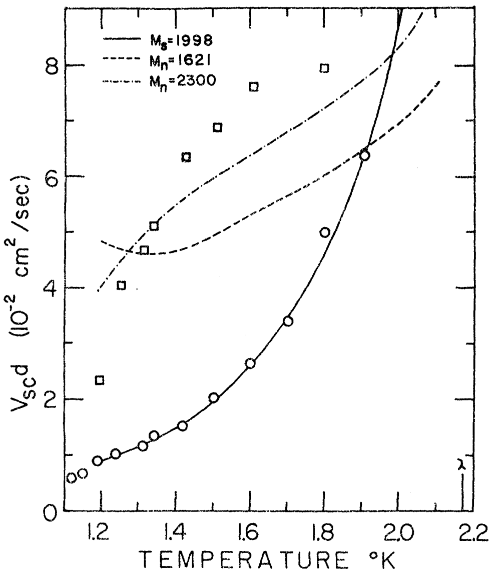

FIG. 2. Temperature dependence of the critical superfluid velocity (Cornelissen and Kramers). Circles: points obtained directly ($V_{sc1}d$); squares: locus of the break in the $\vec{\nabla}T$-vs-$\vec{q}$ curves ($V_{sc2}d$). Lines are the best fit of the mutual friction numbers to the data. $M_n = 1621$ uses Vinen's mutual friction constant in the calculations, and $M_n = 2300$ uses a modified mutual friction constant (see text).

the superfluid velocities V_{sc1} and V_{sc2} corresponding to $\dot{Q}_{c1}$ and $\dot{Q}_{c2}$ are the quantities of interest in the discussion which follows.

Finally for constant bath temperature and increasing heat current, the temperature difference which develops between the heater and the helium-bath free surface can increase until at $\dot{Q}_{c3}$ the relationship (sat denotes saturation)

$$|\vec{\nabla}T| = \rho g \left(\frac{\partial T}{\partial P}\right)_{\text{sat}} \tag{11}$$

is satisfied.[22] $(\partial T/\partial P)_{\text{sat}}$ is related to the latent heat of vaporization through the Clausius-Clapeyron equation, and g is the magnitude of the acceleration due to gravity. At $\dot{Q}_{c3}$, local saturation phenomena, such as the development of cavitation at constrictions in channels[23] or the development of vapor films near heaters,[24,25] is always observed. For this reason, $\dot{Q}_{c3}$ should never be experimentally mistaken for $\dot{Q}_{c1}$ or $\dot{Q}_{c2}$ since the effect of the former is to isolate the heater from the bulk helium bath[26] while the effect of the latter is to increase the temperature gradient in the liquid away from the heater. The discussion which follows is concerned only with $\dot{Q}_{c1}$ and $\dot{Q}_{c2}$ and not $\dot{Q}_{c3}$.

TURBULENCE IN SUPERFLUID HELIUM

In classical hydrodynamics, the discussion of turbulence generally involves a discussion of dimensionless numbers, such as the Reynolds number, which are first derived from the Navier-Stokes equation[27,28] and later used to describe the onset of turbulence in the fluid. Although similar numbers have been derived for superfluid helium,[29,30] the numbers have either been found incapable of describing the V_{sc1} counterflow data over an extended range of temperature[13] or have been found incapable of predicting the onset of two types of turbulence. To further clarify this situation, a new emphasis is now being placed on the description which Eqs. (1) and (2) provide to dynamically similar counterflow patterns (i.e., counterflow patterns in which the fluid has identical flow directions and passes through two geometrically similar bodies in such a manner as to have geometrically similar streamlines[27]) from which a new set of dimensionless numbers will emerge that will describe the onset of turbulence in superfluid helium.

In terms of Eqs. (1) and (2), dynamic similarity is assured if with a suitable choice of units in length, time, and force the equations can be transformed into identical dimensionless forms while describing two geometrically similar but different bodies. For instance, if d (the hydrodynamic diameter, four times the ratio of the area to the perimeter of the channel in which the flow takes place), L (the length of the channel), $\langle V_s\rangle$ (the mean superfluid velocity averaged over the length of the channel), $\langle V_n\rangle$ (the mean normal-fluid velocity averaged over the length of the channel), $\langle V_r\rangle = \langle|\vec{V}_s - \vec{V}_n|\rangle$ (the mean relative fluid velocity averaged over the length of the channel), ΔP_s and ΔP_n (the respective effective pressure differences developed across the length of the channel) are chosen as the reference magnitudes for length, velocity, and pressure, respectively, then the dimensionless quantities[31]

$$\vec{\nabla}^* \equiv d\vec{\nabla} \quad \text{or} \quad \vec{\nabla}^* \equiv L\vec{\nabla},$$
$$\vec{V}_s^* \equiv \vec{V}_s/\langle V_s\rangle, \quad \vec{V}_n^* \equiv \vec{V}_n/\langle V_n\rangle, \quad \vec{V}_r^* \equiv \vec{V}_r/\langle V_r\rangle\ , \tag{12}$$
$$P_s^* \equiv P_s/\Delta P_s\, , \quad P_n^* \equiv P_n/\Delta P_n$$

can be formed. After substituting Eqs. (10) and (12) into Eqs. (1) and (2), dividing both equations by $\eta_n\langle V_n\rangle/d^2$ (a combined operation being required if the addition of the two equations is to remain meaningful in describing the total fluid), and applying the condition of zero mass flow [Eq. (6)], the result is

$$\frac{\rho_s\langle V_s\rangle d}{\eta_n}(\vec{V}_s^*\cdot\vec{\nabla}^*)\vec{V}_s^* = -\frac{d^2\Delta P_s}{L\eta_n\langle V_s\rangle}\vec{\nabla}^* P_s^* - \frac{A\rho_s\rho_n\langle V_r\rangle^3 d^2}{\eta_n\langle V_s\rangle}|\vec{V}_r^*|^2\vec{V}_r^* \tag{13}$$

for the superfluid and

$$\frac{\rho_n\langle V_n\rangle d}{\eta_n}(\vec{V}_n^*\cdot\vec{\nabla}^*)\vec{V}_n^* = -\frac{d^2\Delta P_n}{L\eta_n\langle V_n\rangle}\vec{\nabla}^*P_n^* + \frac{A\rho_s\rho_n\langle V_r\rangle^3 d^2}{\eta_n\langle V_n\rangle}|\vec{V}_r^*|^2\vec{V}_r^* + \vec{\nabla}^{*2}\vec{V}_n \tag{14}$$

for the normal fluid. Dynamic similarity is assured if each of the coefficients of the equations as listed in Table I are constant. The coefficients, except for the coefficients of the $|\vec{V}_r^*|^2\vec{V}_r^*$ terms, have their counterparts in classical hydrodynamics[28,32] provided one realizes that

$$\frac{d^2\Delta P_n}{L\eta_n\langle V_n\rangle} = R_n\left(\frac{d\Delta P_n}{L\rho_n\langle V_n\rangle^2}\right) = \text{const}, \qquad \frac{d^2\Delta P_s}{L\eta_n\langle V_s\rangle} = R_s\left(\frac{d\Delta P_s}{L\rho_s\langle V_s\rangle^2}\right) = \text{const} \tag{15}$$

implies the coefficients of friction λ_s and λ_n are constant. The coefficients of the $|\vec{V}_r^*|^2\vec{V}_r^*$ terms are special only to the superfluid helium problem and will hence be called mutual friction numbers. The mutual friction numbers are similar to Meservey's Gorter number[30] in that they involve the mutual friction force, but there are two mutual friction numbers while there is only one Gorter number, and the mutual friction numbers are a ratio of forces while the Gorter number is a ratio of entropy production rates. The fact that the mutual friction numbers are a ratio of forces is in keeping with the basic fundamentals of classical hydrodynamics[33] while the high degree of symmetry between Eqs. (13) and (14) is in keeping with characteristics of superfluid hydrodynamics.

Counterflow experiments conducted with slits,[34] channels, and wires are all examples of experiments containing dynamically similar flow patterns. However, only the critical-heat data of Brewer and Edwards,[35] Chase[13] (Fig. 1), and Cornelissen and Kramers[18] (Fig. 2) for channels are complete enough to contribute to the present discussion of turbulence. Concerning these data, both London and Zilsel[36] and Tough[29] have observed that values of R_n given by $\dot{Q}_{c1}$ versus T are many times smaller than the 2300 which is typically expected for Reynolds numbers describing flows in channels of circular cross sections in classical hydrodynamics, and that $R_n = \text{const}$ is incapable of describing the temperature dependence of the data. Since R_n and R_s are related through Eq. (6) for counterflow, the observations apply as well to R_s. Attempts to modify the numbers, say, by replacing ρ_n with ρ in R_n have met only limited success[3,13,29] since even though the newly formed R describes the $\dot{Q}_{c1}$ data well at lower bath temperatures, it is quite unsatisfactory for higher bath temperatures. Chase's[37] attempt to reconcile the difficulty by considering "eddy" viscosities seems dubious since turbulence develops from laminar flow in which eddies do not exist. Consequently, it appears the Reynolds numbers cannot relate the counterflow critical-heat data of superfluid helium to the development of turbulence.

From classical hydrodynamics, the Euler numbers N_n and N_s might also be expected to play a negligible role in describing the critical-heat data. This expectation is affirmed by the pressure data of Brewer and Edwards,[2] which indicates a negligible development of pressure across a channel for laminar counterflow, and by a direct calculation of N_n and N_s in which the main contribution to $\vec{\nabla}P_n$ and $\vec{\nabla}P_s$ is assumed to be the London term containing a temperature gradient in keeping with Eq. (8). Hence, the mutual friction numbers are the only remaining numbers which can successfully describe the data.

TABLE I. A summary of the dimensionless numbers which might be found useful in the analysis of the critical-heat data for superfluid helium.

Name	Symbol	Number	Ratio
Normal-fluid equation:			
1. Reynolds number	R_n	$\frac{\rho_n V_n d}{\eta_n}$	Inertial to viscous forces
2. Coefficient of friction	λ_n	$\frac{d\Delta P_n}{L\rho_n V_n^2}$	Pressure to inertial forces
3. Mutual friction number	M_n	$\frac{A\rho_s\rho_n V_r^3 d^2}{\eta_n V_n}$	Mutual friction to viscous forces
Superfluid equation:			
1. Reynolds number	R_s	$\frac{\rho_s V_s d}{\eta_n}$	Inertial to viscous forces
2. Coefficient of friction	λ_s	$\frac{d\Delta P_s}{L\rho_s V_s^2}$	Pressure to inertial forces
3. Mutual friction number	M_s	$\frac{A\rho_s\rho_n V_r^3 d^2}{\eta_n V_s}$	Mutual friction to viscous forces

RESULTS

The mutual friction numbers in terms of the average superfluid velocity are

$$M_s = A\rho_s\rho^3\langle V_s\rangle^2 d^2/\rho_n^2\eta_n, \qquad M_n = A\rho^3\langle V_s\rangle^2 d^2/\rho_n\eta_n. \tag{16}$$

Figures 1 and 2 compare the $V_{sc}d$ versus temperature data of Chase[13] and Cornelissen and Kramers[18] with calculated results obtained from

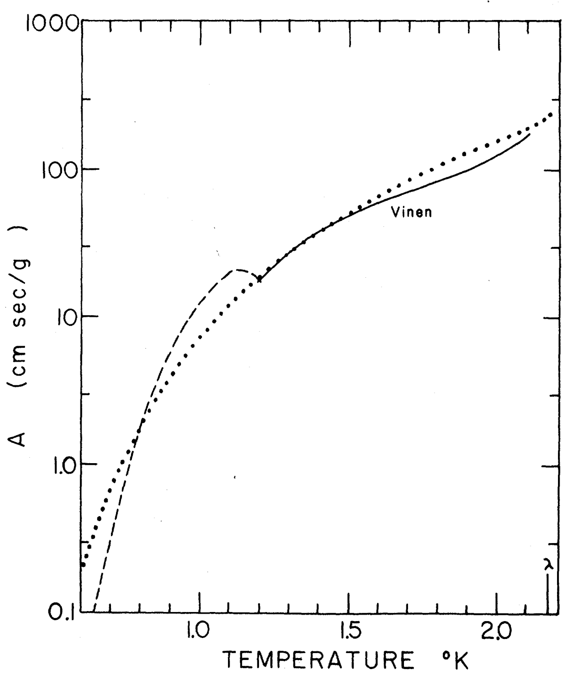

FIG. 3. Temperature dependence of the mutual friction constant. Solid line: Vinen's data; dashed line: derived from a fit of to Cornelissen and Kramers's $V_{sc1}d$ data; dotted line: $\ln A = 5.46 + 0.704 \ln \rho_n/\rho$ (see text).

$M_n = M_{n\,\mathrm{crit}}$ and $M_s = M_{s\,\mathrm{crit}}$, $M_{n\,\mathrm{crit}}$ and $M_{s\,\mathrm{crit}}$ being constants. The calculations were performed for bath temperatures between 1.2 and 2.1 °K, the temperature range covered by Vinen's mutual friction constant[1] (see Fig. 3), and used the values of viscosity obtained by Brewer and Edwards.[11] Table II lists the critical mutual friction numbers found to best represent each set of data along with estimated standard deviations and uncertainties (the standard deviation divided by the square root of the number of data points). From these results the following conclusions can be drawn.

(i) Except for the older and smaller channel data of Brewer and Edwards, the critical mutual friction numbers are of the right order of magnitude to describe the onset of turbulence in superfluid helium.

(ii) For lower bath temperatures, V_{sc1} is best fitted by $M_s = M_{s\,\mathrm{crit}}$ while V_{sc2} is best fitted by $M_n = M_{n\,\mathrm{crit}}$. For higher bath temperatures, the data is best fitted by $M_n = M_{n\,\mathrm{crit}}$ as the curve for $M_s = M_{s\,\mathrm{crit}}$ becomes smeared.

(iii) $M_{n\,\mathrm{crit}}$ and $M_{s\,\mathrm{crit}}$ can be assigned unequal individual values to account for the different temperatures at which the smearing effect of $\dot{Q}_{c2}$ is observed within the different experiments.

(iv) Constant mutual friction numbers imply an inverse dependence of V_{sc} on d in agreement with the critical-heat data summaries provided by Wilkes[38] and Keller and Hammel[39] for large channels.

Concerning the poor fit of $M_n = M_{n\,\mathrm{crit}}$ to the V_{sc2} data of Cornelissen and Kramers, the difficulty may be due to the leveling off of the mutual friction constant as reported by these authors when they reported their data.[18] This leveling-off effect was observed only when V_{sc2} was approached. In an effort to take account of this effect, M_n was recalculated by using the larger of either Vinen's value for or 37 cm sec/g for the mutual friction constant. When this revised M_n was refitted to the data, $M_{n\,\mathrm{crit}}$ was found to be 2300 and the curve for $M_n = M_{n\,\mathrm{crit}}$ was found a somewhat better fit (see Fig. 2). It might be concluded, therefore, that Cornelissen and Kramers not only directly observed strange variations in the mutual friction constant, but also inadvertently recorded these variations in their critical-heat data. The reasons for these variations, however, are not understood since, first of all, the mutual friction force is itself not completely understood and, second, Chase did not observe the effects in his data.

Concerning the data of Chase, it is possible to divide the data up into four flow regimes with each of the flow regimes representing different macroscopic flow characteristics of the liquid. For example, consider the four flow regimes as indicated in Fig. 1. Previous discussions indicate that flow regime I must be characterized by laminar counterflow of the total fluid and that flow regimes II, III, and IV must be characterized by turbulent flow of one or more of the fluid components. Since M_s is a coefficient of the superfluid equation of motion and M_n is a coefficient of the normal-fluid equation of motion, it is reasonable to expect that the superfluid component is turbulent in flow regimes II and III and the normal-fluid component is turbulent in flow regimes III and IV. This interpretation agrees experimentally with the "no clear evidence" for the development of turbulence in

TABLE II. A summary of the values of the mutual friction numbers found to best fit the various data.

Number	Author	Channel size (μ)	Value	Standard deviation	Uncertainty
M_s	Chase	800	3200	±326	±64
M_s	Cornelissen and Kramers	1060	1998	±127	±40
M_s	Brewer and Edwards	108	215	±74	±28
M_s	Brewer and Edwards	52	150	±69	±26
M_n	Chase	800	1160	±111	±22
M_n	Cornelissen and Kramers	1060	1620	±1180	±417

the normal fluid as reported by Vicentini-Missoni and Cunsolo[9] for flow regime II[40] in the presence of superfluid vorticity, and the observations of Allen, Griffith, and Osborne[41] on first the appearance of neither circulation nor bob agitation in flow regime I, then the appearance of circulation with no bob agitation in flow regime II, and then just bob agitation in flow regime III. In addition, characteristics of various bubble states as observed by this author[42] around supercritically heated wires in He II are in qualitative agreement with the interpretation.

On the other hand, other interpretations indicate that the superfluid component should be turbulent in flow regimes III and IV and the normal-fluid component should be turbulent in flow regimes II and III. This follows from the fact that $|\vec{V}_s| > |\vec{V}_n|$ is satisfied near T_λ and $|\vec{V}_n| > |\vec{V}_s|$ is satisfied near 1.2 °K, and that for a given laminar counterflow situation the fluid component with the largest velocity component should be the first component to become turbulent. Unfortunately, this interpretation does not involve mutual friction and does not account for the above cited experimental evidence. When mutual friction is included, conclusions similar to those arrived at in the preceding paragraph can be obtained.[43]

Cornelissen and Kramers[18] have reported additional data on $V_{sc1}d$ down to 0.6 °K. Assuming that all these data can be fitted with the same value of $M_{s\,\mathrm{crit}}$ as that found to fit the $V_{sc1}d$ data above 1.2 °K, the mutual friction constant can be extended down to 0.6°K. Using the experimentally determined viscosity data of Cornelissen and Kramers,[18] the results as recorded in Fig. 3 can be obtained. The value for A of about 1.0 cm sec/g at $T = 0.76$ °K is in agreement with the corresponding value for A reported by Cornelissen and Kramers and the dip in the curve at 1.2 °K is probably due to spurious effects related to a reported change of techniques in the original collection of data. It is observed that the drop in A for lower bath temperatures is not as rapid as ρ_n^2 as might be expected from the formula of Vinen[44]

$$A \propto \beta^3\rho_n^2/\rho^3\kappa \ , \tag{17}$$

where β and κ are constants. In fact $\rho^2 A/\rho_n^2$ at 2.10 °K is 3.16×10^2 cm sec/g while at 0.70 °K, it is 6.55×10^6 cm sec/g. When Fig. 3 is fitted by a linear least-squares fit to an equation of the form

$$\ln A = C + n\ln\rho_n/\rho \ , \tag{18}$$

C is found to be equal to 5.46 ± 0.16 and n is found to be equal to 0.704 ± 0.04. Comparisons between Eqs. (17) and (18) indicate significant differences in the results.

ADDITIONAL CONSIDERATIONS

Discussions so far have concerned only counterflow measurements in large channels with diameters exceeding approximately 10^{-3} cm. When these restrictions are removed, additional variables must be included in the derivation of the mutual friction numbers. For instance, when the counterflow requirement is removed, the mutual friction numbers must be modified to include the net momentum density flux $\vec{j}$ of the fluid,

$$M_n = A\rho_s\rho_n\langle V_r\rangle^3 d^2/\eta_n\langle V_n\rangle \ , \tag{19}$$

$$M_s = A\rho_s^2\rho_n\langle V_r\rangle^3 d^2/\eta_n\langle|\vec{j} - \rho_s\vec{V}_s|\rangle \ . \tag{20}$$

By considering various types of flow patterns, it can be demonstrated that this form of the numbers has limited application.

First, consider flows for which the velocity of the normal fluid is essentially zero and $\vec{j} = \rho_s\vec{V}_s$. For these flows M_s and M_n are undefined and the hydrodynamic properties of the fluid are dominated by the fountain pressure. The Euler numbers may possibly play an important role in the discussion of the data, but reference should be made to the observations made by Kojima *et al.*[45] on related data. One thing for certain, the mutual friction numbers are unsuccessful.

Second, consider flows for which $\vec{V}_n \simeq \vec{V}_s \simeq \vec{V}$ and $\vec{j} = \rho\vec{V}$; $\vec{V}$ is the total fluid velocity. For these flows, M_s and M_n are zero while Staas, Taconis, and van Alphen[3] have found success with

$$R = \rho\langle\vec{V}\rangle d/\eta_n \ . \tag{21}$$

R is the Reynolds number for the total fluid when $\vec{j} = \rho\vec{V}$. From this it can be concluded that some of the data need mutual friction numbers while others need Reynolds numbers to describe the onset of turbulence in superfluid helium and that the choice of numbers depends on the type of flow pattern involved. The critical-heat data must be corrected for the fountain pressure[13] and the net momentum transfer before the mutual friction numbers can be successfully applied. Furthermore, combined effects arising from mutual friction, Reynolds numbers, and London pressure may all contribute in the same experiment to make the flow characteristics of the helium appear very complicated.

When the small channel diameter requirement is removed, corrections must be made to the normal-fluid viscosity for hydrodynamic slip and free molecular flow.[46] The need for these corrections has already been demonstrated by Cornelissen and Kramers[18] for phonons, but corrections for rotons[47] have not yet been developed. Early investigations made by this author have indicated that quantum boundary effects should contribute only to the smallest of the small channel data.

Finally the mutual friction numbers lose their significance up near the λ temperature where the superfluid velocity approaches intrinsic transport limits not related to the mutual friction force nor the normal-fluid viscous force. These limits may arise as a result of either Landau's criterion for critical velocities of second sound,[48] or the result of the dynamic scaling laws for superfluid helium,[49] or the dissipation due to homogeneous nucleation of vortex rings.[50]

CONCLUSION

Success has been achieved in fitting the large channel critical-heat data of superfluid helium with dimensionless mutual friction numbers. The numbers are a ratio of the mutual friction force to the viscous force, in agreement with expectations[51] that the mutual friction force should play a major role in the description of the critical-heat data. The fact that M_s contains a viscous force for the superfluid component is not in contradiction to the two-fluid model since for counterflow, Eq. (6) requires that the superfluid flow with an "effective" viscosity of η_n.

The critical mutual friction numbers play the same role in predicting the onset of turbulence in superfluid helium as the Reynolds number plays in classical hydrodynamics. The mutual friction numbers, however, do not replace the Reynolds number since R describes data for $\vec{V}_n \simeq \vec{V}_s \simeq \vec{V}$ and M_n and M_s describe data for $\vec{V}_n \simeq -\rho_s \vec{V}_s/\rho_n$. Typical values for $M_{s\,\mathrm{crit}}$ and $M_{n\,\mathrm{crit}}$ are scattered, but they generally tend to be somewhat higher than those found by Staas, Taconis, and van Alphen[3] for R_{crit}. Through the use of the mutual friction numbers, the mutual friction constant was found to significantly deviate from Vinen's theory down near 0.65 °K.

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Theory of a Wave Packet in a Laser Cavity

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When describing the behavior of a multimode laser, it sometimes is convenient to express the cavity field in terms of an evolving wave packet that is reflected by the mirrors. We consider here the general relation between the normal-mode description of the field, where mode amplitudes and frequencies characterize the oscillations, and features of the corresponding wave packet. It is shown that in this problem it is natural to represent a packet as a linear superposition of monochromatic waves, integrating over wave number k. To illustrate some of the concepts, an example is treated that relates packet parameters to parameters of a theoretical model for the laser medium.

I. INTRODUCTION

This paper considers features of the electromagnetic field in the cavity of a multimode laser. The optical field is expressed as a superposition of normal-mode oscillations, the wavelength of each standing-wave mode being equal to $2L/n$, where n is a large integer and L is the spacing between the end reflectors. It is shown that such a field can always be described in terms of an evolving wave packet that propagates back and forth between the mirrors. This latter representation is of particular interest when the length of the packet is less than that of the cavity. It is well known that pulses of short duration are in fact obtained when the modes oscillate with a nearly stationary phase relationship. A number of approaches, both of the "passive" and "active" type, have been employed to experimentally achieve the condition of frequency locking.[1]

A central purpose of the paper will be to establish the general relation between the normal-mode representation, where the field is characterized by the complex amplitudes and frequencies of the oscillations, and properties of the corresponding wave packet. Borrowing from the terminology of classical mechanics, this portion of the treatment will be of a kinematical nature rather than a dynamical one, in the sense that no reference will be made to properties of the laser medium. As such, the conclusions should find use in various types of lasers.

In order to elucidate aspects of the physical content of the general results, an example then will be considered wherein parameters of the wave packet are related to actual parameters of a model of a laser medium.

II. GENERAL THEORY

We shall suppose that an optical cavity of the Fabry-Perot type is filled with a uniform active medium that exhibits dispersion. Starting with the formalism used by Lamb,[2] we expand the cavity field in longitudinal-mode eigenfunctions

$$E(z, t) = \textstyle\sum_n A_n(t) \sin(n\pi z/L) \ , \tag{1}$$

where the end reflectors are at $z=0$ and $z=L$. The mode amplitudes $A_n(t)$ are written