Temperature Fluctuations Associated with Gravity Waves at a Vapor-Superfluid Interface

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(Received 1 November 1971)

The existence of temperature oscillations associated with surface gravity waves is shown for an incompressible superfluid in the two-fluid model in equilibrium with its compressible vapor. The properties of the oscillations are consistent with some informally reported observations.

INTRODUCTION

There has been a recent report[1] that very low-frequency temperature oscillations (~2.5 cycles/sec) are observed in Dewars of liquid helium when the temperature is below the λ point. The amplitude of the temperature waves is in the micro-degree range. The frequencies of the waves are those associated with the lowest allowed wavelengths for surface gravity waves[2] [$\omega=(kg)^{1/2}$] but it is not completely clear why a temperature oscillation would be associated with such gravity waves. We show here that if one takes account of the presence of a compressible vapor above a liquid treated as an incompressible liquid in two-fluid hydrodynamics, then a temperature oscillation is expected to accompany a gravity wave. The temperature wave is theoretically expected to be much less strongly damped away from the surface below the λ point and thus would be observable. The predicted properties of the temperature oscillation will be shown to be consistent with the experimental information available. The physical origin of the temperature wave which we find is the temperature variation induced in the compressible vapor by the compressions produced by a surface wave. This source of temperature oscillation is also present for vapor above an ordinary liquid but the oscillations do not propagate as effectively in ordinary liquid (where they propagate by thermal diffusion) as they do in helium II (where they propagate as a compressional wave in the normal-fluid and superfluid components of the two-fluid hydrodynamic system).

The next sections present the calculation and some estimates of the magnitude and other properties of the temperature oscillation. Finally, we discuss the relationship of this to earlier work on liquid helium and point out some remaining questions.

MODEL

We consider a perfect gas in equilibrium[3] with an incompressible liquid which is described by two-fluid hydrodynamics. We suppose that the flow of the fluids is irrotational and we neglect the effects of viscosity and thermal conductivity. We linearize the equations of hydrodynamics to study small oscillations. We assume that the pressure and temperatures are equal at the interface of the gas and the liquid. To take account of the fact that matter is transferred from vapor to liquid during surface wave motions we assume that such transfer takes place at a rate $\sigma(x,y,t)$ g/sec/cm^2 at the point x,y of the surface. $\sigma(x,y,t)$ is determined by the requirement that the pressure and temperature at the surface follow the vapor-pressure curve (see Appendix). Linearizing in the velocities, we then have the following set of equations and boundary conditions: For the vapor the equations of motion are

$$\mathring{\rho}_g=-\rho_g\nabla^2\phi_g\,, \tag{1}$$

$$\rho_g\nabla\dot{\phi}_g=-\nabla P_g-\rho_g g\hat{z}\,, \tag{2}$$

$$\frac{\partial}{\partial t}(\rho_g S_g)=-\rho_g S_g\nabla^2\phi_g\,, \tag{3}$$

expressing conservation of mass, momentum, and entropy and constitutive relations

$$P_g=\rho_g k_B T_g/m\,, \tag{4}$$

$$S_g=(k_B/m)\ln(em/\rho_g\lambda^3)\,. \tag{5}$$

For the liquid the corresponding equations are

$$\rho_n\nabla^2\phi_n+\rho_s\nabla^2\phi_s=0\,, \tag{6}$$

$$\rho_n\nabla\dot{\phi}_n+\rho_s\nabla\dot{\phi}_s=-\nabla P_l-\rho_l g\hat{z}\,, \tag{7}$$

$$\rho_l\dot{S}_l=-\rho_l S_l\nabla^2\phi_n\,, \tag{8}$$

$$\nabla\dot{\phi}_s=-\nabla P_l/\rho_l-g\hat{z}+S_l\nabla T_l\,, \tag{9}$$

where $\vec{v}_g=\nabla\phi_g$ is the velocity of the vapor, ρ_g is the density of the vapor, P_g is the pressure of the vapor, S_g is the entropy per gram of the vapor, T_g is the temperature of the vapor, λ is the thermal wavelength of the vapor, m is the mass of an atom of the vapor, k_B is Boltzmann's constant, g is the acceleration of gravity, $\rho_{n,s}$ is the density of the normal, superfluid, $\vec{v}_{n,s} = \nabla\phi_{n,s}$ is the velocity of the normal, superfluid, S_l is the entropy per gram of the liquid, P_l is the pressure of the liquid, and T_l is the temperature of the liquid.

A dot denotes partial time differentiation, $\hat{z}$ is a unit vector normal to the quiescent surface.

The boundary conditions implied by these assumptions are (see Appendix), for the pressure,

$$[\rho_n\dot{\phi}_n + \rho_s\dot{\phi}_s + \rho_l g\zeta]_0 = [\rho_g\dot{\phi}_g + \rho_g g\zeta]_0 , \qquad (10)$$

where ζ is the surface displacement; taking the time derivative and using

$$\dot{\zeta} = \frac{\sigma}{\rho_g} + \left[\frac{\partial\phi}{\partial z}\right]_0 \qquad (A7)$$

of the Appendix,

$$\sigma = \left(\frac{1}{g}\right)\left\{P'[\dot{T}_g]_0 - \rho_g[\ddot{\phi}_g]_0 - \rho_g g\left[\frac{\partial\phi}{\partial z}\right]_0\right\} \qquad (A15)$$

from the Appendix gives

$$[\rho_n\ddot{\phi}_n + \rho_s\ddot{\phi}_s - \rho_l\ddot{\phi}_g]_0 = -(\Delta\rho/\rho_g)[T_g]_0 \qquad (11)$$

[$P' = (dP/dT)_{vp}$ is the slope of the vapor-pressure curve]. Here $\Delta\rho = \rho_l - \rho_g$ and $[\cdots]_0$ means that the quantities inside the brackets are evaluated at $z = 0$. For the temperature, the boundary condition is

$$[T_g]_0 = [T_l]_0 . \qquad (12)$$

For the velocities, the boundary conditions are worked out in the Appendix:

$$\left[\frac{\partial\phi_n}{\partial z}\right]_0 = \left[\frac{\partial\phi_s}{\partial z}\right]_0 - \frac{\sigma}{\rho_s}\,\frac{\Delta S}{S_l} , \qquad (13a)$$

$$\left[\frac{\partial\phi_n}{\partial z}\right]_0 = \left[\frac{\partial\phi_g}{\partial z}\right]_0 + \sigma\left[\frac{1}{\rho_g}\left(\frac{S_g}{S_l}\right)\left(\frac{1}{\rho_l}\right)\right] , \qquad (13b)$$

in which σ is given by Eq. (A15) of the Appendix. Here $\Delta S = S_g - S_l$.

CALCULATION

Using Eqs. (3) and (5) gives

$$[3/2T_g]\,\dot{T}_g = -\nabla^2\phi_g . \qquad (14)$$

Taking the time derivative of (2) gives

$$\rho_g\ddot{\phi}_g = -\dot{P}_g , \qquad (15)$$

and using (4) together with (14) and (15) implies

$$\ddot{\phi}_g = c_1^2\nabla^2\phi_g ,$$

where $c_1^2 = 5k_BT_g/3m$. This is a standard result describing sound propagation in a perfect gas. We seek a surface wave of form

$$\phi_g(z, x, t) = f(z)\cos(kx - \omega t)$$

and find

$$\frac{d^2f}{dz^2} = \left[k^2 - \left(\frac{\omega}{c_1}\right)^2\right] f$$

or

$$\phi_g = A\exp\{-[k^2 - (\omega/c_1)^2]^{1/2} z\}\cos(kx - \omega t) \qquad (16)$$

by imposing the boundary condition $\phi_g(z \to \infty) = 0$. Using (14) then gives

$$T_g = (-2T\omega/3c_1^2)\,A\exp\{-[k^2 - (\omega/c_1)^2]^{1/2}\}\sin(kx - \omega t) \qquad (17)$$

for the time-varying part of the vapor temperature. Here T is the temperature of the vapor and liquid in the absence of a surface wave.

For the liquid, Eq. (7) gives

$$\nabla P_l + \rho_l g\hat{z} = -\rho_n\nabla\dot{\phi}_n - \rho_s\nabla\dot{\phi}_s . \qquad (18)$$

Inserting this in Eq. (9) gives

$$(\rho_n/\rho)\nabla(\dot{\phi}_s - \dot{\phi}_n) = S_l\nabla T_l . \qquad (19)$$

Taking ∇ of this and using (6) gives

$$\nabla^2\dot{\phi}_s = S_l\nabla^2 T_l , \qquad (20)$$

where we have linearized in time- and space-varying quantities as before. Equation (8) gives

$$S\dot{T}_l = c_2^2\nabla^2\phi_s , \qquad (21)$$

where

$$c_2^2 = \left(\frac{\partial T_l}{\partial S_l}\right)_{\rho_l} S_l^2\left(\frac{\rho_s}{\rho_n}\right) .$$

S_l is the time-independent part of the liquid entropy per gram. Combining (20) and (21) one has

$$\nabla^2 T_l = (1/c_2)^2\,\ddot{T}_l , \qquad (22)$$

which is also a standard result describing second sound in helium.

The program is next to (a) solve Eq. (22) for T_l subject to Eqs. (12) and (17). (b) Then solve Eq. (21) for ϕ_s subject to Eqs. (13b) and (16). (c) Solve Eq. (6) for ϕ_n subject to Eq. (13a). (d) Finally impose the condition of Eq. (11) to determine ω as a function of k.

Step (a): Temperature Oscillation in the Liquid

Write

$$T_l(x, z, t) = T_l(z)\sin(kx - \omega t) \qquad (23)$$

for the time-dependent part of the liquid temperature. Eq. (22) then implies

$$\frac{d^2T_l}{dz^2} = [k^2 - (\omega/c_2)^2]\,T_l(z) \qquad (24)$$

or

$$T_l(z) = T_l(0)\ \exp\{[k^2 - (\omega/c_2)^2]^{1/2} z\}$$

using the condition that

$$T_l(x, z \to -\infty, t) = 0$$

(for the time-dependent part).

Then imposing Eq. (12) and using Eq. (17) we have

$$T_l(x, z, t) = (-2T\omega/3c_1^2) A\ \exp\{[k^2 - (\omega/c_2)^2]^{1/2} z\} \times \sin(kx - \omega t)\,. \quad (25)$$

This equation displays the main result of the paper; the existence of a deeply penetrating temperature wave in the liquid.

Step (b): Velocity Potential for the Two Superfluid

Equation (21) becomes [using (25)]

$$\nabla^2\phi_s = (+2S_l T\,\omega^2/3c_2^2 c_1^2) A\ \exp\{[k^2 - (\omega/c_2)^2]^{1/2} z\} \times \cos(kx - \omega t)\,. \quad (26)$$

This has the general solution

$$\phi_s(x, z, t) = (\alpha_s e^{kz} + (A'/c_2^2/\omega^2)\ \exp\{[k^2 - (\omega/c_2)^2]^{1/2} z\} \times \cos(kx - \omega t))\,, \quad (27)$$

where

$$A' = -2S_l T\,\omega^2 A/3\,c_2^2\,c_1^2\,, \quad (28)$$

and where $\phi_s(x, z \to -\infty, t) = 0$ has been used. α_s is determined from Eqs. (13), giving

$$\alpha_s = \left(\frac{A}{k}\right)\left\{+\gamma\left[k^2 - \left(\frac{\omega}{c_2}\right)^2\right]^{1/2} - \left(\frac{\rho_g}{\rho_l}\right)\left[1 - \frac{\Delta S}{S_l}\left(\frac{\rho_n}{\rho_s}\right)\right]\left[k^2 - \left(\frac{\omega}{c_1}\right)^2\right]^{1/2} + \omega^2\left(1 + \frac{2P'T}{3c_1^2\rho_g}\right)\left[\frac{\Delta\rho + \rho_g(\Delta S/S_l)(\rho_n/\rho_s)}{\rho_l g}\right]\right\}\,, \quad (29)$$

in which

$$\gamma = 2S_l\,T/3c_1^2\,. \quad (30)$$

Step (c): Velocity Potential for the Normal Fluid

Equation (6) is

$$\nabla^2\phi_n = -(\rho_s/\rho_n)\,\nabla^2\phi_s \quad (31)$$

or using (26)

$$\nabla_2\phi_n = (A'\rho_s/\rho_n)\ \exp\{[k^2 - (\omega/c_2)^2]^{1/2} z\}\cos(kx - \omega t)$$

with solution

$$\phi_n = (\alpha_n e^{kz} - (\rho_s c_2^2/\rho_n\omega^2) A'\ \exp\{[k^2 - (\omega/c_2)^2]^{1/2} z\} \times \cos(kx - \omega t)\,. \quad (32)$$

α_n is determined by Eq. (13a) to be

$$\alpha_n = (A/k)\,\{-\gamma(\rho_s/\rho_n)[k^2 - (\omega/c_2)^2]^{1/2} - (\rho_g/\rho_l)(1 + \Delta S/S_l)\,[k^2 - (\omega/c_1)^2]^{1/2} + (\omega^2/g)(1 + 2P'T/3c_1^2 g)(\Delta\rho/\rho_l - (\Delta S/S_l)(\rho_g/\rho_l)\}\,. \quad (33)$$

Step (d): ω vs k for Gravity Waves

Equation (11) gives

$$-(\alpha_n\rho_n + \alpha_s\rho_s) + \rho_l A = -\Delta\rho\xi A\,, \quad (34)$$

where we have set $\xi = 2P'T/3\rho_g c_1^2$ by use of Eqs.(17), (27), and (32). Then using Eqs. (29) and (33) and setting

$$y = [k^2 - (\omega/c_1)^2]^{1/2} k$$

gives

$$\omega^2/gk = \frac{\rho_g y + \rho_l + \Delta\rho\xi}{\Delta\rho(1+\xi)}\,. \quad (35)$$

In the case that $\rho_g \ll \rho_l$ we set $\rho_g/\rho_l = \epsilon$ and get to first order in ϵ that

$$\omega^2 = gk\left(1 + \frac{(1+y)}{(1+\xi)}\,\epsilon\right) \quad \text{if } \rho_g/\rho_l \ll 1\,, \quad (36)$$

which reduces correctly to $\omega^2 = gk$ when $\epsilon = 0$. To check that the result is correct when $\sigma = 0$, we note that in the case $\sigma = 0$ we have from Eq. (A15) that

$$\sigma = (A/g)\,\{\rho_g(1+\xi)\,\omega^2 + \rho_g g[k^2 - (\omega/c_1)^2]^{1/2}\} = 0\,, \quad (37)$$

or when $(\omega/c_1)^2 \ll k^2$ that

$$\omega^2/gk = -1/(\xi + 1)\,. \quad (38)$$

Inserting this into (35) and solving again for ω^2/gk gives

$$\omega^2 = gk[\Delta\rho/(\rho_g + \rho_l)] \text{ if } \sigma = 0 \quad (39)$$

in agreement with the standard result for two fluids without evaporation or compressibility.[2]

DISCUSSION OF TEMPERATURE OSCILLATION IN LIQUID

In this section we first consider the order of magnitude implications of Eq. (25) which displays the existence of a temperature wave in the liquid. We use parameters relevant to the experimental case of interest.[1] In that case $k \approx 1\ \text{cm}^{-1}$ (5-in. Dewar) and $\omega \approx 15\ \text{sec}^{-1}$ ($\nu \approx 16$ Hz) while $c_2 = 2000$ cm/sec. Therefore, we have $(\omega/c_2) \approx \frac{1}{100}\ \text{cm}^{-1} \ll k$. Thus the temperature oscillation in Eq. (25) damps out in a distance of the order of centimeters. We can estimate the magnitude of the temperature oscillation in terms of the amplitude A by use of $T \approx 2\ ^\circ\text{K}$, $c_1 \approx 1/\sqrt{5} \times 10^4$ cm/sec and Eq. (25), giving an amplitude of $\Delta T \approx 7\times10^{-6} A$ °K. A is related to the amplitude of the surface wave by

$$v_{gz} \simeq k\phi_g \quad \text{and} \quad \omega\zeta \simeq v_{gz} + \sigma/\rho_g .$$

Using

$$\sigma \simeq \rho_g A(\omega^2/g + k) \simeq 2\rho g A k$$

then gives

$$\zeta \simeq 3Ak/\omega \quad \text{or} \quad \Delta T \simeq \tfrac{1}{3}\times 10^{-4}\,|\,\zeta\,|\ \text{°K}$$

when T is in °K and ζ is in cm. Thus a microdegree oscillation corresponds to a surface oscillation of amplitude 3×10^{-2} cm. In the experimental arrangement, gravity waves of this amplitude might be excited quite easily by stray laboratory vibrations.[1] Thus the predicted amplitude of temperature waves is probably large enough to account qualitatively for the observations.

The approximations involved in going from Eq. (35) to $\omega \simeq (gk)^{1/2}$ involve assuming that

$$(\omega/c_1 k)^2 \simeq 5\times10^{-6} \ll 1$$

and

$$\rho_g/\rho_l = \epsilon \simeq \tfrac{1}{20} \ll 1 .$$

Thus the predicted frequency is $\omega \simeq (gk)^{1/2}$ for this case to within about 5% approximation. This is consistent with the observed frequency of the temperature oscillations. The largest corrections to $\omega = (gk)^{1/2}$ are those of order ϵ in Eq. (36). We expect ω to be about 5% *higher* than in the absence at the vapor. The result (39) with no evaporation gives on the contrary a frequency lower than $(gk)^{1/2}$. The equation also predicts a possibly observable temperature dependence of ω.

To understand why the temperature oscillation might not be observed above the λ point we consider the corresponding calculation with an ordinary incompressible liquid replacing the two-fluid liquid. If the thermal conductivity is ignored, then there is no temperature wave at all. Including thermal conductivity gives

$$T_l = -(2T\omega A/3c_1^2)\sin\{kx - \omega t + kz[1+(\omega/Dk^2)^2]^{1/2}\sin[\tfrac{1}{2}\tan^{-1}(\omega/Dk^2)]\} \times \exp(-kz\{[1+(\omega/Dk^2)^2]^{1/2}\cos[\tfrac{1}{2}\tan^{-1}(\omega/Dk^2)]\}),$$

where[4]

$$D = \kappa/c_v\rho_l \simeq 10^{-3}\ \text{cm}^2/\text{sec}$$

is the thermal diffusion constant for liquid helium above T_λ. κ is the thermal conductivity and c_v is the specific heat. With $\omega \sim 15\ \text{sec}^{-1}$ and $k\sim 1\ \text{cm}^{-1}$ we have $(\omega/Dk^2)\sim 10^4$ so that

$$T_l \simeq (-2T\omega A/c_1^2)\exp(-\omega z/\sqrt{2}Dk) \times \sin(kx-\omega t) + \omega z/\sqrt{2}Dk).$$

This disturbance is damped in a distance

$$\sqrt{2}Dk/\omega \simeq 10^{-4}\ \text{cm}$$

and would therefore not be seen.[5] Thus this calculation is consistent with the observed disappearance of the temperature oscillations above T_λ.

RELATION TO THIRD SOUND, RIPPLONS, AND RECENT WORK

The work of Atkins[6] predicting third sound is related to the present calculation in the following way. Atkins implicitly includes normal-fluid viscosity by fixing the normal fluid. He does not include pressure variations in the vapor but he does include evaporation effects. With these differences, his calculation is the shallow-water version of the standard calculation describing surface gravity waves[7] (using two-fluid hydrodynamics instead of one-fluid hydrodynamics). Our calculation is the two-fluid hydrodynamic version of the standard calculation for two fluids in contact in the deep-water limit (the kh, $kh' \gg 1$ limit of problem 2, Sec. 12 of Ref. 2), with the added proviso that the upper fluid be compressible and with evaporation included. The other difference in the two calculations is that in the shallow-water limit (Atkins[4]) the force of gravity is replaced by the van der Waals force.

The present calculation is also closely related to earlier work of Kuper[8] and Atkins[9] on gravity waves called ripplons by Kuper. In the paper by Atkins,[9] the interest is in the effect of these waves on surface tension. Atkins claimed that the surface tension is only affected by the short-wavelength surface waves in which gravity plays a negligible role. Kuper's paper was concerned with critical velocities. He considered bulk gravity waves but applied the results to a film, a procedure which Atkins pointed out was incorrect. Neither of these papers considered the vapor and neither suggested that a temperature variation would be associated with surface waves on the bulk fluid. In the terminology of Kuper's paper, we can say that the result of this calculation is that consideration of the compressible vapor above the fluid shows that temperature oscillations are associated with ripplons on the surface of bulk superfluid helium.

Arkhipov[10] has also treated the film problem. He did not obtain a temperature variation because he included neither the vapor (as done here) nor the locking of the normal third (as done by Atkins[6]) nor the evaporation effects (also done by Atkins[6] and here).

Recently, work by Meinhold-Heerlein and Loos[11] appears to take most of these features into account in studying the problem of sound waves incident on a vapor-liquid boundary. We cannot make a detailed comparison of this work with the present calculation because a detailed account of the calculation has not appeared. The approach to entropy balance appears, however, to be different.

Finally, the recent work of Seiden[12] has treated the bulk liquid surface oscillation including viscosity but neglecting the vapor. He finds a temperature oscillation linear in the viscosity in the absence of the vapor which he finds to be "very weak."

FURTHER QUESTIONS AND DISCUSSION

The range that the temperature oscillations penetrate the liquid is shown by (25) to depend sharply on the temperature near T_λ through the velocity of second sound $c_2(T)$. This should be an experimentally verifiable point which could be used to check the theory quantitatively. The depth dependence of the oscillations would not be hard to monitor by use of a small thermometer at various depths and at various temperatures below but near T_λ. A second feature of the theory which should be quite easy to check experimentally is the temperature dependence of the oscillation frequency implied by Eq. (36). In the case of interest this reads

$$\omega^2 \simeq gk(1+\rho_g/\rho_l).$$

The correction term is of order 5% (2% in the frequency) and is temperature dependent near T_λ.

It is straightforward to extend the calculation to liquid helium of finite depth. Other possible extensions of interest include adding the effect of normal-fluid viscosity and thermal conductivity and of rotational flow. The neglect of thermal conductivity and viscosity in both the liquid and the vapor seems justified, both because they are small and because the phenomenon of interest occurs suddenly at the λ point and therefore seems unlikely to depend essentially on these quantities.

ACKNOWLEDGMENTS

It is a pleasure to thank Bill Zimmermann for his patience in several discussions and for pointing out an error in an initial version of this calculation. Ed Ney is thanked for discussions of the experiments.

APPENDIX: BOUNDARY CONDITIONS

We suppose that a mass $\sigma(x,y,t)$ per unit area per unit time is deposited from the gas to the liquid at the surface at the point x, y of the surface and that at the same time entropy transfer σ_s per unit area per unit mass takes place from the gas to the liquid. We may include these effects in the equations of motion through use of δ functions. Alternatively, by integrating the resultant equations of motion over appropriate surfaces, we may take the effects of σ, σ_s into account in the boundary conditions. We proceed from the first formulation to the second. The equations of mass continuity including σ are

$$\dot{\rho}_g = -\rho_g \nabla\cdot \vec{v}_g - \sigma\,\delta(z-\zeta)\,, \tag{A1}$$

$$\rho_n \nabla\cdot\vec{v}_n + \rho_s \nabla\cdot\vec{v}_s = \sigma\,\delta(z-\zeta)\,. \tag{A2}$$

For entropy conservation we have

$$\frac{\partial}{\partial t}(\rho_g S_g) = -\rho_g S_g \nabla^2\phi_g - \sigma_s\,\delta(z-\zeta)\,, \tag{A3}$$

$$\rho_l \frac{\partial S_l}{\partial t} = -\rho_l S_l \nabla^2\phi_n + \sigma_s\,\delta(z-\zeta)\,. \tag{A4}$$

Evaporation corrections to the momentum-conservation equations can be shown to arise only in second order in the velocities. One therefore has

$$\frac{\partial \vec{j}_g}{\partial t} = -\nabla P_g - \rho_g g\hat{z}\,, \tag{A5}$$

$$\frac{\partial \vec{j}_l}{\partial t} = -\nabla P_l - \rho_l g\hat{z} \tag{A6}$$

for momentum conservation to lowest order in the velocities.

We integrate each of the Eqs. (A1)–(A4) over a pill box defined as in Fig. 1(a) for the liquid equations and as in Fig. 1(b) for the gas equations, assuming $\sqrt{a} \gg \epsilon > \zeta$ in each case. We get, linearizing as before,

$$\dot{\zeta} = \frac{\sigma}{\rho_g} + \left[\frac{\partial\phi_g}{\partial z}\right]_0\,, \tag{A7}$$

$$\dot{\zeta} = \frac{\sigma}{\rho_l} + \left(\frac{\rho_n}{\rho_l}\right)\left[\frac{\partial\phi_n}{\partial z}\right]_0 + \left(\frac{\rho_s}{\rho_l}\right)\left[\frac{\partial\phi_s}{\partial z}\right]_0 \tag{A8}$$

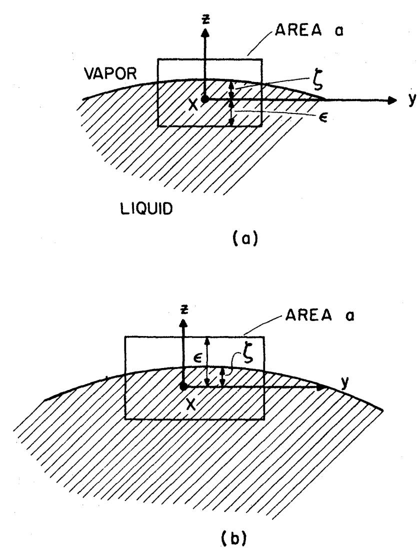

FIG. 1. Integration volumes used in the Appendix.

from the continuity equations,

$$\dot{\zeta} = (\sigma_s/\rho_g S_g) + \left[\frac{\partial \phi_g}{\partial z}\right]_0 , \qquad \text{(A9)}$$

$$\dot{\zeta} = \left(\frac{\sigma_s}{\rho_l S_l}\right) + \left[\frac{\partial \phi_n}{\partial z}\right]_0 \qquad \text{(A10)}$$

from the entropy equations, and

$$\rho_g[\dot{\phi}_g]_0 = -[P]_0 - \rho_g g \zeta , \qquad \text{(A11)}$$

$$\rho_n[\dot{\phi}_n]_0 + \rho_s[\dot{\phi}_s]_0 = -[P]_0 - \rho_l g \zeta \qquad \text{(A12)}$$

from the momentum equations. Equating the pressures in (A11) and (A12) then gives Eq. (10).

We use (A11) to impose the condition that the temperature and pressure stay on the vapor-pressure curve at the surface: Taking the time derivation of (A11) and linearizing

$$[\dot{P}]_0 = \rho_g[\dot{\phi}_g]_0 + \rho_g g \dot{\zeta} , \qquad \text{(A13)}$$

but $[\dot{P}]_0$ involves only perturbations, so we have

$$\rho_g[\ddot{\phi}_g]_0 + \rho_g g \dot{\zeta} = P'[\dot{T}_g]_0 . \qquad \text{(A14)}$$

Combining this with (A7) gives an equation for σ:

$$\sigma = \left(\frac{1}{g}\right)\left\{P'[\dot{T}_g]_0 - \rho_g[\ddot{\phi}_g]_0 - \rho_g g \left[\frac{\partial \phi_g}{\partial z}\right]_0\right\} . \qquad \text{(A15)}$$

To determine σ_s we compare equations (A7) and (A9) giving $\sigma_s = S_g \sigma$. Then combining (A8) and (A10) gives

$$\left[\frac{\partial \phi_n}{\partial z}\right]_0 = \left[\frac{\partial \phi_g}{\partial z}\right]_0 + \sigma\left[\frac{1}{\rho_g} - \left(\frac{S_g}{S_l}\right)\left(\frac{1}{\rho_l}\right)\right] , \qquad \text{(A16)}$$

and using (A8) we have

$$\left[\frac{\partial \phi_n}{\partial z}\right]_0 = \left[\frac{\partial \phi_s}{\partial z}\right]_0 + \frac{\sigma}{\rho_s}\left[1 - \frac{S_g}{S_l}\right] . \qquad \text{(A17)}$$

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[5] Note that the calculation of the dispersion relation for surface waves would differ in the case of an ordinary fluid with $\kappa \neq 0$ because heat conduction across the boundary is not included in our boundary conditions.

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