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Bremsstrahlung Rate and Spectra from a Hot Gas ($Z = 1$)*

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An interpolation between the quadrupole and extreme relativistic electron-electron bremsstrahlung contribution to the emission rate and spectra from a Maxwell-Boltzmann electron gas is carried out. These results are then added to the "exact" electron-ion results already in the literature to give the total emission for the temperature region 1 keV $\leq kT_e \lesssim 0.3$ MeV. It is found that quadrupole corrections are sufficient until the temperatures reach $kT_e \simeq 100$ keV. At higher temperatures all multipoles must be included in order to describe the emission accurately.

I. INTRODUCTION

In a previous publication,[1] the quadrupole electron-electron (e-e) contribution to the bremsstrahlung spectrum of a hot Maxwell gas was given. It was found that sizable corrections to the short-wavelength portion of the spectrum occur when $kT_e \gtrsim 20$ keV. There also exists in the literature an integration of the Bethe-Heitler cross-section for electron-ion (e-i) bremsstrahlung over a relativistic Maxwell-Boltzmann distribution of electrons.[2] A simple formula[3] was given for including corrections to the well-known dipole spectrum. For e-i bremsstrahlung, successively higher-order multipoles will be down by a factor kT_e/mc^2 from the previous order, whereas the factor will be $(kT_e/mc^2)^2$ for e-e bremsstrahlung, since only the 2^{2n} ($n = 1, 2, \ldots$) poles contribute.

One of the important applications of this work is in the field of observational x-ray astronomy. It is standard practice, nowadays, to fit the spectrum of an x-ray source with a black body, thermal bremsstrahlung, or synchrotron model and thereby determine the electron temperature from the data. In this way a temperature of 8.1×10^7 °K (7 keV) was measured for SCO X-1[4] and a temperature of 3.5×10^8 °K (30 keV) was measured for CYG X-1.[5] Measurements also show a strong variation in intensity over periods ranging from fractions of a second to months.

In this paper, an interpolation between the quadrupole and extreme relativistic e-e bremsstrahlung rate and spectra is carried out.

In Sec. II the method of interpolation is discussed and graphs of the e-e spectra are given for several different temperatures. In Sec. III, the e-e and e-i contributions are added together to obtain the total bremsstrahlung spectra and rate over the temperature region 1 keV $\leq kT_e \lesssim 0.3$ MeV. Results are presented in the form of graphs and a table, and the e-e and e-i contributions are separately compared with the dipole result as well as the total emission.

II. ELECTRON-ELECTRON INTERPOLATION

The Born cross section for e-e bremsstrahlung is extremely complicated and has only recently been evaluated numerically at several angles and energies for the electron-positron case.[6] On the other hand, integrations of the quadrupole[1] and extreme-relativistic[7] cross sections over a Maxwell-Boltzmann distribution already exist in the literature. We choose to interpolate, at several fixed frequencies, between these two results, to obtain the spectra for all temperatures.

The quadrupole spectrum is given by[1]

$$P^{e\text{-}e}_{\text{quad}}(h\nu, kT_e)\,d\nu = (kT_e/mc^2)\,B(\lambda)\,P^{e\text{-}i}_{\text{dipole}}(h\nu, kT_e)\,d\nu\ , \quad (1)$$

where

$$\lambda = h\nu/kT_e$$

and

$$P^{e\text{-}i}_{\text{dipole}}(h\nu, kT_e)\,d\nu = \frac{16}{3}\left(\frac{2}{\pi m kT_e}\right)^{1/2} \alpha\, r_0^2 n_e^2 m c^2 \times e^{-\lambda/2} K_0(\lambda/2)\,\frac{d\nu}{\nu}\ . \quad (2)$$

$K_0(x)$ is the modified Bessel function of the second kind. In the above equations we have set $n_e = Z n_i$ and $Z = 1$. The function $B(\lambda)$ is plotted in Ref. 1. A fit for the function $B(\lambda)$ is given in the Appendix.

The extreme-relativistic spectrum is given by[7]

$$P^{e\text{-}e}_{\text{er}}(h\nu, kT_e)\,d\nu = 2\alpha\, r_0^2 n_e^2 c\,\frac{d\nu}{\nu}\, e^{-\lambda} \left\{\frac{28}{3} + 2\lambda + \frac{\lambda^2}{2} + 2\left(\frac{8}{3} + \frac{4}{3}\lambda + \lambda^2\right)\right.$$

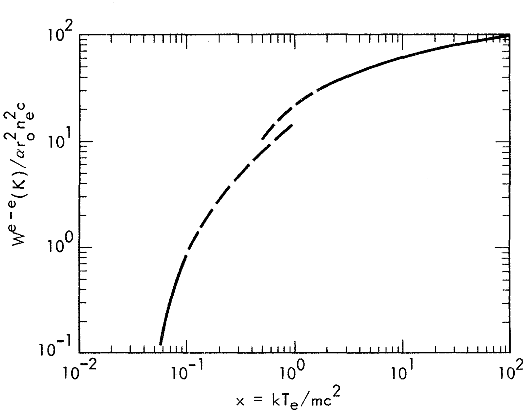

FIG. 1. $e-e$ emission rate at $K = 100$ keV.

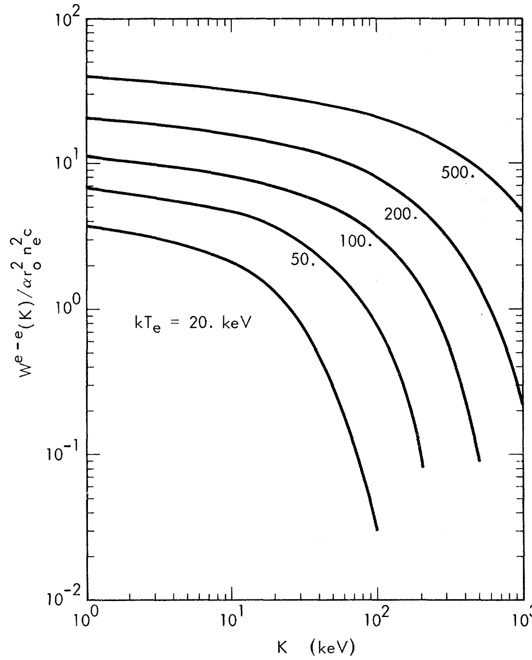

FIG. 2. e-e bremsstrahlung spectra.

$$\left.\times\left[\ln\left(\frac{2kT_e}{mc^2}\right) - 0.577\right] - e^{\lambda}\,\text{Ei}(-\lambda)\left(\frac{8}{3} - \frac{4}{3}\lambda + \lambda^2\right)\right\}\ , \quad (3)$$

where Ei$(-x)$ is the exponential integral. Fits for the functions $K_0(x)$ and Ei$(-x)$ are given in the Appendix.

The interpolation is carried out by plotting Eqs. (1) and (3) for fixed frequency as a function of temperature. The quadrupole result is assumed to be accurate for $kT_e \leq 50$ keV and the extreme-relativistic result is assumed to be accurate for $kT_e \geq 1$ MeV. For 50 keV $\leq kT_e \lesssim 1$ MeV, the two curves are connected with an "eyeball fit." The quadrupole and extreme-relativistic curves for $K = 100$ keV are shown in Fig. 1. The energy spectrum is defined by

$$W(K, T_e)\,dK \equiv h\nu P(h\nu, kT_e)\,d\nu\ , \quad (4)$$

where $K = h\nu$, is plotted vs $x = kT_e/mc^2$. This procedure was carried out for K = 1, 5, 10, 25, 50, 75, 100, 150, 200, 300, 400, 600, 800, and 1000 keV. The resulting energy spectra for kT_e = 20, 50, 100, 200, and 500 keV are plotted vs K (in keV) in Fig. 2.

Corrections to the quadrupole spectrum become significant for $kT_e \gtrsim 100$ keV, in the short-wavelength region $0.5 \lesssim K/kT_e \lesssim 5$. More than half of the energy is emitted in this region. For $K < 0.5\ kT_e$, corrections to the quadrupole spectrum be-

TABLE I. Bremsstrahlung energy spectra ($Z=1$).

$kT_e = \{23, 53, 100, 200\}$ keV

$h\nu$(keV)	Dipole	$e-i$ total	$e-e$ total	$W(K)/\alpha r_0^2 n_e^2 c$
1	80.0	80.0	4.0	84.0
	64.0	67.0	6.8	73.8
	52.0	57.0	11.1	68.1
	41.5	47.7	20.4	68.1
10	26.0	26.0	2.3	28.3
	30.0	31.5	4.7	36.2
	28.0	32.0	8.1	40.1
	25.0	32.0	15.8	47.8
20	14.0	14.0	1.5	15.5
	20.0	21.1	3.9	25.0
	21.0	23.8	6.8	30.6
	20.0	25.8	14.1	39.9
50	2.6	2.7	0.42	3.12
	7.8	8.7	2.21	10.91
	12.0	13.3	5.10	18.40
	13.5	17.65	11.07	28.72
100	0.215	0.235	0.05	0.285
	2.4	2.80	0.96	3.76
	5.6	6.65	3.07	9.72
	8.2	11.75	8.09	19.84
200	0.0025	0.0025	0.00025	0.0027
	0.26	0.36	0.13	0.49
	1.6	2.07	1.45	3.52
	3.8	6.16	4.90	11.06
300	...	...	...	...
	0.032	0.052	0.023	0.075
	0.50	0.72	0.51	1.23
	2.0	3.56	3.11	6.67
400	...	...	...	...
	0.004	0.007	0.003	0.01
	0.15	0.27	0.20	0.47
	1.10	2.10	2.04	4.14
600	...	...	...	...
	...	...	...	...
	0.017	0.034	0.038	0.072
	0.36	0.76	0.70	1.46

come important when $kT_e \gtrsim 150$ keV. As we will see in Sec. III, the e-e contribution is comparable to the e-i contribution in the short-wavelength region for $kT_e \gtrsim 100$ keV (for $Z=1$).

For the e-i problem, the extreme-relativistic spectrum is accurate to within 10% for $kT_e = 5$ MeV.[8] For the e-e problem, the extreme-relativistic spectrum was assumed to be accurate for $kT \gtrsim 1$ MeV. The nonrelativistic spectra was assumed to be accurate for $kT \lesssim 50$ keV since only 2^{2n} ($n=1, 2, \ldots$) poles contribute.

III. BREMSSTRAHLUNG SPECTRA AND TOTAL RATE (e-i+e-e)

If we now add together the electron-ion[2] and electron-electron results, including all multipole contributions, we obtain the emission spectra for a Maxwell gas with a temperature ranging from several keV to several hundred keV. Table I gives the dimensionless quantity $[W(K)/\alpha r_0^2 n_e^2 c]$ where $W(K)dK$ is the energy emitted per cm^3 per sec between K and $K+dK$ by e-i and e-e bremsstrahlung ($Z=1$). The dipole, total e-i, total e-e, and total emission (e-i+e-e) are listed separately for comparison. The four numbers in each box are for electron temperatures $kT_e = 23$, 53, 100, and 200 keV. For fully ionized gases with $Z>1$, Table I may be used if the e-i total is multiplied by Z.

The energy spectra (e-e+e-i) for $kT_e = 23$, 100, and 200 keV are plotted in Fig. 3. The dipole spectra for 23 and 100 keV are included for comparison.

Table I shows that the e-e contribution is always greater than the e-i multipole corrections to the dipole spectrum for all temperatures and frequencies. At temperatures $kT_e \gtrsim 100$ keV, the e-e contribution is greater than one-half the total e-i contribution in the short-wavelength portion of the spectrum.

From Fig. 3 the "flattening" of the spectrum at high temperatures is particularly noticeable from a comparison of the 23- and 200-keV spectra.

The bremsstrahlung rate of energy loss, defined as

$$W(T_e) \equiv \int_0^\infty dK\,[W^{e\text{-}e}(K, T_e) + W^{e\text{-}i}(K, T_e)] \ , \qquad (5)$$

where $W^{e\text{-}e}(K, T_e)$ and $W^{e\text{-}i}(K, T_e)$ are defined according to Eq. (4), is found by interpolating between the nonrelativistic and extreme-relativistic rates. For the e-i case, these rates are given by[1,8]

$$W_{\mathrm{nr}}^{e\text{-}i}(T_e) = \frac{32}{3}\left(\frac{2kT_e}{\pi mc^2}\right)^{1/2} \alpha Z^2 r_0^2 n_e n_i c mc^2 \qquad (6)$$

and

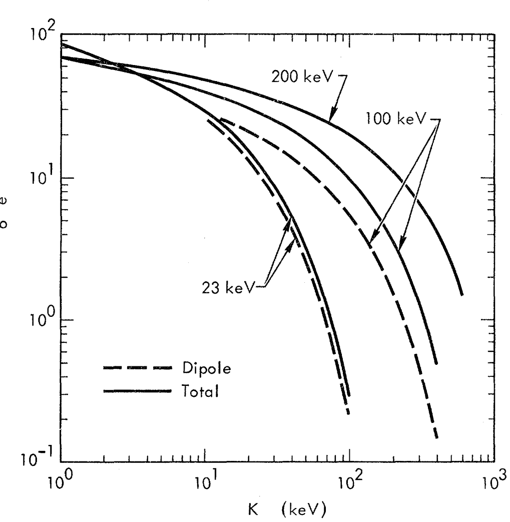

FIG. 3. Bremsstrahlung spectra ($e-i+e-e$) ($Z=1$).

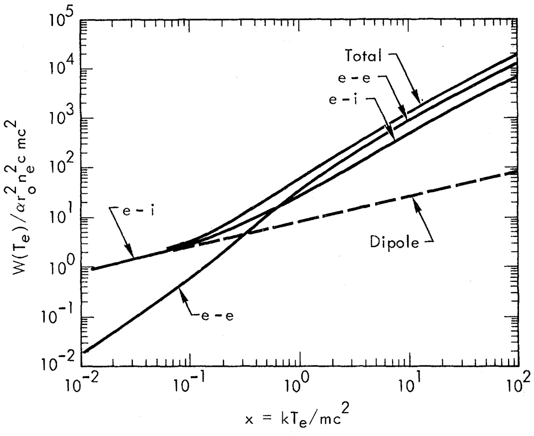

FIG. 4. Bremsstrahlung rate ($Z=1$).

$$W_{er}^{e-i}(T_e) = 12\alpha Z^2 r_0^2 n_e n_i ckT_e \times\left[\frac{3}{2} + \ln\left(\frac{2kT_e}{mc^2}\right) - 0.577\right] . \quad (7)$$

For the e-e case, the rates are[1,7]

$$W_{nr}^{e-e}(T_e) \simeq 32\left(\frac{2kT_e}{\pi mc^2}\right)^{1/2} \alpha r_0^2 n_e^2 ckT_e \quad (8)$$

and

$$W_{er}^{e-e}(T_e) = 24\alpha r_0^2 n_e^2 cKT_e \times\left[\frac{5}{4} + \ln\left(\frac{2kT_e}{mc^2}\right) - 0.577\right] . \quad (9)$$

Interpolating between $kT_e = 50$ keV and 1 MeV, we obtain e-i and e-e rates for all temperatures. The results are plotted in Fig. 4 vs $x = kT_e/mc^2$. The dimensionless quantity plotted is $W(T_e)/(\alpha r_0^2 \times n_e^2 cmc^2)$ where we have again set $n_e = Zn_i$ and $Z = 1$. The very rapid deviation from the dipole rate above $kT_e \sim 30$ keV is obvious. The e-e rate climbs rapidly until it crosses the e-i rate at a temperature $kT_e \simeq 350$ keV. At very high temperatures, $kT_e \gtrsim 50$ MeV, the e-e rate becomes twice the e-i rate ($Z = 1$).

In order to correctly describe the emission at temperatures $kT_e \gtrsim 300$ keV, the process of pair creation should be included. For temperatures $kT_e \lesssim 5$ keV, the Elwert factor (Coulomb correction) should be included in the cross section to correctly describe the emission.[1]

APPENDIX: FUNCTIONS $K_0(x)$, Ei$(-x)$, and $B(\lambda)$

Starting from the standard polynomial expansion for $K_0(x)$,[9] the formula

$$K_0(x) \simeq \ln\left(\frac{2}{x}\right)\left[1 + 3.52\left(\frac{x}{3.60}\right)^2\right] - 0.577 + 0.62\left(\frac{x}{2}\right)^2, \quad x < 1.7 \quad \text{(A1)}$$

was found to be accurate by comparison with tables of $e^x K_0(x)$. The standard asymptotic form

$$K_0(x) \simeq e^{-x}\left(\frac{1.57}{x}\right)^{1/2}\left(1 - \frac{0.125}{x} + \frac{0.07}{x^2}\right) \quad \text{(A2)}$$

was used for $x > 2$.

For the exponential integral, the expansions[9]

$$\text{Ei}(-x) \simeq \ln x + 0.577 - x + 0.25x^2 - 0.055x^3, \quad x \lesssim 1 \quad \text{(A3)}$$

and

$$\text{Ei}(-x) \simeq -\frac{e^{-x}}{x}\left[\frac{x^2 + 2.33x + 0.25}{x^2 + 3.33x + 1.68}\right], \quad x \geq 1 \quad \text{(A4)}$$

were used.

The fit used for $B(\lambda)$ was[10]

$$B(\lambda) = 0.85 + 1.35\sqrt{x} + 0.38x \quad . \quad \text{(A5)}$$

The CURVE routine[11] was used for plotting the nonrelativistic and extreme-relativistic spectra and rates.

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