# Asymptotic Retarded van der Waals Forces Derived from Classical Electrodynamics with Classical Electromagnetic Zero-Point Radiation

Timothy H. Boyer
*Department of Physics, City College of the City University of New York, New York, New York 10031*
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The large-separation asymptotic forms for the retarded van der Waals forces between a neutral polarizable particle and a conducting wall, and between two neutral polarizable particles, are derived from classical electromagnetism under the assumption that the universe contains fluctuating classical electromagnetic radiation with a Lorentz-invariant spectrum (classical electromagnetic zero-point radiation). These forces were first calculated by Casimir and Polder from quantum electrodynamics, and then recalculated by Casimir using ideas of zero-point energy. The present calculation involves purely classical electromagnetic attractions between classical oscillators driven by fluctuating classical radiation.

## I. INTRODUCTION

In 1948 Casimir and Polder[1] calculated the van der Waals force between two neutral polarizable particles to fourth order in quantum electrodynamics. They found that because of retardation effects, the interparticle potential fell off with distance faster than the London[2] $r^{-6}$, actually going asymptotically to $r^{-7}$. This result for the retarded van der Waals force has been recalculated[3-5] a number of times within attempts to obtain an understanding of the physical mechanism for the asymptotic behavior. The present paper presents yet another recalculation in the context of a basically simple classical picture.

Very shortly after his work with Polder, Casimir[3] presented an interpretation of the asymptotic $r^{-7}$ potential based upon ideas of quantum zero-point energy. The present writer, essentially as an outgrowth from Casimir's ideas, has been led to propose[6-9] that many of the phenomena that are presently regarded as quantum mechanical in nature may be explained in terms of classical physics when we include the possibility that the universe contains random classical electromagnetic radiation with a Lorentz-invariant spectrum.[10] Planck's constant is introduced as the multiplicative constant fixing the scale of the classical radiation spectrum. This hypothesis of classical electromagnetic zero-point radiation is a valid possibility within the context of classical electromagnetism, and it leads to entirely classical derivations of the blackbody radiation spectrum,[6] of fluctuations usually associated with photon statistics,[7] of the third law of thermodynamics,[8] and of rotator specific heats.[9] It is tempting to suggest that the zero-point radiation may provide the basis for the fluctuations in Nelson's derivation[11] of the Schrödinger equation from Newtonian mechanics plus a particle random walk.

Very recently, it has been shown[12] that the unretarded London–van der Waals potential ($r^{-6}$) between two polarizable particles may be derived from classical electromagnetism including classical electromagnetic zero-point radiation. In the present paper, we will give the derivation of the asymptotic retarded potential ($r^{-7}$). In the future, we hope to obtain the full fourth-order van der Waals force and indeed to prove the equivalence to all orders in perturbation theory between the quantum electrodynamic calculations of van der Waals forces and those from classical electromagnetism including classical electromagnetic zero-point radiation.

## II. BASIC ANALYSIS FOR FORCES IN ASYMPTOTIC REGION OF LARGE SPATIAL SEPARATIONS

The polarizable particles experiencing van der Waals forces are regarded as harmonic dipole oscillators satisfying Newton's second law of motion

$$m\ddot{\vec{x}} = -m\omega_0^2\vec{x} + e\vec{E}(\vec{r},t) + \tfrac{2}{3}(e^2/c^3)\dddot{\vec{x}}, \qquad (1)$$

where $\vec{x}$ is the displacement of the oscillator of charge $e$ and mass $m$, $\omega_0$ is the natural frequency of the oscillator, $\vec{E}(\vec{r},t)$ is the electric field at the position of the oscillator, and the term $\frac{2}{3}(e^2/c^3)\dddot{\vec{x}}$ gives the particle radiation reaction. The electric field $\vec{E}$ in space is taken as that due to all the charged particles present, plus the random classical zero-point radiation field

$$\vec{E}_0(\vec{r},t) = \mathrm{Re}\sum_{\lambda=1}^{2}\int d^3k\,\hat{\epsilon}(\vec{k},\lambda)\,\mathfrak{h}(\vec{k},\lambda) \times \exp\{i[\vec{k}\cdot\vec{x} - \omega t + \theta(\vec{k},\lambda)]\}, \qquad (2)$$

with

$$\hat{\epsilon}(\vec{k},\lambda)\cdot\hat{\epsilon}(\vec{k},\lambda') = \delta_{\lambda\lambda'}, \quad \omega = ck \qquad (3)$$

and

$$\pi^2 \mathfrak{h}^2(\vec{k}, \lambda) = \tfrac{1}{2}\hbar\omega \; . \tag{4}$$

The fluctuations are treated with the random-phase convention of Planck[13] and of Einstein and Hopf.[14]

The essential mechanism for van der Waals forces is quite simple. The fluctuating zero-point radiation causes random polarization of the particles, and these then interact through their classical electromagnetic fields. When the particles are close together, their mutual interactions can be handled through the electrostatic fields of the induced dipoles. The natural frequencies $\omega_0$ and radiation damping of the oscillators play a crucial role (see Ref. 12) in arriving at exactly the London result.

However, when the polarizable particles are very far apart, it is only the very low frequencies of oscillation which we expect to play a significant role. For high frequencies (small wavelengths), there will be canceling effects between adjacent frequencies if the particles are very far apart, since the slight phase shifts will build up over the large distances. Thus in the asymptotic region $r\to\infty$, we expect only the frequencies near $\omega \cong 0$ to affect the interactions of polarizable particles. Although the expressions we employ are conveniently expressed in terms of integrals over all frequencies, only the low-frequency parts actually contribute in the asymptotic region of large separation.

Now the analysis for low frequencies is particularly simple, since we may neglect the term $m\ddot{x}$ and the radiation reaction involving $\dddot{x}$. We have the oscillator in phase with the driving radiation

$$\vec{x} = (e/m\omega_0^2)\,\vec{E}(\vec{r}, t) \; , \tag{5}$$

$$\vec{p} = e\vec{x} = \alpha\vec{E} \; , \tag{6}$$

where

$$\alpha = e^2/m\omega_0^2 \tag{7}$$

is the static polarizability of the particle.

The force on an electric dipole $\vec{p}$ in an electric field $\vec{E}$ and magnetic field $\vec{B}$ is given by

$$\vec{F} = (\vec{p}\cdot\nabla)\vec{E} + \dot{\vec{p}}\times\vec{B} \; . \tag{8}$$

If the dipole is induced by the electric field as in Eq. (6), then

$$\vec{F} = (\alpha\vec{E}\cdot\nabla)\vec{E} + \alpha\dot{\vec{E}}\times\vec{B} \; , \tag{9}$$

where $\alpha$ is the polarizability. The associated energy of the particle for $\alpha =$ const is

$$\mathcal{E} = -\tfrac{1}{2}\alpha\vec{E}^2 \quad , \quad \langle\vec{F}\rangle = -\nabla\langle\mathcal{E}\rangle . \tag{10}$$

It is this last expression that we use as the basis for calculations of the potentials $U(R)$ for particles separated by a distance $R$. We write

$$U(R) = \mathcal{E}(R) - \mathcal{E}(\infty) \; , \tag{11}$$

subtracting the comparison energy $\mathcal{E}(\infty)$ when the polarizable particles are removed to spatial infinity.

## III. POLARIZABLE PARTICLE AND CONDUCTING WALL

Following the example set by Casimir and Polder,[1] we first calculate the second-order attraction of a polarizable particle to a perfectly conducting wall, and then later calculate the attraction between two polarizable particles.

We consider a polarizable particle a distance $R$ along the $x$ axis from a conducting wall in the $yz$ plane. The random electromagnetic zero-point radiation is reflected from the conducting wall and hence satisfies (with $k_x < 0$ only)

$$\vec{E}(\vec{r}, t) = \mathrm{Re}\sum_{\lambda=1}^{2}\int d^3k\, \mathfrak{h}(\vec{k},\lambda)\left(\hat{\epsilon}\exp[i(\vec{k}\cdot\vec{x}-\omega t+\theta)] + (2\hat{i}\,\hat{i}\cdot\hat{\epsilon}-\hat{\epsilon})\exp\{i[(-2\hat{i}\hat{i}\cdot\vec{k}+\vec{k})\cdot\vec{x}-\omega t+\theta]\}\right)$$

$$= \sum_{\lambda=1}^{2}\int d^3k\, \mathfrak{h}(\vec{k},\lambda)2[\hat{i}\,\epsilon_x\cos k_x x\cos(k_y y+k_z z-\omega t+\theta) - (\hat{j}\,\epsilon_y+\hat{k}\,\epsilon_z)\sin k_x x\sin(k_y y+k_z z-\omega t+\theta)] . \tag{12}$$

Substituting the value for $\vec{E}(\vec{r}, t)$ at the point $(R, 0, 0)$ into Eq. (10) for the energy $\mathcal{E}$,

$$\mathcal{E}(R) = -\tfrac{1}{2}\alpha\left\langle\sum_{\lambda=1}^{2}\sum_{\lambda'=1}^{2}\int d^3k\int d^3k'\, 4\,\mathfrak{h}(\vec{k},\lambda)\,\mathfrak{h}(\vec{k}',\lambda') \times[\epsilon_x\epsilon_x'\cos k_x R\cos k_x' R + (\epsilon_y\epsilon_y'+\epsilon_z\epsilon_z')\sin k_x R\sin k_x' R]\cos(-\omega t+\theta)\cos(-\omega' t+\theta)\right\rangle \; , \tag{13}$$

where the brackets indicate the averages over time

$$\langle\cos[-\omega t+\theta(\vec{k},\lambda)]\cos[-\omega' t+\theta(\vec{k}',\lambda')]\rangle = \tfrac{1}{2}\delta_{\lambda\lambda'}\delta^3(\vec{k}-\vec{k}') \; , \tag{14}$$

$$\langle\sin[-\omega t+\theta(\vec{k},\lambda)]\cos[-\omega' t+\theta(\vec{k}',\lambda')]\rangle = 0 \; . \tag{15}$$

Now integrating over all $k_x$,

$$\mathcal{E}(R) = -\tfrac{1}{2}\alpha\sum_{\lambda=1}^{2}\int d^3k\,\tfrac{1}{2}(\hbar\omega/\pi^2) \times[\epsilon_x^2\cos^2 k_x R + (\epsilon_y^2+\epsilon_z^2)\sin^2 k_x R] \; . \tag{16}$$

This integral for the energy is divergent at high frequencies. However, our interest lies only in the change in the energy with the distance $R$ of the polarizable particle from the conducting plate. Hence we subtract the energy at some fixed comparison point, chosen conveniently as $R\to\infty$,

$$U(R)=\mathcal{E}(R)-\mathcal{E}(\infty)$$

$$=-\tfrac{1}{2}\alpha\sum_{\lambda=1}^{2}\int d^3k\,\tfrac{1}{2}(\hbar\omega/\pi^2)\,[\epsilon_x{}^2(\cos^2k_xR-\tfrac{1}{2})+(\epsilon_y{}^2+\epsilon_z{}^2)(\sin^2k_xR-\tfrac{1}{2})]\,, \quad (17)$$

where we have written

$$\cos^2k_xR\to\tfrac{1}{2},\quad \sin^2k_xR\to\tfrac{1}{2} \quad (18)$$

as $R\to\infty$. Except for the difference between box and free-space normalization, this potential is the same as the electric terms in Eq. (12) of Ref. 5. The integrations are easily carried out as indicated in that paper, giving

$$U(R)=-(3/8\pi)(\hbar c/R^4)\,, \quad (19)$$

which is exactly the Casimir-Polder result.

## IV. TWO POLARIZABLE PARTICLES

The asymptotic potential between two polarizable particles is obtained in analogous fashion. We consider two polarizable particles $A$ and $B$, where $A$ is situated at the origin of coordinates and $B$ lies to the right a distance $R$ along the $x$ axis.

The electric field acting on particle $B$ consists of the classical zero-point radiation $\vec{E}_0$ of Eq. (2), and also of the electric dipole field $\vec{E}_{\vec{p}}$ from particle $A$. The electric field of a dipole $\vec{p}\,e^{-i\omega t}$ oscillating with frequency $\omega$ is

$$\vec{E}_{\vec{p},\omega}(\vec{r},t)=\mathrm{Re}[\vec{p}\,\mathcal{F}-\hat{n}\,(\hat{n}\cdot\vec{p})\,\mathcal{G}\,]\,e^{-i\omega t}\,, \quad (20)$$

with

$$\mathcal{F}=k^3[(kr)^{-1}+i(kr)^{-2}-(kr)^{-3}\,]\,e^{ikr}\,, \quad (21)$$

$$\mathcal{G}=k^3[(kr)^{-1}+3i(kr)^{-2}-3(kr)^{-3}\,]\,e^{ikr}\,. \quad (22)$$

The potential between the polarizable particles is found from the energy of particle $B$ in the field of particle $A$:

$$\mathcal{E}(R)=-\tfrac{1}{2}\,\alpha_B[\vec{E}_0(\vec{r}_B,t)+\vec{E}_{\vec{p}A}(\vec{r}_B,t)\,]^2\,. \quad (23)$$

Again subtracting the divergent energy when the particles are far apart, we find

$$U(R)=\mathcal{E}(R)-\mathcal{E}(\infty)\,, \quad (24)$$

and keeping terms to lowest order,

$$U(R)=-\alpha_B\vec{E}_0(\vec{r}_B,t)\cdot\vec{E}_{\vec{p}A}(\vec{r}_B,t)\,, \quad (25)$$

with

$$\vec{p}_A=\alpha_A\vec{E}_0(\vec{r}_A,t)\,. \quad (26)$$

Thus we arrive at

$$U(R)=-\alpha_A\,\alpha_B\,\Big\langle\sum_{\lambda=1}^{2}\sum_{\lambda'=1}^{2}\int d^3k\int d^3k'\,\mathfrak{h}(\vec{k},\lambda)\mathfrak{h}(\vec{k}',\lambda')\cos(k_xR-\omega t+\theta)$$
$$\times\{\hat{\epsilon}\cdot\hat{\epsilon}'[(k'^2/r)\cos(k'R-\omega't+\theta')-(k'/r^2)\sin(k'R-\omega't+\theta')-(1/r^3)\cos(k'R-\omega't+\theta')]$$
$$-\epsilon_x\epsilon'_x[(k'^2/r)\cos(k'R-\omega't+\theta')-(3k'/r^2)\sin(k'R-\omega't+\theta')-(3/r^3)\cos(k'R-\omega't+\theta')]\}\Big\rangle. \quad (27)$$

The time averaging involves terms such as

$$\langle\cos(k_xR-\omega t+\theta)\cos(k'R-\omega't+\theta')\rangle=\cos k_xR\cos k'R\,\langle\cos(-\omega t+\theta)\cos(-\omega't+\theta')\rangle$$
$$+\sin k_xR\sin k'R\,\langle\sin(-\omega t+\theta)\sin(-\omega't+\theta')\rangle-\cos k_xR\sin k'R\,\langle\cos(-\omega t+\theta)\sin(-\omega't+\theta')\rangle$$
$$-\sin k_xR\cos k'R\,\langle\sin(-\omega t+\theta)\cos(-\omega't+\theta')\rangle=(\cos k_xR\cos kR+\sin k_xR\sin kR)\,\tfrac{1}{2}\,\delta_{\lambda\lambda'}\delta^3(\vec{k}-\vec{k}')\,. \quad (28)$$

However, the term in $\sin k_xR$ is odd in $k_x$ and will vanish in the integration over $k_x$. The potential becomes

$$U(R)=-\alpha_A\,\alpha_B\sum_{\lambda=1}^{2}\int d^3k\,\tfrac{1}{2}\,\frac{\hbar\omega}{\pi^2}\left[\left(\frac{k^2}{r}\cos kR-\frac{k}{r^2}\sin kR-\frac{1}{r^3}\cos kR\right)-\epsilon_x{}^2\left(\frac{k^2}{r}\cos kR-\frac{3k}{r^2}\sin kR-\frac{3}{r^3}\cos kR\right)\right]\tfrac{1}{2}\cos k_xR\,. \quad (29)$$

Comparing this with Eq. (23) of Ref. 5, we see that we have (except for the change of normalization) exactly the terms of that equation involving the electric polarizability. Carrying out the integrations as indicated there, we arrive at the Casimir-Polder result

$$U(R) = -\frac{23\,\alpha_A \alpha_B \hbar c}{4\pi R^7} \quad . \tag{30}$$

## V. SUMMARY

If one assumes that the universe contains random fluctuating classical radiation, then neutral polarizable particles are continually being polarized, and accordingly are continually emitting and absorbing radiation. In the asymptotic region where distances are large, only low-frequency field fluctuations will influence the attractions between neutral polarizable particles. In this low-frequency domain, it is easy to calculate the dipole moments induced by the zero-point radiation and then to obtain the mutual forces between polarizable particles. Assuming a Lorentz invariant spectrum of classical fluctuating radiation with the scale set by Planck's constant, the results obtained for the attraction between a polarizable particle and a perfectly conducting wall, and between two neutral polarizable particles are in agreement with the quantum electrodynamic results of Casimir and Polder.

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# Space-Charge-Controlled Diffusion in an Afterglow*

M. A. Gusinow and R. A. Gerber
*Sandia Laboratories, Albuquerque, New Mexico* 87115
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Calculations of space-charge-controlled diffusion of electrons and positive ions in an isothermal afterglow are presented. In particular, the transition from electron-ion ambipolar diffusion to free diffusion of the electrons and ions is investigated. The results are in qualitative agreement with the experiment of Gerber, Gusinow, and Gerardo insofar as predicting the general features of the transition from electron-ion ambipolar diffusion to free diffusion. In addition, the results substantiate the general behavior implicitly predicted by the more elaborate steady-state calculations of Allis and Rose.

## INTRODUCTION

This work reports calculations of space-charge-controlled diffusion of electrons and positive ions in an isothermal afterglow. In particular, the transition from electron-ion ambipolar diffusion to free diffusion of the electrons and ions is investigated.

Allis and Rose[1] (hereafter designated as AR) laid the foundations for this work in 1954 when they calculated the ionization rate necessary to maintain a steady-state discharge. Their result of interest here is the effective diffusion coefficients of electrons and (implicitly) positive ions as functions of the electron density in the discharge. In their conclusions it was pointed out that if the ionization frequency in a steady-state discharge is associated with the electron loss rate in an afterglow, then the aforementioned effective diffusion coefficient should describe the electron loss rate in an afterglow.

To date there have been several attempts to test the AR theory.[2,3] These did not give quantitative agreement with theory. However, considering that the AR results were for hydrogen (in the steady state) and the experiments used helium[2] and neon[3] this is not surprising.

There have been many experiments in which a mass analyzer has been used to measure the ion decay rate (inferred from the ion wall current) during an afterglow. In this work we are concerned primarily with the experiments of Gerber *et al.*[4]