

Reduction Method and Distortion Potentials for Many-Particle Scattering Equations*

Yukap Hahn^{†‡} and K. M. Watson

*Physics Department and Lawrence Berkeley Laboratory, University of California,
Berkeley, California 94720*

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A general procedure is formulated by which the hierarchy of many-particle scattering equations of the Faddeev types can be reduced systematically to matrix equations of lower dimensions. As illustrations, such reduced sets are derived explicitly for the three-particle system by rearranging different components of the total Green's function. The resulting equations iterate with connected kernels *only* in the explicit channels. Using the reduction method, the connection between the various versions of scattering equations derived earlier have been exhibited. The effect of implicit channels may be taken into account noniteratively using the channel projection operators. It is also shown that an arbitrary set of distortion potentials may be introduced for the purpose of minimizing the coupling between the rearrangement channels, thus improving the convergence of the multiple-scattering series.

I. INTRODUCTION

Much effort has been made¹⁻³ in recent years to apply the Faddeev equations⁴ (FE) to actual physical three-particle systems in atomic and nuclear problems, with moderate success in some cases. Although the FE provide a mathematically consistent description of the three-particle scattering problems,³⁻⁶ it has become increasingly clear that they are often not the most convenient set of equations to apply. Some of the practical difficulties are (i) the absence of any *direct* distortion potential terms; the distortion effect is generated solely by the coupling to other rearrangement channels. This seems to place too much emphasis on the coupling terms. (ii) The presence of three-coupled equations, each with infinite sets of inelastic channels, makes actual applications extremely difficult, although the set treats all three rearrangement channels symmetrically. For systems with more than three particles, the set of equations is almost impossible to solve. (iii) The convergence of the final amplitudes, as the input two-particle amplitudes are improved, is very slow in many cases. This is of course directly related to the problems (i) and (ii).

In the present paper, we consider the question of what are the most convenient sets of equations to solve by relaxing the mathematical rigor of the original FE and obtaining a simpler theory which can be more readily applied to many-particle elastic, inelastic, and rearrangement processes. If we denote the initial and final channels of interest to be *explicit* and all the rest of the channels as *implicit*, then the reduced matrix equations that we obtain are such that the iteration kernels are compact *only* for the explicit channels. This is in contrast with the FE which provide connected kernels in *all* the rearrangement channels. The dis-

cussion below is based on the assumption that, when the implicit channels are treated by some noniterative approximations, the loss of rigor for the implicit channels then may not be serious.

A brief summary of the result of Faddeev is given in Sec. II, and we define notations. Several crucial steps in the derivation of the FE are pointed out, which will be useful later on. In particular, we note certain arbitrariness in the separation of the total Green's function and the scattering functions into several components, and this feature is used to reduce the FE into sets of two-coupled equations for particular explicit channels. We obtain in Sec. III the sets which are symmetric as well as asymmetric in the channel labels. One of the reduced forms turns out to be identical to that obtained earlier by Watson^{7,8} and also by Lovelace.⁶ In order to improve further the applicability of the equations, we consider ways to introduce distortion potentials into the FE and the reduced sets so as to minimize the coupling effect among the rearrangement channels. This is described in some detail in Sec. IV, again following the original procedure of decomposing the total Green's function, as has been done in Secs. II and III. The treatment of the implicit channels using the channel projection operators is also briefly discussed.⁹ Finally, a general reduction method is outlined in Sec. V, which allows one to immediately generalize the preceding results to systems involving more than three particles in a very trivial way.

Throughout the paper, we formulate the problem in terms of pairwise potentials and Green's functions in configuration space. Description of the problem and various possible formulations in terms of the differential scattering equations and scattering wave functions seems to be more direct and simple. It would, of course, involve a straightforward algebra in rewriting our result in the

forms of integral equations and two-particle amplitudes. However, we see no apparent practical advantage in doing so, at least at this stage of the treatment.⁸ A sample derivation is given in the Appendix as an illustration.

Thus, the procedure outlined above may circumvent some of the difficulties (i)–(iii) and facilitate applications of the Faddeev-Watson-type equations. A preliminary study to determine the potentials Y_1 and Y_2 of (4.9) in the case of high-energy proton-hydrogen charge-exchange collision is being carried out,¹⁰ and the result will be reported on elsewhere. The main result of the paper is contained in (4.9), (4.10), (4.11), and (4.17) for the breakup reactions.

II. FADDEEV EQUATIONS

We briefly summarize the results of Faddeev⁴ which are relevant to our discussion and point out several essential steps in the derivation of the FE which will be useful later on. For definiteness, we consider the elastic scattering of the particle 1 by a bound state of (2+3), with the coupling to other channels 2+(1+3) and 3+(1+2) taken into account explicitly. The elastic amplitude is given by

$$f_{e1} = (\Phi_1 | V_1 | \Psi_1), \quad (2.1)$$

where

$$\begin{aligned} H &= H_i + V_i, \quad i = 1, 2, 3 = \text{channel labels} \\ (H_i - E)\Phi_i &= 0, \end{aligned} \quad (2.2)$$

with

$$H_i = H_0 + V_{jk}, \quad V_i = V_{ij} + V_{ik} \quad (ijk \text{ cyclic}). \quad (2.3)$$

The total wave function is

$$\Psi_1 = \Phi_1 + G V_1 \Phi_1, \quad (2.4)$$

where

$$\begin{aligned} G^{(+)} &= (E^{(+)} - H)^{-1} = G_0 + G_0 V G, \quad G_0^{(+)} = (E^{(+)} - H_0)^{-1}, \\ V &\equiv V_0 = V_{12} + V_{13} + V_{23} = H - H_0. \end{aligned} \quad (2.5)$$

In order to simplify notations, we will drop the (+) in G 's and E without loss of clarity. Following Faddeev, we set

$$G = G_0 + \sum_{i=1}^3 G^{(i)}, \quad (2.6)$$

with

$$G^{(i)} \equiv G_0 V_{jk} G \quad (ijk \text{ cyclic}). \quad (2.7)$$

Substituting (2.6) into (2.7) and rearranging the terms, we obtain

$$\begin{aligned} G^{(i)} &= G_0 V_{jk} G_0 + G_0 V_{jk} G^{(i)} + G_0 V_{jk} (G^{(j)} + G^{(k)}) \\ &= G_i V_{jk} G_0 + G_i V_{jk} (G^{(j)} + G^{(k)}), \end{aligned} \quad (2.8)$$

where

$$G_i = (E^{(+)} - H_i)^{-1} = G_0 + G_0 V_{jk} G_i. \quad (2.9)$$

On the other hand, the form (2.6) for G immediately suggests that Ψ_1 be separated into components as

$$\Psi_1 = \Phi_1 + G_0 V_1 \Phi_1 + \sum_i G^{(i)} V_1 \Phi_1 \equiv \sum_i \Psi^{(i)}, \quad (2.10)$$

where

$$\begin{aligned} \Psi^{(1)} &= \Phi_1 + G^{(1)} V_1 \Phi_1, \\ \Psi^{(2)} &= G_0 V_{13} \Phi_1 + G^{(2)} V_1 \Phi_1, \\ \Psi^{(3)} &= G_0 V_{12} \Phi_1 + G^{(3)} V_1 \Phi_1. \end{aligned} \quad (2.11)$$

Combining (2.8) and (2.11), we can rewrite $\Psi^{(i)}$ in the forms

$$\Psi^{(1)} = \Phi_1 + G_1 V_{23} (\Psi^{(2)} + \Psi^{(3)}), \quad (2.12)$$

$$\Psi^{(2)} = G_2 V_{13} (\Psi^{(1)} + \Psi^{(3)}), \quad (2.13)$$

$$\Psi^{(3)} = G_3 V_{12} (\Psi^{(1)} + \Psi^{(2)}). \quad (2.14)$$

Thus, we finally have the FE in the differential form

$$(H_i - E) \Psi^{(i)} = -V_{jk} (\Psi^{(j)} + \Psi^{(k)}). \quad (2.15)$$

The above derivation is, of course, well known, and the salient properties of (2.15) have been clarified by many workers so that we do not repeat them here. However, we point out several essential steps involved in reaching the result (2.15). First of all, the identification (2.6) is a necessary step in the derivation of the FE, but such a separation of G is rather *arbitrary*. Different separations of G will naturally lead to different sets of coupled equations, and will be studied in more detail later. Secondly, (2.15) does not contain any distortion potentials on the left-hand side. Instead, all the scattering in the i th channel is brought about through its coupling to the j th and k th rearrangement channels. This apparent overemphasis of the coupling to rearrangement channels seems to be one of the major difficulties in actual applications. In fact, one rarely evaluates amplitudes beyond the second term in the multiple-scattering expansion, except in some special cases.

As the number of particles involved in the collision increases, the scattering equations become rapidly more complicated. The compactness of the iteration kernels of their solution would be of little practical help, because we cannot iterate too much. Before the iteration, we first have to *reduce* the equations to a manageable level of complexity. If we denote the initial and final states as *explicit* channels and the rest as *implicit*, then the simplest possibility may be to treat the explicit channels by a set of coupled equations, while the implicit channels are to be handled in some non-iterative way so that the noncompactness of the kernels for implicit channels would be the least of our problems. This point will be elaborated on in

Sec. III.

In the present paper, we will refer to the compactness of the iteration kernels and the connectedness properties interchangeably, with the value of $i\epsilon$ in G 's held at some small *nonzero* value throughout the calculation.³⁻⁵

Finally, it is noted that (2.15) and the result to follow can be rewritten in terms of two- and three-particle transition operators, as usually has been done.¹¹ However, we express our result in the form of coupled differential equations in order to avoid sometimes tedious but straightforward manipulations. The differential form is especially convenient for a configuration-space treatment of the problem, and also brings out clearly the structures involved.

III. REDUCED FADDEEV EQUATIONS

In this section, we consider the various possibilities of reducing the FE (2.15) to sets of two-coupled equations. We can proceed either with (2.15) and eliminate one of the channels, or go back to (2.6) and (2.10). Both will be considered.

A. Direct Reductions

We start with (2.15) and eliminate, e. g., the third channel as in the reaction $1 + (3 + 2) \rightarrow (1 + 3) + 2$. Using (2.14), we have

$$(H_1 - E) \Psi^{(1)} = -V_{23} \Psi^{(2)} - V_{23} G_3 V_{12} \Psi^{(1)} - V_{23} G_3 V_{12} \Psi^{(2)}, \quad (3.1)$$

$$(H_2 - E) \Psi^{(2)} = -V_{13} \Psi^{(1)} - V_{13} G_3 V_{12} \Psi^{(1)} - V_{13} G_3 V_{12} \Psi^{(2)}. \quad (3.2)$$

Therefore, we may define the new functions

$$\tilde{\Psi}^{(1)} \equiv \Psi^{(1)} + G_3 V_{12} \Psi^{(1)}, \quad \tilde{\Psi}^{(2)} \equiv \Psi^{(2)} + G_3 V_{12} \Psi^{(2)} \quad (3.3)$$

and obtain

$$\begin{aligned} [H_1 + V_{12} - E] \tilde{\Psi}^{(1)} &= -V_{23} \tilde{\Psi}^{(2)}, \\ [H_2 + V_{12} - E] \tilde{\Psi}^{(2)} &= -V_{13} \tilde{\Psi}^{(1)}, \end{aligned} \quad (3.4)$$

with

$$\Psi_1 = \tilde{\Psi}^{(1)} + \tilde{\Psi}^{(2)}. \quad (3.5)$$

Obviously, there are other possible choices for $\tilde{\Psi}^{(i)}$ which lead to different forms of scattering equations as long as (3.5) is maintained—such as the form $\tilde{\Psi}^{(i)} = \Psi^{(i)} + G_3 V_{12} \Psi^{(j)}$. On the other hand, it is important to note that (3.4) can also be obtained at an earlier stage by redefining the Green's function G . Since (2.6) was a rather arbitrary choice, we may set with equal validity

$$\begin{aligned} G &= G_3 + G_3 V_3 G = G_3 + G_3 V_{31} G + G_3 V_{32} G \\ &\equiv G_3 + \tilde{G}^{(1)} + \tilde{G}^{(2)}. \end{aligned} \quad (3.6)$$

Now, following the similar steps that led to (2.15), we recover (3.4). In the following discussion, we prefer this latter procedure of redefining G and Ψ_1 .

When (3.4) is iterated, we immediately find that the iteration kernels for the explicit channels 1 and 2 are compact, i. e., connected in the form $g_1 V_{13} g_2 V_{23}$, while the channel 3 component is not connected, with the kernels $G_2 V_{12}$ and $G_1 V_{12}$. This is the price we pay when (2.15) is simplified to (3.4) $[g_i = (E^{(+)} - H_0 - V_i)^{-1}]$. The basic assumption of the present approach is that the effect of non-compact kernels for the implicit channels may not be serious as long as we avoid the iterative solutions for this part.

B. Multiple-Scattering Equation

Instead of the rearrangement scattering considered in Sec. IIIA, we now turn to the elastic scattering $1 + (2 + 3) \rightarrow 1 + (2 + 3)$ and try to eliminate the channel 1. For this purpose, we write

$$G = G_1 + G_1 V_{12} G + G_1 V_{13} G \equiv G_1 + G^{(2)A} + G^{(3)A}, \quad (3.7)$$

where

$$\begin{aligned} G^{(2)A} &\equiv G_1 V_{12} G = g_2 V_{12} G_1 + g_2 V_{12} G^{(3)A}, \\ G^{(3)A} &= g_2 V_{13} G_1 + g_3 V_{13} G^{(2)A}, \end{aligned} \quad (3.8)$$

with

$$\begin{aligned} g_2 &= (E^{(+)} - H_0 - V_2)^{-1} = G_1 + G_1 V_{12} g_2, \\ g_3 &= (E^{(+)} - H_0 - V_3)^{-1} = G_1 + G_1 V_{13} g_3. \end{aligned} \quad (3.9)$$

Now, Ψ_1 may be written in the form

$$\Psi_1 = -\Phi_1 + \Psi^{(2)A} + \Psi^{(3)A}, \quad (3.10)$$

with

$$\begin{aligned} \Psi^{(2)A} &\equiv \Phi_1 + G_1 V_{13} \Phi_1 + G^{(3)A} V_1 \Phi_1 \\ &= \Phi_1 + g_3 V_{13} \Psi^{(3)A}, \end{aligned} \quad (3.11)$$

$$\Psi^{(3)A} = \Phi_1 + G_1 V_{12} \Phi_1 + G^{(2)A} V_1 \Phi_1 = \Phi_1 + g_2 V_{12} \Psi^{(2)A}.$$

Therefore, we get

$$\begin{aligned} (H_2 + V_{23} - E) (\Psi^{(2)A} - \Phi_1) &= -V_{13} \Psi^{(3)A}, \\ (H_3 + V_{23} - E) (\Psi^{(3)A} - \Phi_1) &= -V_{12} \Psi^{(2)A}. \end{aligned} \quad (3.12)$$

The set of equations (3.12) may be compared with that derived by Watson⁷ in his multiple-scattering theory. It is given in our notation as

$$\Psi_1 = \Phi_1 + G_1 t_2 \chi_2 + G_1 t_3 \chi_3, \quad (3.13)$$

where

$$\chi_2 = \Phi_1 + G_1 t_3 \chi_3, \quad \chi_3 = \Phi_1 + G_1 t_2 \chi_2 \quad (3.14)$$

and

$$t_2 = V_{12} + V_{12} G_1 t_2, \quad t_3 = V_{13} + V_{13} G_1 t_3. \quad (3.15)$$

Noting that

$$G_1 t_2 = g_2 V_{12} \text{ and } G_1 t_3 = g_3 V_{13}, \quad (3.16)$$

we immediately have

$$\Psi_1 = \Phi_1 + g_2 V_{12} \chi_2 + g_3 V_{13} \chi_3 = -\Phi_1 + \chi_2 + \chi_3, \quad (3.17)$$

with the identifications

$$\chi_2 = \Psi^{(2)A}, \quad \chi_3 = \Psi^{(3)A}; \quad (3.18)$$

thus, (3.14) is identical to (3.12). A similar form was also given later by Lovelace.⁶

C. Alternative Reductions

Evidently (3.6) and (3.7) are not the only possible choices of G , and we study several other possibilities here.

(i) If we write G as

$$\begin{aligned} G &= G_0 + G_0 V G = G_0 + G_0 V_1 G + G_0 V_{23} G \\ &\equiv G_0 + G^{(1)B} + G^{(2)B}, \end{aligned} \quad (3.19)$$

then we obtain

$$\begin{aligned} G^{(1)B} &= G_1 V_{23} G_0 + G_1 V_{23} G^{(2)B}, \\ G^{(2)B} &= g_1 V_1 G_0 + g_1 V_1 G^{(1)B}. \end{aligned} \quad (3.20)$$

Now let us define

$$\Psi_1 = \Phi_1 + G V_1 \Phi_1 \equiv \Psi^{(1)B} + \Psi^{(2)B}, \quad (3.21)$$

with

$$\begin{aligned} \Psi^{(1)B} &\equiv \Phi_1 + G^{(1)B} V_1 \Phi_1 = \Phi_1 + G_1 V_{23} \Psi^{(2)B}, \\ \Psi^{(2)B} &= G_0 V_1 \Phi_1 + G^{(2)B} V_1 \Phi_1 = g_1 V_1 \Psi^{(1)B}, \end{aligned} \quad (3.22)$$

and we obtain

$$\begin{aligned} (H_1 - E) \Psi^{(1)B} &= -V_{23} \Psi^{(2)B}, \\ (H_2 + V_{12} - E) \Psi^{(2)B} &= -V_1 \Psi^{(1)B}. \end{aligned} \quad (3.23)$$

(ii) Instead of (3.19), if we take

$$G = G_0 + G_0 V_2 G + G_0 V_{13} G \equiv G_0 + G^{(1)C} + G^{(2)C}, \quad (3.24)$$

the resulting equations would be

$$\begin{aligned} (H_1 + V_{12} - E) \Psi^{(1)C} &= -V_2 \Psi^{(2)C}, \\ (H_2 - E) \Psi^{(2)C} &= -V_{13} \Psi^{(1)C}. \end{aligned} \quad (3.25)$$

(iii) A slightly different form of Eqs. (3.12) can be obtained in which V_{23} does not appear on the left-hand sides. For this purpose, we write

$$\begin{aligned} G &= G_1 + G_1 V_2 G + G_1 V_3 G - 2G_1 V_{23} G \\ &= \bar{G}_1 + G^{(2)D} + G^{(3)D}, \end{aligned} \quad (3.26)$$

where

$$\bar{G}_1 = (E^{(+)} - H_0 + V_{23})^{-1} \neq G_1 = (E^{(+)} - H_0 - V_{23})^{-1}.$$

Then, with

$$\Psi_1 = \Phi_1 + \Psi^{(2)D} + \Psi^{(3)D}, \quad (3.27)$$

where

$$\begin{aligned} \Psi^{(2)D} &= \bar{G}_1 V_{13} \Phi_1 + G^{(2)D} V_1 \Phi_1, \\ \Psi^{(3)D} &= \bar{G}_1 V_{12} \Phi_1 + G^{(3)D} V_1 \Phi_1, \end{aligned} \quad (3.28)$$

we obtain

$$\begin{aligned} (H_2 - E) \Psi^{(2)D} &= -V_3 \Psi^{(3)D} - V_{13} \Phi_1, \\ (H_3 - E) \Psi^{(3)D} &= -V_2 \Psi^{(2)D} - V_{12} \Phi_1, \end{aligned} \quad (3.29)$$

which would be useful for elastic scattering.

(iv) Returning to the rearrangement process $1 + (2 + 3) \rightarrow 2 + (1 + 3)$, we try to derive a more symmetric form than (3.23) and (3.25), in the process of eliminating the third channel. For this purpose, we write

$$\begin{aligned} G &= G_0 + G_0 V_1 G + G_0 V_2 G - G_0 V_{12} G \\ &= \bar{G}_3 + \bar{G}_3 V_1 G + \bar{G}_3 V_2 G \equiv \bar{G}_3 + G^{(1)F} + G^{(2)F}, \end{aligned} \quad (3.30)$$

where

$$\bar{G}_3 = (E^{(+)} - H_0 + V_{12})^{-1} \neq G_3.$$

$G^{(1)F}$ can be written as

$$\begin{aligned} G^{(1)F} &\equiv \bar{G}_3 V_2 G = G_1 V_2 \bar{G}_3 + G_1 V_2 G^{(2)F}, \\ G^{(2)F} &\equiv \bar{G}_3 V_1 G = G_2 V_1 \bar{G}_3 + G_2 V_1 G^{(1)F}. \end{aligned} \quad (3.31)$$

The total wave function is separated into two parts as

$$\Psi_1 = \Psi^{(1)F} + \Psi^{(2)F}, \quad (3.32)$$

where

$$\begin{aligned} \Psi^{(1)F} &\equiv \Phi_1 + G^{(1)F} V_1 \Phi_1 = \Phi_1 + G_1 V_2 \Psi^{(2)F}, \\ \Psi^{(2)F} &\equiv \bar{G}_3 V_1 \Phi_1 + G^{(2)F} V_1 \Phi_1 = G_2 V_1 \Psi^{(1)F}. \end{aligned} \quad (3.33)$$

Thus, we finally have the desired coupled equations⁹

$$(H_1 - E) \Psi^{(1)F} = -V_2 \Psi^{(2)F}, \quad (H_2 - E) \Psi^{(2)F} = -V_1 \Psi^{(1)F}. \quad (3.34)$$

It is thus increasingly clear from these examples that certain regular patterns are emerging in the way of reducing the original matrix equations. The arbitrariness in the choice of G and Ψ can lead to a variety of sets of equations. In particular, we advocate the use of (3.29) and (3.34) for the elastic and rearrangement collisions, respectively. In the Appendix, we give the T -matrix version of (3.34) as an illustration. All the other systems of equations presented here can be converted in a similar fashion.

D. Breakup Channels

We briefly consider the reaction $1 + (2 + 3) \rightarrow 1 + (2 + 3)'$ in which the final state is described asymptotically in terms of the eigenstates of $H_0 = H - V$, rather than the distorted three-particle continuum states of H_i , for example. The latter possibility has the attractive feature that the initial and final

states are orthogonal, but we do not consider it here. With the decomposition of G as

$$G = G_0 + G_0 V G = G_0 + G_0 V G + G_0 V_1 G - G_0 V_1 G \\ = G'_0 + G'_0 V G + G'_0 V_1 G \equiv G'_0 + G^{(0)H} + G^{(1)H}, \quad (3.35)$$

where

$$G'_0 = G_0 - G_0 V_1 G'_0, \quad (3.36)$$

we have

$$G^{(0)H} = G_0 V_1 G'_0 + G_0 V_1 G^{(1)H}, \\ G^{(1)H} = G_1 V G'_0 + G_1 V G^{(0)H}. \quad (3.37)$$

Thus, we define

$$\Psi = \Phi^{(1)H} + \Psi^{(0)H}$$

with

$$\Psi^{(1)H} = \Phi_1 + G^{(1)H} V_1 \Phi_1, \\ \Psi^{(0)H} = G'_0 V_1 \Phi_1 + G^{(0)H} V_1 \Phi_1, \quad (3.38)$$

we get the coupled equations

$$(H_1 - E) \Psi^{(1)H} = -V \Psi^{(0)H}, \quad (H_0 - E) \Psi^{(0)H} = -V_1 \Psi^{(1)H}. \quad (3.39)$$

It is especially useful to rewrite (3.39) in the uncoupled form

$$[H_1 + V G_0 V_1 - E] \Psi^{(1)H} = 0, \quad \Psi^{(0)H} = G_0 V_1 \Psi^{(1)H} \quad (3.40)$$

or

$$[H_0 + V_1 G_1 V - E] \Psi^{(0)H} = -V_1 \Phi_1. \quad (3.41)$$

The form (3.40) would be especially useful in setting up a coupled-equations procedure to estimate $\Psi^{(1)}$, and evaluate the amplitude

$$f_{01} = (\Phi_0 | V | \Psi_1). \quad (3.42)$$

E. Connectedness Properties

In their reduced forms derived in this section, the effect of the third channel is not treated in a mathematically rigorous fashion at the same level as with the original FE (2.15). The binding potential V_{12} for the third channel appears, e.g., in (3.34), only in the coupling terms V_1 and V_2 on the right-hand side. However, as in (3.3), $\Psi^{(1)}$ and $\Psi^{(2)}$ of (3.34) should have components of the third channel if they are open. However, in so far as its effect on the channels 1 and 2 are concerned, (3.34) may be sufficient to determine the correct $\Psi^{(i)}$. This point will become even more plausible by the introduction of projection operators, as will be shown in Sec. IV.

Thus, the form (3.34) will iterate with compact kernels in so far as the explicit channels 1 and 2 are concerned, while the rest of the implicit channels will not exhibit such compactness property. How-

ever, we will partially remedy this situation by the use of the channel projection operators. (We have used the compactness and connectedness interchangeably in the above discussion, with the convention that the scattering energy is kept complex with small $+i\epsilon$ throughout the calculation.)

IV. DISTORTION POTENTIALS

As discussed earlier, the set of equations (2.15) does not contain any direct distortion potentials in the left-hand sides. Consequently, the scattering of (2+3) and 1, for example, is brought about by its coupling to the other two rearrangement channels. This in turn implies that we have to iterate at least once to get the first distortion effect. We believe that this feature is partially responsible for the slow convergence of the iteration series in some cases and makes the application of (2.15) difficult. The situation is the same with (3.29) and (3.34). On the other hand, the sets (3.12), (3.23), and (3.25) contain spurious distortions on the left-hand sides which have to be cancelled by the higher-order terms; the problem becomes especially serious if the two-particle potentials are either very long ranged or very singular. It would, therefore, be extremely desirable from practical point of view to reformulate the theory of Secs. II and III such that some distortion potentials could be introduced. This will not only minimize the dependence of the scattering on the coupling to rearrangement channels, but would also improve the convergence property of the multiple-scattering expansion.

For simplicity of discussion, we consider explicitly (3.23) and show how the distortion potentials Y could be introduced in the left-hand side of the equations. The procedure we follow is the same as that used in Secs. II and III, i.e., to go back to Ψ_1 and redefine G and $\Psi^{(i)}$ in some suitable fashion.

Thus, in place of (3.19), we set

$$G = G_0 + G_0(V_1 - Y_1)G + G_0(V_{23} - Y_2^3)G \\ + G_0 Y_1 G + G_0 Y_2^3 G \\ = \tilde{G}_0 + \tilde{G}_0(V_1 - Y_1)G + \tilde{G}_0(V_{23} - Y_2^3)G \\ \equiv \tilde{G}_0 + G^{(2)J} + G^{(1)J}, \quad (4.1)$$

where

$$\tilde{G}_0 = (E^{(+)} - H_0 - Y_1 - Y_2^3)^{-1}.$$

The $G^{(i)J}$ in (4.1) may be rewritten in the form

$$G^{(1)J} \equiv \tilde{G}_0(V_{23} - Y_2^3)G \\ = \tilde{G}_1(V_{23} - Y_2^3)\tilde{G}_0 + \tilde{G}_1(V_{23} - Y_2^3)G^{(2)J}, \quad (4.2)$$

$$G^{(2)J} \equiv \tilde{G}_0(V_1 - Y_1)G \\ = \tilde{G}_2(V_1 - Y_1)\tilde{G}_0 + \tilde{G}_2(V_1 - Y_1)G^{(1)J}, \quad (4.3)$$

where

$$\begin{aligned}\tilde{G}_1 &= (E^{(+)} - H_0 - V_{23} - Y_1)^{-1}, \\ \tilde{G}_2 &= (E^{(+)} - H_0 - V_1 - Y_{23})^{-1}.\end{aligned}\quad (4.4)$$

By defining $\Psi^{(i)J}$ in terms of $G^{(i)J}$ and rearranging the terms, we arrive at the equations

$$\begin{aligned}(H_1 + Y_1 - E) \Psi^{(1)J} &= - (V_{23} - Y_2^3) \Psi^{(2)J}, \\ (H_2 + V_{12} + Y_2^3 - E) \Psi^{(2)J} &= - (V_1 - Y_1) \Psi^{(1)J},\end{aligned}\quad (4.5)$$

which is the desired modification of (3.19). The distortion potentials Y_1 and Y_2^3 are completely arbitrary so far. Obviously, the compactness of the iteration kernels in the particular explicit channels depends on the behavior of Y_1 and Y_2^3 in the asymptotic channel region.

Various different choices of Y reproduce the result obtained earlier; for example, the choices

$$Y_1 = 0 \text{ and } Y_2^3 = -V_{12} \quad (4.6)$$

reduce (4.5) to (3.34), while

$$Y_1 = V_{12} \text{ and } Y_2^3 = 0 \quad (4.7)$$

give (3.4). Furthermore, the choices

$$Y_1 = Y_1 \text{ and } Y_2^3 = Y_2 - V_{12} \quad (4.8)$$

lead to a new form⁹ of (3.34),

$$\begin{aligned}(H_1 + Y_1 - E) \Psi^{(1)K} &= - (V_2 - Y_2) \Psi^{(2)K}, \\ (H_2 + Y_2 - E) \Psi^{(2)K} &= - (V_1 - Y_1) \Psi^{(1)K}.\end{aligned}\quad (4.9)$$

This is the most general set of equations for the processes $1 + (2+3) \leftrightarrow (1+3) + 2$, such as e^+H , nd , and pd scatterings, and also e^+H . For the elastic scattering in the channel 1, we can modify (3.29) into a form

$$\begin{aligned}(H_2 + Y_2 - E) \Psi^{(2)L} &= - (V_3 - Y_2) \Psi^{(3)L} - V_{13} \Phi_1, \\ (H_3 + Y_3 - E) \Psi^{(3)L} &= - (V_2 - Y_2) \Psi^{(2)L} - V_{12} \Phi_1.\end{aligned}\quad (4.10)$$

The form (4.10) may be convenient for the systems such as πd , in which the πn and πp channels are explicitly considered. Similar distortion-potential modifications can be introduced also to the Watson equations (3.12) and to (3.23) and (3.25). In particular, the original FE can be modified to a form

$$(H_i + Y_i - E) \Psi^{(i)M} = - (V_{jk} - Y_j^k) (\Psi^{(j)M} + \Psi^{(k)M}), \quad (4.11)$$

where

$$Y_i = Y_i' + Y_i^k \quad (ijk \text{ cyclic}). \quad (4.12)$$

In (4.11), we have to require that all Y_i' should vanish in *all* three channels so that the compactness property is preserved. Similarly, in (4.9), Y_1 and Y_2 are to be chosen such that they decay both in the explicit channels 1 and 2. For the set (4.10), we are again to choose Y_2 and Y_3 such that they decay

both in the implicit channels 2 and 3. Furthermore, their form can be chosen such that the coupling to the other rearrangement channels may be minimized. There are several approximate procedures to obtain optimal determination of the Y 's, and this has been discussed previously,⁹ to which we refer for the details.

Now, return to (4.9) and consider briefly how we are going to solve the problem for $\Psi^{(1)K}$ and $\Psi^{(2)K}$. Incidentally, (4.9) is not restricted to the three-particle systems, but H_1 and H_2 can accommodate more complex clusters so long as the problem is to be treated in terms of *two* explicit rearrangement channels. Therefore, in general, attempts to obtain an iterative solution of (4.9) will result in the noncompact kernels in all possible implicit channels of the problem. A powerful technique which at least partially circumvents this problem is the use of channel projection operators.

We define the projection operators P_i and Q_i , $i = 1$ and 2 , such that

$$[P_i, H_i] = 0, \quad P_i^2 = P_i^\dagger = P_i, \quad (4.13)$$

$$Q_i = 1_i - P_i, \quad Q_i P_i = 0;$$

but, in general,

$$[P_i, P_j] \neq 0, \quad i \neq j. \quad (4.14)$$

Then, (4.9) can be rewritten in the forms

$$\begin{aligned}P_i(H_i + Y_i - E)P_i \Psi^{(i)K} + P_i(V_j - Y_j)P_j \Psi^{(j)K} \\ = - P_i Y_i Q_i \Psi^{(i)K} - P_i(V_j - Y_j)Q_j \Psi^{(j)K}\end{aligned}\quad (4.15)$$

and

$$\begin{aligned}Q_i(H_i + Y_i - E)Q_i \Psi^{(i)K} + Q_i(V_j - Y_j)Q_j \Psi^{(j)K} \\ = - Q_i Y_i P_i \Psi^{(i)K} - Q_i(V_j - Y_j)P_j \Psi^{(j)K}.\end{aligned}\quad (4.16)$$

For channels with two clusters, we may construct P_i in terms of the cluster wave functions, in which case, all the quantities in (4.15) are *completely connected in all channels*. Unfortunately, this is not the case with (4.16), and the problem is then to make a careful treatment of (4.16) for $Q_i \Psi^{(i)}$. Here, the usefulness of Y_i and P_i become apparent: If P_i contains the initial channel, then the "reasonable" choice of P_i and Y_i could make the right-hand side of (4.15) small. For such a choice, we simply have to solve a well-defined set of coupled equations to obtain $P_i \Psi^{(i)K}$ and the desired amplitudes.

In an exactly analogous way, we can introduce the appropriate channel projection operators into (4.10) for the elastic scattering and also into the FE in the modified form (4.11). The coupled equations for the breakup reactions (3.39) can also be modified to include distortions. Thus, we have, e.g.,

$$\begin{aligned} (H_1 + Y_1 - E)\Psi^{(1)R} &= -(V - Y_0)\Psi^{(0)R}, \\ (H_0 + Y_0 - E)\Psi^{(0)R} &= -(V_1 - Y_1)\Psi^{(1)R}. \end{aligned} \quad (4.17)$$

Finally, we should point out the possibility of reducing the effect of "spurious" long-range coupling to the breakup channels in (4.11). As stressed by Noyes,¹ for some E above the breakup threshold, some components of $\Psi^{(j)}$ and $\Psi^{(k)}$ may have outgoing waves such that the coupling terms decay only as R_i^{-1} , as their respective asymptotic regions are approached. This in turn makes the configuration-space treatment of the FE more difficult.

First of all, we stress the fact that such a spurious long-range effect *does not* appear as long as the total energy E is below the lowest breakup threshold. Secondly, when Y_i and Y_j^k are chosen such that the coupling terms of the form $P_i(V_{jk} - V_j^k) \times (Q_j\Psi^{(j)} + Q_k\Psi^{(k)})$ are made small, the spurious effect may be minimized sufficiently. However, there are no satisfactory ways to eliminate such an effect completely, except perhaps by choosing a very peculiar form of Y_j^k in the asymptotic region.

V. GENERAL REDUCTION METHOD

Based on the detailed discussions given in Secs. III and IV, we can now generalize the procedure to reduce the N -particle scattering equations of the Faddeev- and multiple-scattering type to sets of matrix equations of small dimensions and also to introduce some effective distortion potentials in a "consistent" way. Such method was presented earlier⁹ for the rearrangement collisions from a more intuitive point of view, based on the ansatz that the total Ψ can always be written in several components, each of which satisfies some parts of the original scattering equations. Since we have shown above that such ansatz is valid and the result derived from it is the same as those presented here, we recapitulate the reduction method. It makes the derivation of all the equations completely trivial, and there is no need of going back to G and readjusting $G^{(i)}$ and $\Psi^{(i)}$.

Consider the N -particle scattering system which may be described by the matrix equations of the Faddeev type with the dimensions $\frac{1}{2}N(N-1)$. Instead, we try to construct a simpler set of equations of dimension I , where I is a small number representing the explicit channels of interest here. The question we are asking is how to construct the set of equations such that its iteration kernels possess compactness properties in all I channels and also contain distortion potentials Y_c , $c = 1, 2, \dots, I$. The channels are taken here to mean those asymptotic states which are eigenstates of the distinct channel Hamiltonian H_c . We denote the $I \times I$ matrix equation as

$$D\Psi = -B\Phi, \quad (5.1)$$

with

$$\Psi_i = \sum_{c=1}^I \Psi^{(c)} + \bar{\delta}_{Ii} \Phi_i, \quad (5.2)$$

where D and B are matrix operators, while Ψ is a vector, consisting of elements $\Psi^{(c)}$, $\bar{\delta}_{Ii} = 1 - \delta_{Ii}$, and $\delta_{Ii} = 0$ if $I \supset i$ and $\delta_{Ii} = 1$ if $I \ni i$. The elements $D_{cc'}$ and $B_{cc'}$ are obtained in the following way: (a) The diagonal terms D_{cc} are given by

$$D_{cc} = H_c + Y_c - E, \quad (5.3)$$

where

$$Y_c = \sum_{c'=1, (c' \neq c)}^I Y_{c'}^{c'}. \quad (5.4)$$

$Y_{c'}^{c'}$ in (5.4) are the distortion potentials which vanish in *all* the I asymptotic channel regions. (b) For each column c , the off-diagonal elements $D_{c'c}$ can be chosen such that

$$\sum_{c' \neq c}^I D_{c'c} = H - H_c - Y_c. \quad (5.5)$$

Other than this, $D_{c'c}$ are completely arbitrary, including some zeros. Note that (5.4) and (5.5) are designed so that

$$\sum_{c'=1}^I D_{c'c} = H - E \quad (5.6)$$

for all $c = 1, 2, \dots, I$. (c) Finally, $B_{c'c}$ are chosen as follows:

$$\begin{aligned} B_{cc'} &= 0 \quad \text{for } c \neq c', \\ B_{cc} &= (H_c - H_0) \bar{\delta}_{Ii}, \end{aligned} \quad (5.7)$$

where H_0 is the free- N -particle Hamiltonian. The entire B term is designed to incorporate the case such as (3.12) in which the initial channel component is eliminated from the coupled equations. (d) If we want to reduce the $I \times I$ equations obtained in the above sequence to $(I-1) \times (I-1)$ dimensional equations, it simply requires the addition of the elements $D_{c'c}$ of the row being eliminated to any of the other rows *columnwise*, and then the dropping of that particular column altogether. This will preserve the sum (5.6).

The explicit forms of the reduced matrix equations presented in Secs. III-V provide examples of this reduction procedure.⁹

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APPENDIX: SCATTERING OPERATORS FOR (3.34)

The various systems of coupled equations derived in this paper can be converted to sets of amplitude equations. As an example we consider here ex-

plicitly Eq. (3.34) for the elastic scattering. The amplitude is given by

$$f_{11} = f_{e1} = (\Phi_1 | V_1 | \Psi_1) \equiv (\Phi_1 | T | \Phi_1), \quad (\text{A1})$$

where

$$T_{11} \Phi_1 \equiv T \Phi_1 = V_1 \Psi_1 = V_1 (\Psi^{(1)} + \Psi^{(2)}). \quad (\text{A2})$$

If we define $T^{(i)}$ by

$$T = T^{(1)} + T^{(2)} \quad (i = 1, 2), \quad (\text{A3})$$

with

$$T^{(i)} \Phi_1 = V_1 \Psi^{(i)} \quad (\text{A4})$$

and

$$\tilde{T}^{(2)} \Phi_1 = V_2 \Psi^{(2)}, \quad (\text{A5})$$

then we obtain a set of coupled equations

$$T^{(1)} = V_1 + V_1 G_1 \tilde{T}^{(2)}, \quad \tilde{T}^{(2)} = V_2 G_2 T^{(1)}, \quad (\text{A6})$$

and

$$T^{(2)} = V_1 G_2 T^{(1)}. \quad (\text{A7})$$

The potentials V_1 and V_2 in (A6) and (A7) can be eliminated in a trivial way by writing

$$\begin{aligned} t_1 &= V_1 + V_1 G_1 t_1 \equiv V_1 \Omega_1, \\ t_2 &= V_2 + V_2 G_2 t_2 \equiv V_2 \Omega_2. \end{aligned} \quad (\text{A8})$$

Thus, (A6) and (A7) become, respectively,

$$\Omega_1 T^{(1)} = t_1 + t_1 G_1 \tilde{T}^{(2)}, \quad \Omega_2 \tilde{T}^{(2)} = t_2 G_2 T^{(1)}, \quad (\text{A9})$$

and

$$T^{(2)} = \Omega_1^{-1} t_1 G_2 T^{(1)}. \quad (\text{A10})$$

Since the off-shell t_i 's are involved in (A9) and (A10), we have not gained anything by rewriting (A6) and (A7) in the above forms: For given t_1 , t_2 , and G_0 , we can always construct G_1 , V_1 , and Ω_1 , with exactly the same amount of physical contents.

Obviously, we can further decompose t_1 into the set which involves T_i defined by

$$T_i = V_{jk} + V_{jk} G_0 T_i. \quad (\text{A11})$$

Thus, e. g., we find

$$t_1 = t_1^{(2)} + t_1^{(3)},$$

where

$$t_1^{(2)} = \tilde{T}_2 + \tilde{T}_2 G_1 t_1^{(3)}, \quad t_1^{(3)} = \tilde{T}_3 + \tilde{T}_3 G_1 t_1^{(2)}, \quad (\text{A12})$$

and

$$\tilde{T}_2 = T_2 + T_2 G_0 T_1 G_0 \tilde{T}_2, \quad \tilde{T}_3 = T_3 + T_3 G_0 T_1 G_0 \tilde{T}_3. \quad (\text{A13})$$

The procedure is then to solve first for \tilde{T}_i of (A13) and get $t_i^{(j)}$ of (A12). Then (A9) and (A10) will give $T^{(1)}$ and $T^{(2)}$.

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†On leave from the Physics Department, University of Connecticut, Storrs, Conn. 06268.

‡Participating guest.

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