although not treated exactly, were included in the approximation. A first-order transition resulted.

Experiments of He^4 [11] and He^3 [12] absorbed on graphite do show a large peak in the heat capacity at densities of one-third. The evidence appears to indicate that the transition is of second order. It must be noted, however, that irrespective of whether the model considered should evince a second-order transition, corrections to the exact results of the classical model due to the inherent quantum-mechanical nature of the helium system must be investigated.

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Theory of a Charged Boson Gas Using the Random-Phase Approximation*

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The Brandow boson many-body theory is used to calculate the generalized dielectric constant within the random-phase approximation. The generalized dielectric constant obtained this way agrees with the Bogoliubov theory in the high-density limit and thus yields all of the well-known Bogoliubov results. The momentum distribution function for $\vec{k} \neq 0$ is also calculated. The treatment is fully number conserving.

I. INTRODUCTION

In a previous paper,[1] hereafter referred to as I, a study was made on a charged boson gas using the analogy to a fermion system with high spin degeneracy.[2] The noteworthy feature of the theory is that one introduces the concept of a hole in the boson system and thereby treats the dynamics of the condensate in a proper way. The number of particles is conserved without the aid of the chemical potential.[3] The well-known Brueckner-Goldstone cluster expansion is directly applied without any modification to the condensation operators. The result of I shows an exact agreement with the Bogoliubov theory of Foldy[4] both for the ground-state energy and the condensation fraction.

In this paper, this study is extended to the elementary excitation with the same Hamiltonian as used in I. In Sec. II the irreducible polarization part is calculated and then the generalized dielectric constant is obtained, which agrees again with the Bogoliubov theory in the high-density limit. This time-dependent dielectric constant yields all of the known results of Bogoliubov theory. In Sec. III we supplement I by calculating the momentum distribution function for $\vec{k} \neq 0$.

II. DIELECTRIC CONSTANT AND RELATED FUNCTIONS

The Hamiltonian is

$$H=\sum_{\vec q}\epsilon(q)a^\dagger_{\vec q}a_{\vec q}+\tfrac{1}{2}\sum_{\vec q}{}'v(q)[a^\dagger_{\vec q}a^\dagger_0a_{\vec q}a_0+a^\dagger_0a^\dagger_{\vec q}a_0a_{\vec q}+a^\dagger_0a^\dagger_0a_{\vec q}a_{-\vec q}+a^\dagger_{-\vec q}a^\dagger_{\vec q}a_0a_0]\,, \quad (1)$$

with $\epsilon(q)=\hbar^2q^2/2m$ and $v(q)=(4\pi e^2/\Omega)(1/q^2)$. Here Ω is the volume and N will represent the total number of particles. The prime indicates the absence

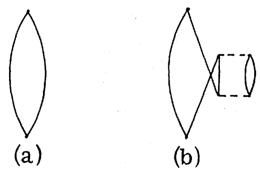

FIG. 1. Diagram shown in (a) is the simplest possible polarization part $-i\Pi_0$. For example, the diagram shown in (b) is not considered in this paper.

of the term with $q=0$. We regard $a^\dagger_{\vec{k}}$ and $a_{\vec{k}}$ as Fermi operators with N spin degeneracies, spin variable being suppressed in k. Then one can introduce the Green's function and draw diagrams to represent the perturbation terms. For a justification of this view, we refer to Appendix A of Ref. 2.

We first need the free-particle Green's function $G_0(\vec{k},t)$ defined in the Heisenberg picture by

$$G_0(\vec{k},t) = -i\langle 0|\, T\{a_{\vec{k}}(t)a^\dagger_{\vec{k}}(0)\}\,|0\rangle \quad , \tag{2}$$

where $|0\rangle$ is the noninteracting N-particle ground state and T is the time-ordering operator. The Fourier transform of Eq. (2) is readily found to be

$$G_0(\vec{k},\omega) = \frac{\eta_{\vec{k}}}{\omega - \epsilon(k) + i\delta} + \frac{1-\eta_{\vec{k}}}{\omega - i\delta} \quad , \tag{3}$$

with

$$\begin{aligned}\eta_{\vec{k}} &= 1 \quad \text{if } \vec{k}\neq 0\\ &= 0 \quad \text{if } \vec{k}=0 \; .\end{aligned}$$

Here δ is the usual positive infinitesimal introduced to make the switching of the interaction adiabatic.

With Eq. (3) let us calculate the irreducible polarization part $-i\Pi_0(k,\,\omega)$, which is shown in Fig. 1(a). It is given by

$$-i\Pi_0(k,\omega) = -Ni^2 \sum_{\vec{q}} \times \int \frac{d\omega'}{2\pi}\, G_0(\vec{k}+\vec{q},\omega+\omega')G_0(\vec{q},\omega') \; . \tag{4}$$

Here N comes from the sum of the spin coordinate in the Fermi loop. The integration is performed easily by closing the contour either above or below. We find

$$\Pi_0(k,\omega) = \frac{-2N\epsilon(k)}{[\omega-\epsilon(k)+i\delta][\omega+\epsilon(k)-i\delta]} \; . \tag{5}$$

Equation (5) is the simplest possible approximation to the exact irreducible polarization part. Following the practice in the electron gas, we call it the random-phase approximation.

We recall the general screening effect of the many-body medium. The interaction between two particles of the system is not simply the direct interaction force. One has to also include the virtual-polarization effect of the medium. In this fashion, one can introduce a time-dependent effective potential and thus a time-dependent dielectric constant. We proceed to calculate the effective potential V_{eff} first. The perturbation expansion is shown in Fig. 2. It satisfies a simple Dyson's equation.[5] Consequently, V_{eff} can be expressed in terms of $\Pi_0(k,\omega)$:

$$V_{\text{eff}}(k,\omega) = v(k)/\,[1+v(k)\Pi_0(k,\omega)] \; . \tag{6}$$

The generalized dielectric constant $\epsilon(k,\omega)$ is given by

$$\begin{aligned}1/\epsilon(k,\omega) &= V_{\text{eff}}(k,\omega)/v(k)\\ &= \frac{\omega^2-\epsilon^2(k)}{2[\epsilon^2(k)+\omega_p^2]^{1/2}}\left[\frac{1}{\omega-[\epsilon^2(k)+\omega_p^2]^{1/2}+i\delta} - \frac{1}{\omega+[\epsilon^2(k)+\omega_p^2]^{1/2}-i\delta}\right]\end{aligned} \tag{7a}$$

and

$$\epsilon(k,\omega) = \frac{\omega^2-\epsilon^2(k)-\omega_p^2}{2\epsilon(k)} \times \left[\frac{1}{\omega-\epsilon(k)+i\delta} - \frac{1}{\omega+\epsilon(k)-i\delta}\right] \; , \tag{7b}$$

where $\omega_p = [2N\epsilon(k)\,v(k)]^{1/2} = (4\pi\rho e^2\hbar^2/m)^{1/2}$, the classical plasma frequency (with $\hbar=1$). Important properties of Eq. (7) are shown in Fig. 3, which should be compared with a similar figure for the electron gas that appears in Ref. 6.

It is interesting to calculate the ground-state energy from the dielectric constant given by Eq. (7). The result should be equal to the ground-state energy obtained by evaluating the sum of the ring diagrams.[1] This expectation is borne out as follows. The ground-state energy E is given by[6]

$$E = -\int_0^{e^2} \frac{d(e'^2)}{e'^2}\left[\frac{N}{2}\, v(r=0;e'^2) + \frac{\Omega}{2}\int \frac{d\vec{q}}{(2\pi)^3}\,\frac{d\omega}{2\pi}\,\text{Im}\,\frac{1}{\epsilon(q,\omega)}\right] \; . \tag{8}$$

On the other hand, from Eq. (7) and using the relation

$$\frac{1}{\omega\pm i\delta} = \frac{P}{\omega} \mp i\delta(\omega) \; ,$$

$$\text{Im}\,\frac{1}{\epsilon(q,\omega)} = \frac{\pi\omega_p^2}{2[\epsilon^2(q)+\omega_p^2]^{1/2}}\,\{\delta(\omega-[\epsilon^2(q)+\omega_p^2]^{1/2}) + \delta(\omega+[\epsilon^2(q)+\omega_p^2]^{1/2})\} \; . \tag{9}$$

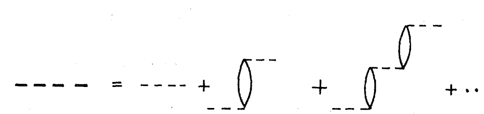

FIG. 2. Perturbation expansion for the effective potential.

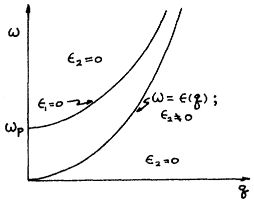

FIG. 3. Here ϵ_1 is the real part of $\epsilon(q, \omega)$, while ϵ_2 is the imaginary part.

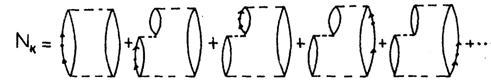

FIG. 4. Diagram series represents the perturbation expansion for the expected number of particles with momentum $\hbar\vec{k}$. The dot represents the single particle vertex of $a_{\vec{k}}^\dagger a_{\vec{k}}$.

Substituting Eq. (9) into Eq. (8) and writing $v(r=0)$ as $[\Omega/(2\pi)^3]\int d\vec{q}\, v(q)$, one can quite easily carry out the e'^2 integration and obtain

$$E/N = -\frac{1}{2}\int \frac{d\vec{q}}{(2\pi)^3}\frac{4\pi e^2}{q^2} - \int \frac{d\vec{q}}{(2\pi)^3}\frac{1}{2\rho}\left\{[\epsilon^2(q) + 4\pi\rho\hbar^2 e^2/m]^{1/2} - \epsilon(q)\right\}. \tag{10}$$

Now by writing the momentum variable q in the unit of $(4\pi m e^2\rho/\hbar^2)^{1/4}$ and the density ρ in terms of r_s which is defined by $\rho = (3/4\pi)(r_s\hbar^2/me^2)^{-3}$, we find

$$\frac{E}{N} = \frac{2}{\pi}3^{1/4} r_s^{-3/4} \times \int_0^\infty dq\, q^2\left[\left(\frac{q^4}{4}+1\right)^{1/2} - \frac{q^2}{2} - \frac{1}{q^2}\right] \mathrm{Ry} \quad , \tag{11}$$

which agrees with the earlier mentioned result of I.

The condition $\epsilon(k, \omega) = 0$ yields

$$\omega_k = [\epsilon^2(k) + \omega_p^2]^{1/2} \quad , \tag{12}$$

which represents the excitation mode, the plasmon. Equation (12) is the well-known Bogoliubov spectrum[7] and obeys the Feynman relation.[8]

Equation (7) agrees with the Bogoliubov theory of Pines[9] in the high-density limit. Thus it will sufficient to remark that Eq. (7) yields all the related functions $S(k, \omega)$ (dynamic form factor), $S(k)$ (static form factory), $\chi(k, \omega)$ (linear density response function), etc., in agreement with the Bogoliubov theory.[8,9] We will just write the results:

$$S(k,\omega) = [N\epsilon(k)/\omega_k]\,\delta(\omega - \omega_k)\,,$$
$$S(k) = \epsilon(k)[\epsilon^2(k) + \omega_p^2]^{-1/2}\,, \tag{13}$$
$$\chi(k,\omega) = \frac{2N\epsilon(k)}{(\omega - \omega_k + i\delta)(\omega + \omega_k - i\delta)} \quad .$$

The static screened interaction[10] is

$$V_{\text{eff}}(k, \omega = 0) = v(k)\epsilon^2(k)/[\epsilon^2(k) + \omega_p^2] \tag{14}$$

and the average induced density fluctuation in the system because of a static impurity of charge $-e$ brought to the origin is[11]

$$\langle \rho(k,\omega)\rangle = \frac{\omega_p^2}{\epsilon^2(k) + \omega_p^2}\, 2\pi\delta(\omega) \quad . \tag{15}$$

As Pines has discussed in several places, the f sum rule is exactly satisfied and so is the conductivity sum rule.[12] The perfect screening effect in the long-wavelength limit is obvious from Eq. (7).

III. MOMENTUM DISTRIBUTION FUNCTION FOR $\vec{k} \neq 0$

In I, we calculated the expected number of particles in the condensate at the ground state. The result shows

$$f_0 = N_0/N = 1 - 0.2114 r_s^{3/4}\,. \tag{16}$$

That is, in the high-density limit, the majority of the particles are in the condensate. Nevertheless, it is interesting to find the fraction of particles above the condensate. The expected number is given by

$$N_{\vec{k}} = \langle a_{\vec{k}}^\dagger a_{\vec{k}}\rangle \quad . \tag{17}$$

The expectation value is obtained by the sum of all possible Goldstone ground-state energy diagrams which in addition to the particle interaction in the perturbation include $a_{\vec{k}}^\dagger a_{\vec{k}}$. The single-particle vertex due to $a_{\vec{k}}^\dagger a_{\vec{k}}$, however, should appear only once.[13] The single-particle vertex cannot appear in the diagram before the particle interaction produces an excited particle of momentum $\hbar\vec{k}$. The perturbation expansion is shown in Fig. 4. One can sum these terms using the same trick as used in I. The result shows

$$f_{\vec{k}} = N_{\vec{k}}/N = \frac{r_s^{3/4}}{3^{1/4}4\pi^2}\left(\frac{k^2/2 + 1/k^2}{(k^4/4+1)^{1/2}} - 1\right) \quad . \tag{18}$$

It is interesting to note that

$$f_0 + \sum_{\vec{k}} f_{\vec{k}} = 1$$

follows naturally.

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Interference and Correlations in Photon and Electron Optics*

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The quantum theory of interference effects is given. Three experiments of particular interest are treated: (i) Young-type experiments, (ii) production of beats in photocurrents, and (iii) optical modulation of electron beams. These have been of considerable interest in recent experiments using lasers. We show that they can be accounted for by the same simple quantum-mechanical principle. In the first case, we include the effect of interference in single-photon processes and correlations in multiphoton processes. Here the method of Weisskopf and Wigner is used, and useful distinction between the concept of interference and correlation is proposed. We further emphasize the dynamical aspect of the theory. In the last two cases, we use the standard method of quantum electrodynamics. Here we view the actual experimental setup as being described by a single dynamical process. The description involves the use of nonstationary states as the relevant states to describe the successive excitation. It is shown that all the three phenomena analyzed are closely related.

The term "interference" was probably introduced into physics by Young[1] in the discussion of his famous two-pinhole experiment, although the concept was known earlier in connection with surface waves on water. Since Young, interference in the sense of the superposition of electromagnetic waves has played a central role in optics. With the introduction of wave mechanics, the application of the concept of interference has been greatly widened, and effects which can be accounted for by the superposition of quantum probability amplitudes are included in the roster interference effects. This widened use of the term interference leads to ambiguity in applications to optics, because probability amplitudes exist in the configuration space of a system which can be many-dimensional in contrast to light-wave amplitudes which exist in three-dimensional physical space. As one would expect from this dimensional distinction, the ambiguity comes in when processes involving more than one quantum of the relevant field are in question; a single quantum is described in three-dimensional physical space while two or more quanta require a higher-dimension configuration space.

It is our aim to analyze the experimental and theoretical aspects of optical phenomena from the quantum-mechanical point of view. In this way we hope to show how some previously unrelated effects are really closely related, to make a useful distinction between the terms "interference" and "correlation" in optics, and to point out the usefulness of considering free particles which are not in eigenstates of energy and momentum as constituents of optical systems.

In Sec. I the original form of Young's experiment is reviewed in both classical and quantum terms. In Sec. II similar experiments, called pseudo-Young experiments, in which the illuminated slits are replaced by atomic sources, are analyzed when the final state contains one, two, or very many photons. In Sec. III the production of beats in photoelectron is described in completely quantum terms showing that there should be a beat component of the photocurrent proportional to the over-