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Charge Dependence of Ionization Energy Loss for Relativistic Heavy Nuclei

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The ionization energy loss of relativistic heavy nuclei is calculated using the exact Mott cross section for close collisions. Deviations from the Bloch formula are computed in some typical cases and found to be significant for nuclei with $Z \gtrsim 20$.

I. INTRODUCTION

Methods for calculating ionization energy loss of charged particles traversing matter predate the beginning of quantum mechanics. The original classical calculation by Bohr[1] was followed by Bethe's quantum-mechanical treatment[2] and Williams' method of impact parameters.[3] Bloch[4] has indicated the connection between the classical and quantum-mechanical methods, and Mott[5] has demonstrated the equivalence of Bethe's method and the method of impact parameters. Elaborations and extensions of this early work to include such things as shell corrections and the density effect have been given, and there are review articles which give a complete summary of all this work.[6]

One result of these calculations is that the average energy loss of a particle of charge Ze and velocity β is proportional to Z^2 when $Z/137\beta \ll 1$. When this condition is not satisfied, Bloch's formula[7] gives a more general Z dependence which reduces to the classical result given by Bohr[1] when $Z/137\beta \gg 1$. Bloch's correction is present in the nonrelativistic case and can be thought of as the result of a modification of the minimum scattering angle below which no energy transfer takes place.[8]

If nuclear collisions are ignored, charged particles lose energy primarily by collisions with atomic electrons. It is convenient to divide these collisions into "distant collisions," in which electron-binding effects are included, and "close collisions," in which the binding energy of the electrons is ignored.[9] In the relativistic case, there are corrections to the Born approximation for close collisions of electrons with heavy nuclei ($20 < Z < 120$) which are not included in Bloch's original calculation. In this paper, these corrections are computed numerically using the exact Mott cross section[10] transformed to the frame in which the electron is initially at rest. The usual treatment of distant collisions is assumed to be valid here (see Sec. II B for a discussion of this assumption). The total energy loss is computed as a function of Z and compared with Bloch's formula in some typical cases.

II. ENERGY-LOSS CALCULATION

A. Close Collisions

In order to obtain the energy loss of a heavy particle in a collision with a free electron, we first calculate the partial-wave form of the Mott cross section for elastic scattering of an electron of velocity β from a nucleus of charge Ze,[11]

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm Mott} = \lambda\!\!\!^{-2}\,[q^2(1-\beta^2)]\,|F|^2 \csc^2\tfrac{1}{2}\theta + |G|^2 \sec^2\tfrac{1}{2}\theta] ,$$

where the functions F and G are defined as follows:

$$F = F_0 + F_1, \quad G = G_0 + G_1 ,$$

$$F_0 = \tfrac{1}{2} i \frac{\Gamma(1-iq)}{\Gamma(1+iq)} \exp(iq \ln \sin^2 \tfrac{1}{2}\theta) ,$$

$$G_0 = (-iq \cot^2 \tfrac{1}{2}\theta) F_0 ,$$

$$F_1 = \tfrac{1}{2} i \sum_{l=0}^{\infty} [lD_l + (l+1) D_{l+1}] (-1)^l P_l(\cos\theta) ,$$

$$G_1 = \tfrac{1}{2} i \sum_{l=0}^{\infty} [l^2 D_l - (l+1)^2 D_{l+1}] (-1)^l P_l(\cos\theta) ,$$

$$D_l = \frac{e^{-i\pi l}}{l+iq} \frac{\Gamma(l-iq)}{\Gamma(l+iq)} - \frac{e^{-i\pi\rho_l}}{\rho_l+iq} \frac{\Gamma(\rho_l - iq)}{\Gamma(\rho_l + iq)} ,$$

$$\rho_l^2 = l^2 - (\beta q)^2 , \quad q = \left(\frac{Ze^2}{\hbar c \beta}\right) .$$

$2\pi\lambda\!\!\!^-$ is the de Broglie wave length and θ is the electron scattering angle. After the electron is scattered, its kinetic energy T in the frame in which it was initially at rest can be obtained by a Lorentz transformation,

$$\begin{aligned} T &= \gamma[\gamma mc^2 - \beta(\gamma\beta mc^2)\cos\theta] - mc^2 \\ &= 2mc^2\gamma^2\beta^2 \sin^2\tfrac{1}{2}\theta = T_{\rm max} \sin^2\tfrac{1}{2}\theta , \end{aligned} \tag{1}$$

where $\gamma^{-2} = 1 - \beta^2$ and $T_{\rm max}$ is the maximum energy transferable to the electron of mass m. From this it follows that

$$d\Omega = 2\pi \sin\theta\, d\theta = 4\pi\, dT/T_{\rm max} .$$

So that the cross section $d\sigma/dT$ for production of an electron of energy T by the scattering of a heavy particle from an electron at rest is given by

$$\frac{d\sigma}{dT} = \frac{4\pi}{T_{\rm max}} \frac{d\sigma}{d\Omega} ,$$

where the angle θ which corresponds to a given T is obtained from Eq. (1).

If the D's are expanded to first order in q, one obtains the second Born approximation for the cross section

$$\left(\frac{d\sigma}{dT}\right)_{\rm SB} = \frac{2\pi Z^2 r_0^2 mc^2}{\beta^2 T^2} \left\{ 1 - \beta^2\left(\frac{T}{T_{\rm max}}\right) + \frac{Z\pi\beta}{137}\left(\frac{T}{T_{\rm max}}\right)^{1/2}\left[1 - \left(\frac{T}{T_{\rm max}}\right)^{1/2}\right]\right\} .$$

The first Born approximation $(d\sigma/dT)_{\rm FB}$ is obtained by neglecting the last term in this expression (or setting the D's equal to zero). The contribution to the energy loss per unit path length dE/dx due to the close collisions is then given by

$$\left(\frac{dE}{dx}\right)_{\rm close} = N \int_\eta^{T_{\rm max}} T \left(\frac{d\sigma}{dT}\right) dT , \tag{2}$$

where η is some energy above which the electron binding energy can be neglected, and N is the number of electrons per cm^3.

We have computed $d\sigma/d\Omega$ using the Euler transformation and the angular transformations described by Sherman[11] to improve the large- and small-angle convergence, respectively, and have obtained agreement with previous published results.[11,12] We have then calculated the integral in Eq. (2) using a standard Newton-Coates quadrature formula, with an error of less than 1%.[13]

Conditions for the validity of the Mott cross section are[14]:

(i) point-charge nucleus, $R_A \ll h/\beta\gamma mc$, where h is Planck's constant and $R_A \sim 0.5\,A^{1/3}\,e^2/mc^2$ is the approximate radius of a nucleus with atomic mass number A;

(ii) infinitely heavy nucleus $2\gamma(m/M) \ll 1$, where m/M is the ratio of the electron to nuclear mass;

(iii) nucleus with negligible spin effects, $(2\beta\gamma/Z)^2(T/T_{\rm max})(m/M)^2 \ll 1$ for all T within the range of the integral in Eq. (2);

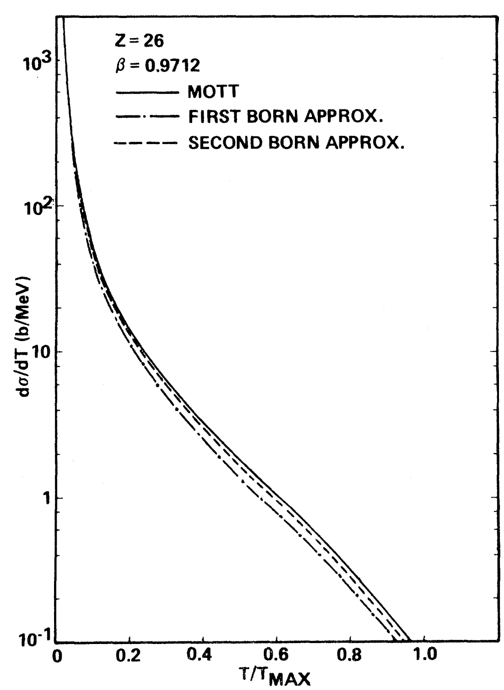

FIG. 1. Comparison of the first Born, second Born, and exact differential cross sections for $Z = 26$ and $\beta = 0.9712$.

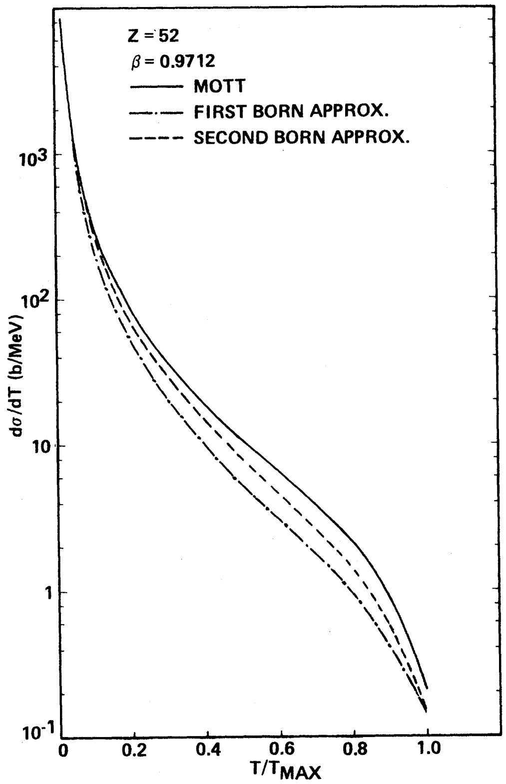

FIG. 2. Same as Fig. 1 except $Z = 52$.

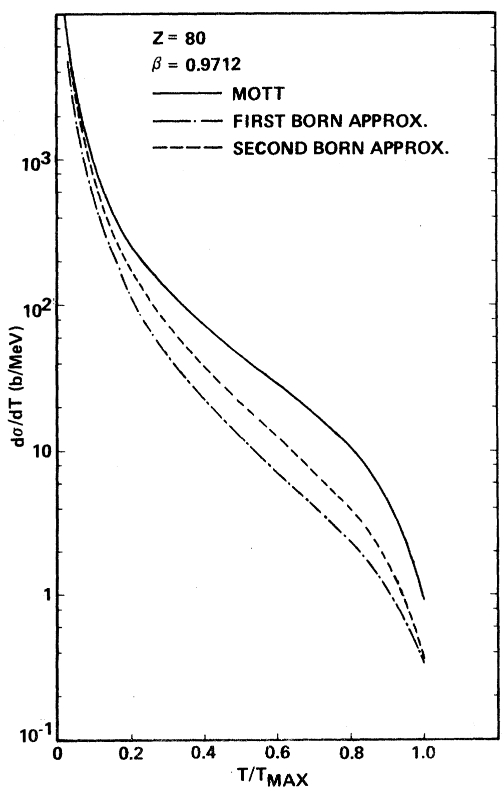

FIG. 3. Same as Fig. 1 except $Z = 80$.

(iv) stripped nucleus—according to Ref. 10, this is satisfied for energies above ~1 GeV/nucleon and absorbers with atomic number much greater than one;

(v) radiative corrections are negligible—according to Jankus,[15] this is the case for $\gamma < 100$ for the integral in Eq. (2).

Condition (v) appears to be the most restrictive upper limit on γ. When condition (iv) is not satisfied, then one can safely neglect the screening effect of the captured electrons provided $\frac{1}{137} Z \ll 2\beta\gamma (T/T_{max})^{1/2}$.[14]

B. Distant Collisions

It has been assumed here that the contribution to the energy loss from distant collisions (energy transfer $< \eta$) is given by

$$\left(\frac{dE}{dx}\right)_{dist} = \frac{2\pi N Z^2 r_0^2 mc^2}{\beta^2} \left\{ \ln \frac{2m\beta^2 c^2 \eta}{I^2(1-\beta^2)} - \beta^2 + 2\left[\Psi(1) - \mathrm{Re}\Psi\left(1 + i\frac{Z}{137\beta}\right)\right] - \frac{2C}{z} - \delta \right\}, \quad (3)$$

where I is the mean excitation potential of the absorber with atomic number z, r_0 is the classical electron radius, and ReΨ denotes the real part of the logarithmic derivative of the Γ function Ψ. This result is obtained from Bloch's expression for $(1/\Delta Z)[\Delta T_A + (\Delta T_B)_\beta]$ [Eqs. (20a) and (44a) of

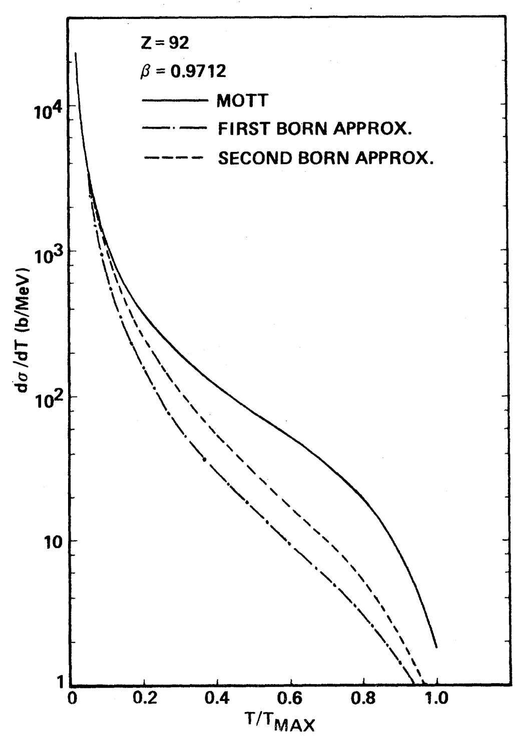

FIG. 4. Same as Fig. 1 except $Z = 92$.

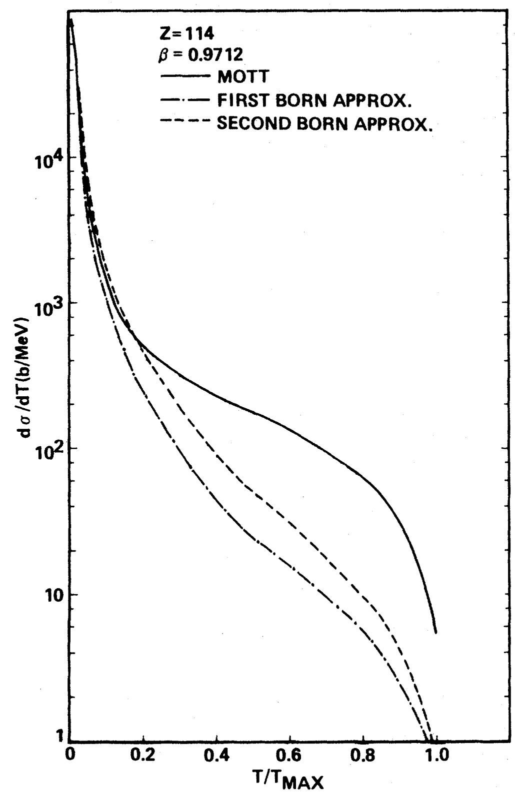

FIG. 5. Same as Fig. 1 except $Z = 114$.

Ref. 4] with Williams' relativistic correction [Eq. (28) of Ref. 3] added to ΔT_A.[16] η is related to Bloch's θ_0 by the equation $\theta_0 = 2(\eta/T_{\max})^{1/2}$, which

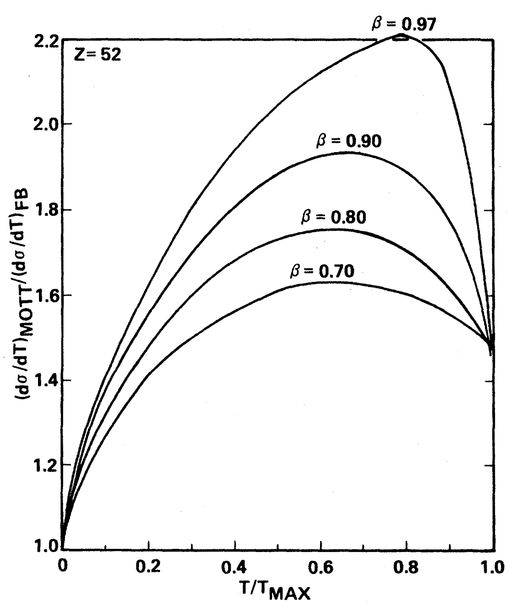

FIG. 6. Ratio of exact to first Born diffential cross section for $\beta = 0.7$, 0.8, 0.9, and 0.97 and $Z = 52$.

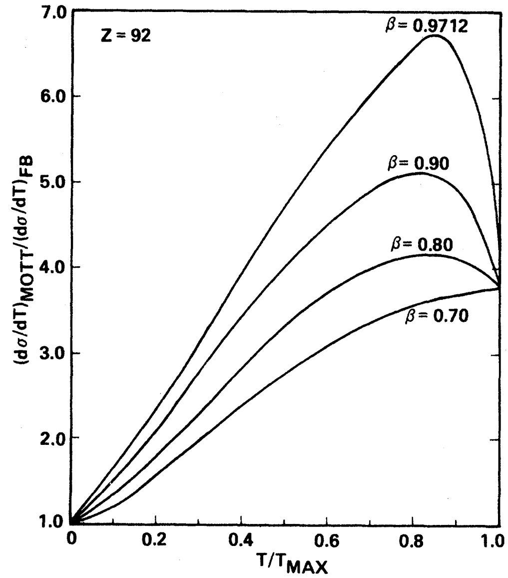

FIG. 7. Same as Fig. 6 except $Z = 92$.

follows from Eq. (1) for small θ. The last two terms are included to represent shell corrections and the density effect, respectively.[17,18]

The terms containing Ψ in Eq. (3) arise from collisions in which electron binding effects are neglected but which result in small energy transfer. When one constructs a wave packet to represent the incident particle, one finds that the range of momenta present will lead to a modification of the energy loss. In the limit $Z/137\beta \ll 1$, this effect vanishes and Eq. (3) reduces to Bethe's expression [Eq. (1) of Ref. 9].

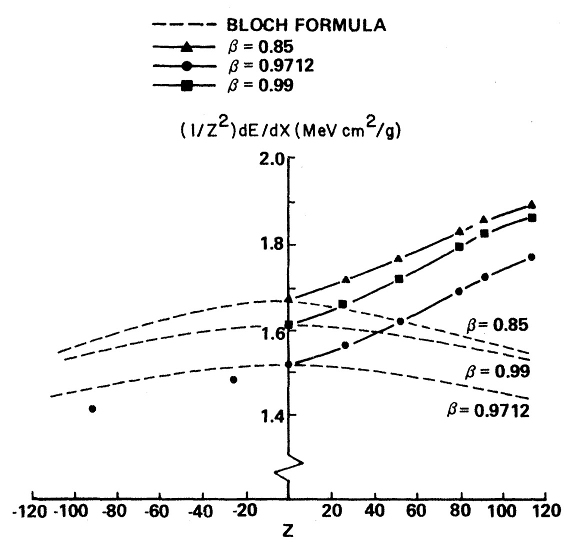

FIG. 8. Comparison of $(1/Z^2)\, dE/dx$ calculated using the Mott and first-Born differential cross section for close collisions with argon as the absorbing medium.

If $(d\sigma/dT)_{FB}$ is inserted in Eq. (2) [corresponding to Bloch's $(1/\Delta Z)(\Delta T_B)_\alpha$], and the equation $T_{max} = 2mc^2\beta^2\gamma^2$ is used, then the sum of Eq. (2) and Eq. (3) (without last two correction terms) gives Northcliffe's Eqs. (4) and (6), which is presumably Bloch's result with the correct relativistic correction.

The conditions given by Bloch for the validity of Eq. (3) without the last two correction terms are: (i) $\beta_0/\beta \ll 1$, where β_0 is an average atomic electron velocity [Eq. (16) of Ref. 4] and (ii) $(Z/137\beta)(\beta_0/\beta) \ll 1$ [Eq. (21), of Ref. 4]. The C/z term must be included as a correction when condition (i) is not satisfied; it has been shown that this term is important for $X \lesssim 100$ where $X = (137\beta)^2/z$.[19] Condition (ii) is necessary in order to neglect higher-order terms in the expression for distant collisions. It has been suggested[20] that there is indeed a correction term proportional to Z^3 for distant collisions which vanishes for $X \gg 1$. In addition, we must require that (iii) η must be much less than T_{max} and much greater than the electron-binding energy. The single scattering Born approximation is still valid for the very low-energy transfer involved in Eq. (3), as we will show in Sec. III, so that one can combine Eqs. (2) and (3) with the result being independent of η.

III. RESULTS

A. Scattering Cross Sections

In Figs. 1–5, we have plotted $d\sigma/dT$ as a function of T/T_{max} for $Z = 26$, 52, 80, 92, and 114 and an incident energy of 3 GeV/nucleon ($\beta = 0.9712$). These Z values were chosen as a representative sample over the range in which corrections to the Born approximation are expected to be significant. The first and second Born approximations are also included. One sees that for T near T_{max} the Mott cross section differs from the first Born by as much as a factor of 10 for $Z = 114$ down to a factor of 1.3 for $Z = 26$. For small T, however, the difference approaches zero, as mentioned previously. In fact, if we use the small-angle approximation to the Mott cross section,[12] we find

$$\left(\frac{d\sigma}{dT}\right)_{SA} = \frac{2\pi Z^2 r_0^2 mc^2}{\beta^2 T^2}\left[1 + \frac{\pi Z\beta}{137}\left(\frac{T}{T_{max}}\right)^{1/2}\cos\gamma\right],$$

where

$$\cos\gamma = \mathrm{Re}\,\frac{\Gamma(\frac{1}{2} - iq)\Gamma(1+iq)}{\Gamma(\frac{1}{2} + iq)\Gamma(1-iq)}$$

is tabulated in Ref. 12 and is a decreasing function of q. If this is compared with $(d\sigma/dT)_{SB}$ we see that for small T and $q \simeq 1$, $(d\sigma/dT)_{SA} < (d\sigma/dT)_{SB}$; this is evident in Fig. 5. This is the reason for Semikoz's[21] overestimate of the corrections to the Born approximation. In Figs. 6 and 7, $(d\sigma/dT)_{Mott}/(d\sigma/dT)_{FB}$ is plotted for $\beta = 0.7$, 0.8, 0.9, 0.97 and $Z = 52$ and 92.

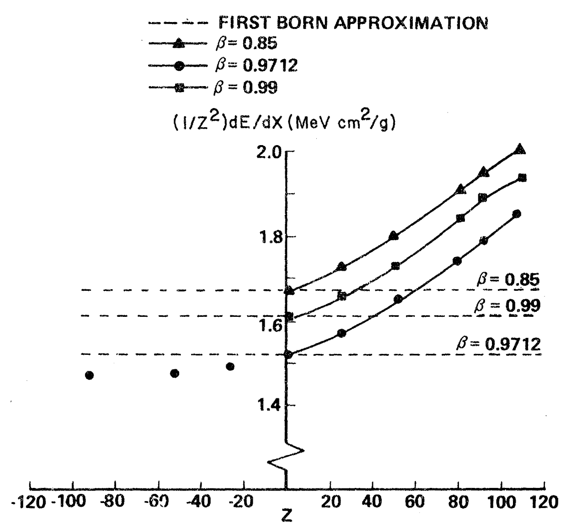

FIG. 9. Comparison of $(1/Z^2)\ dE/dx$ calculated using the Mott and first Born differential cross section without the Bloch correction with argon as the absorbing medium.

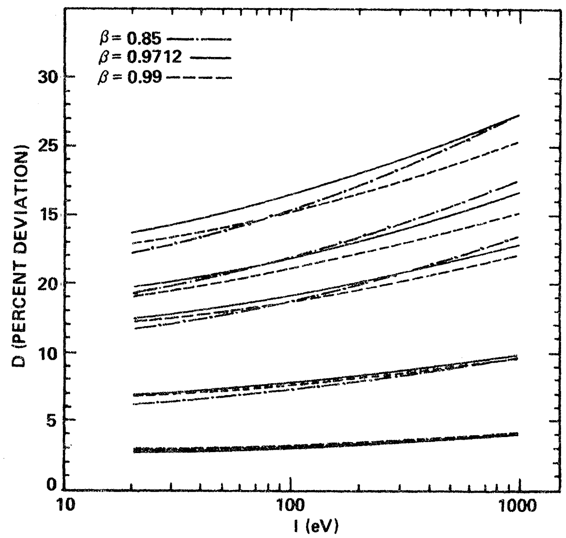

FIG. 10. Percentage deviation of Bloch formula from dE/dx calculated using the Mott formula as a function of the mean excitation potential I for $Z = 26$, 52, 80, 92, and 114.

B. Average Energy Loss

In Fig. 8, we have plotted $(1/Z^2)(dE/dx) = (1/Z^2)[(dE/dx)_{close} + (dE/dx)_{dist}]$ as a function of Z for $\beta = 0.85$, 0.9712, 0.99 and $I = 210$ eV (argon).[22] We have chosen argon as the absorbing medium here because of the possible application of this type of calculation to cosmic-ray energy-loss measurements using gas ionization chambers.[23] Shell corrections and the density effect can be safely neglected here.[24] For $\eta \sim 10^4$ or 10^5 eV, the result is independent of η. One sees that the deviations from Bloch's formula range from about 3% for iron up to 20% for $Z = 114$. The deivations for negative Z are much less, probably because when the cross section is expanded as a series in Z, it becomes an alternating series over most of the energy range for negative Z. Figure 9 shows $(1/Z^2)(dE/dx)$ when the Bloch correction [terms containing Ψ in Eq. (3)] is omitted. The energy loss of positively charged nuclei exceeds that of negatively charged nuclei by up to 25% for $Z = \pm 114$.

In Fig. 10 we have plotted $[(dE/dx) - (dE/dx)_B]/(dE/dx)_B = D$ as a function of I for various Z's and β's where $(dE/dx)_B$ is calculated using $(d\sigma/dT)_{FB}$ in Eq. (2). The deviation D from the Bloch formula is a slowly increasing function of I and thus absorber atomic number z. This figure only applies when the last two terms in Eq. (3) can be neglected. The density effect would in general decrease the contribution of Eq. (3) to the total-energy loss and thus increase the deviation D. We do not intend to give a comprehensive tabulation of results for various media but only to show that, in general, for relativistic particles with $Z \gtrsim 20$ the deviations from the Born approximation are significant and should be included in energy-loss calculations.

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NMR in Nondilute Solid ^{3}He-^{4}He Mixtures

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An explanation of the anomalous NMR specific heat first seen by Garwin and Reich in nondilute solid ^{4}He-^{3}He mixture is given in terms of the energy of interaction of ^{4}He-^{4}He.

The purpose of this note is to put forth a tentative suggestion for the explanation of some of the unusual NMR data on ^{3}He-^{4}He mixtures. The data to which the explanation is addressed is that of Garwin and Reich,[1] Bernier and Landesman,[2] and Reich and Yu[3] which suggests the existence of an anomalous specific heat due to the presence of ^{4}He impurities at concentrations up to 1% in ^{3}He. There are two things in the data that must be accounted for: (i) the basic interaction among the excitations