Collision Times and Kinetic Theory for Superfluid Helium†

Wayne M. Saslow*
Department of Physics, University of Pittsburgh, Pittsburgh, Pennsylvania 15213
(Received 2 June 1971; revised manuscript received 11 November 1971)

We have found that the low-temperature collision-time kinetic theories of Khalatnikov and Chernikova and of Disatnik are not in agreement with all of the predictions of superfluid hydrodynamics. In particular, they overestimate, by a factor of about 30, the attenuation of hydrodynamic first sound. In addition, they use a hydrodynamic collision time to describe both the hydrodynamic and collisionless regimes, causing disagreement with many-body calculations valid for the collisionless regime. We have constructed a kinetic theory which is in agreement with superfluid hydrodynamics, and we have analyzed the existing kinetic theories in order to find the origins of the disagreement with hydrodynamics and with the many-body theory. To obtain correct transport coefficients (and therefore correct attenuation of hydrodynamic first sound) it is found necessary to ensure that the system relaxes to local, rather than to static, equilibrium. We calculate the transport coefficients within a collision-time model, and point out that for generality one must employ two hydrodynamic collision times τ_l and τ_t (with $\tau_l \geq \tau_t$) associated with longitudinal and transverse processes, respectively. From this we show that ζ_2 is not necessarily equal to zero. In addition, we discuss difficulties involved in extending the theory to higher frequencies, and present a physical argument which requires the use of a wide-angle collision time to define the (low-frequency) hydrodynamic regime and a small-angle collision time to define the (high-frequency) collisionless regime. This reconciles the disagreement between kinetic theory and many-body calculations upon the collision time which defines the collisionless regime. Various experiments are interpreted on this basis, thereby eliminating certain discrepancies with theory. Measurements in the low-temperature hydrodynamic regime are shown to require relatively large chambers (with linear dimensions $\gtrsim 10$ cm).

I. INTRODUCTION

Recent theoretical investigations on the irreversibility of static heat flow in He II,[1] and on the high-frequency shear response of He II,[2] have caused us to reexamine the kinetic theory of He II at low temperatures ($T \lesssim 0.6$ K). Two such theories have appeared in the literature, that by Khalatnikov and Chernikova (KC),[3] and that by Disatnik (D).[4] These theories, although in agreement with one another for the hydrodynamic regime, are not in agreement with all of the predictions of hydrodynamics. In particular, they overestimate, by a factor of about 30, the attenuation of hydrodynamic first sound. In addition, they use a hydrodynamic collision time to describe both the hydrodynamic and collisionless regimes, causing disagreement with many-body calculations valid for the collisionless regime. The primary motivations for this paper are to construct a kinetic theory in agreement with hydrodynamics, and to analyze the existing kinetic theories in order to find the origins of the disagreement with hydrodynamics and with the many-body theory. We also discuss the applicability of collision-time models to the kinetic theory of He II, we reinterpret various experiments in terms of the relevant collision times, and we discuss the possibility of experiments in the low-temperature hydrodynamic regime.

Although, so far, all experiments on sound in He II at low temperatures have been conducted in the nonhydrodynamic regime,[5-14] in principle, the hydrodynamic regime also can yield much useful information. In particular, measurement of first- and second-sound attenuation can give the longitudinal relaxation time τ_l associated with the viscosity η_n. If such measurements can be made, they will provide a test for the calculation by Landau and Khalatnikov of the transverse phonon-phonon relaxation time.[15] We remark that until recently no such calculation had been done for the phonon-phonon relaxation time associated with longitudinal relaxation.[16] We will show that $\tau_l \geq \tau_t$.

Much effort has already been expended on the calculation of the sound velocity and attenuation in the nonhydrodynamic regime.[3,4,17-29] There has been some success, in particular the work of Refs. 26 and 28, but even with all this work it has not been possible to explain all of the data (especially see Refs. 13 and 14). Therefore, it may be of some value to study the velocity and attenuation of sound in a regime which presumably we understand, and then to try to extend the theory beyond this regime. Since hydrodynamics should be valid at low enough frequencies, it would be advisable to start with a theory correct in the hydrodynamic regime. Kinetic theory, with the appropriate collision integral, is expected to give us a theory valid for all frequencies. We have outlined such a theory and have applied it to low frequencies,

since thereby much useful information can be obtained. As will be discussed there are certain difficulties with the collision term (even when using a phenomenological collision term) which one must confront in order to extend the theory to higher frequencies.

In Sec. II we present a discussion of kinetic theory in He II. We apply this in Sec. III to the hydrodynamic regime in order to calculate the transport coefficients. In Sec. IV we discuss the work of KC and of Disatnik. Section V contains a discussion of the collision time to be used in a phenomenological collision-time model and an analysis of various experiments in light of this discussion. In Sec. VI we consider the possibility of low-temperature measurements in the hydrodynamic regime. In Sec. VII we summarize our results.

II. KINETIC THEORY FOR He II

In this section we present and discuss the kinetic theory of excitations for He II at low temperatures. In the literature these equations are usually assumed, and discussion of them is minimal. However, an appreciation of the meaning of these equations, at a simple but basic level, is necessary to an understanding of the relationship between kinetic theory and superfluid hydrodynamics.

At low temperatures, He II can be described as consisting of a ground state and a set of well-defined Bose excitations, for which one can write a kinetic equation:

$$\frac{\partial n_p}{\partial t} + \frac{\partial n_p}{\partial \vec{r}} \cdot \frac{\partial \vec{r}}{\partial t} + \frac{\partial n_p}{\partial \vec{p}} \cdot \frac{\partial \vec{p}}{\partial t} = \frac{\partial n_p}{\partial t}\bigg|_{\text{collisions}} \equiv J[n_p] \ . \tag{1}$$

Here n_p is the occupation number for, and $\partial \vec{r}/\partial t = \vec{v}_p$ is the velocity of, an excitation of momentum p. If a disturbance of frequency ω and wave vector $\vec{k}$ is applied to the system, the occupation number can be conveniently described by breaking it up into three terms, although formally it can be obtained from one expression:

$$n_p = \left[\exp\left(\frac{\epsilon_p + \vec{p}\cdot(\vec{v}_s - \vec{v}_n) - \mu_p}{K_B T}\right) - 1\right]^{-1} \ . \tag{2}$$

Here ϵ_p is the energy of the excitation, computed in the superfluid rest frame from the local pressure (P) and temperature (T), and $\vec{v}_s$ and $\vec{v}_n$ are the superfluid and normal fluid velocities, respectively. This corresponds to oscillating local equilibrium, with a large static equilibrium value (e.g., P_0) dominating the small oscillating value (e.g., P'). The deviation from local equilibrium for each excitation is described by μ_p, the chemical potential, which is zero for nonconserved bosons in local equilibrium. Equation (2) can be expanded in the small oscillating quantities $P', T', \vec{v}_s', \vec{v}_n'$, and μ_p' to yield

$$n_p = n_p^{(0)} + \delta n_p^{(1e)} + n_p' \ . \tag{3}$$

Here $n_p^{(0)}$ is the static equilibrium value,

$$\begin{aligned}\delta n_p^{(1e)} &= \frac{\partial n_p}{\partial \epsilon_p}\left(\frac{\partial \epsilon_p}{\partial T} T' + \frac{\partial \epsilon_p}{\partial P} P'\right) + \frac{\partial n_p}{\partial T} T' \\ &\quad + \frac{\partial n_p}{\partial \vec{v}_s}\cdot \vec{v}_s' + \frac{\partial n_p}{\partial \vec{v}_n}\cdot \vec{v}_n' \\ &= \frac{\partial n_p}{\partial \epsilon_p}\left(\frac{\partial \epsilon_p}{\partial T} T' + \frac{\partial \epsilon_p}{\partial P} P' \right. \\ &\quad \left. - \frac{\epsilon_p}{T} T' + \vec{p}\cdot(\vec{v}_s' - \vec{v}_n')\right)\end{aligned} \tag{4}$$

and

$$n_p' = \frac{\partial n_p}{\partial \mu_p}\ \mu_p' = -\frac{\partial n_p}{\partial \epsilon_p}\ \mu_p' \ . \tag{5}$$

In the above equations the absence of a superscript denotes the static equilibrium values. (Note that $\vec{v}_s = \vec{v}_s'$, $\vec{v}_n = \vec{v}_n'$.) For low frequencies we expect that $n_p' \ll \delta n_p^{(1e)}$; for high frequencies no distinction is necessary, since the collision integral (depending only on n_p') is negligible. With a given model for collisions of the excitations, one can in principle follow the ratio $n_p'/\delta n_p^{(1e)}$ as ω is increased.

In the above there are five oscillating unknown variables: The ratios (taken with respect to P') of the quantities $T', \vec{v}_s', \vec{v}_n'$, and μ_p', and the wave vector k, which is related to ω by some unknown function $\omega(\vec{k})$ that must be determined. The Boltzmann equation yields information about μ_p'. In addition, there are four other equations: conservation of mass, the superfluid equation of motion, conservation of momentum, and conservation of energy. This set of equations permits a solution of the problem. The last two equations, conservation of momentum and conservation of energy, can be derived from the Boltzmann equation. In the kinetic theories of Refs. 3 and 4, two other conditions are used, distinct from energy and momentum conservation; to $O(\rho_n/\rho)$ they reproduce the correct hydrodynamic velocities.

When energy dissipation (proportional to the square of the temperature and velocity gradients) can be neglected, energy conservation can be transformed to entropy conservation. For completeness, we write down the four equations explicitly, neglecting such nonlinear dissipative terms[30]:

momentum conservation,

$$\begin{aligned}\rho_n \frac{\partial \vec{v}_n'}{\partial t} + \rho_s \frac{\partial \vec{v}_s'}{\partial t} + \vec{\nabla} P' &= -\frac{\partial}{\partial t}\langle n' \vec{p}\rangle \\ &\quad - \rho \vec{\nabla}\left\langle n' \frac{\partial \epsilon}{\partial \rho}\right\rangle - \vec{\nabla}\cdot\langle n' \vec{p}\vec{v}\rangle \ ;\end{aligned} \tag{6}$$

mass conservation,

$$\frac{\partial \rho'}{\partial t} + \rho_n \vec{\nabla}\cdot\vec{v}_n' + \rho_s \vec{\nabla}\cdot\vec{v}_s' = -\vec{\nabla}\cdot\langle n'\vec{p}\rangle \ ; \tag{7}$$

superfluid motion,

$$\frac{\partial \vec{v}_s}{\partial t} + \frac{1}{\rho}\vec{\nabla}P' - \frac{S}{\rho}\vec{\nabla}T' = -\vec{\nabla}\left\langle n' \frac{\partial \epsilon}{\partial \rho}\right\rangle ; \quad (8)$$

energy conservation,

$$\frac{\partial S'}{\partial t} + S\vec{\nabla}\cdot\vec{v}_n' = -\frac{1}{T}\left(\frac{\partial}{\partial t}\langle n'\epsilon\rangle + \vec{\nabla}\cdot\langle n'\epsilon\vec{v}\rangle\right) . \quad (9)$$

In Eqs. (6)–(9) ρ_n, ρ_s, and $\rho(=\rho_n+\rho_s)$ are the normal fluid, superfluid, and total densities, and S $(\equiv\rho\sigma)$ is the entropy per unit volume. An angular bracket about a quantity means an integral over phase space:

$$\langle F\rangle = (2\pi\hbar)^{-3}\int F_p d^3p . \quad (10)$$

Also, n', ϵ, $\vec{p}$, and $\vec{v}$ in the $\langle\ \rangle$ brackets refer to n_p', etc.

The angle bracketed quantities represent the effect of those excitations not in equilibrium. They contribute to the momentum in Eqs. (6) and (7), and to the energy in Eq. (9). Dissipative processes arise from the stress tensor $\langle n'\vec{p}\vec{v}\rangle$, the additional "pressure" $\langle n'\partial\epsilon/\partial\rho\rangle$ (these produce the first and second viscosities), and from the heat-flow vector $\langle n'\epsilon\vec{v}\rangle$ (this produces effects analogous to thermal conduction). With the Boltzmann equation [Eq. (1)], Eqs. (6)–(9) are sufficient to determine the response of the system when subjected to a disturbance of frequency ω. For convenience we rewrite the Boltzmann equation[30]:

$$\frac{\partial n_p}{\partial \epsilon_p}\left[-\vec{v}_p\cdot\vec{\nabla}(\vec{p}\cdot\vec{v}_n') - \frac{\epsilon_p}{T}\vec{v}_p\cdot\vec{\nabla}T' + \frac{\partial \epsilon_p}{\partial t} + \vec{p}\cdot\left(\frac{\partial \vec{v}_s'}{\partial t} - \frac{\partial \vec{v}_n'}{\partial t}\right) - \frac{\epsilon_p}{T}\frac{\partial T'}{\partial t}\right] = J[n_p] - \frac{\partial n_p'}{\partial t} - \vec{v}_p\cdot\vec{\nabla}n_p' . \quad (11)$$

We note that $J[n_p]$ depends upon all phonon occupation numbers. Since $J[n_p]=0$ for all $n_p=n_p^{(0)}+n_p^{(1e)}$, it is a linear functional of $\{n_p'\}$, and from now on we write it as $J[n_p']$. Thus we have separated the Boltzmann equation, as we did the conservation equations, into a left-hand side depending upon local equilibrium quantities and a right-hand side depending upon nonequilibrium quantities. We remark that the inclusion of $\partial n_p'/\partial t$ and $\vec{v}_p\cdot\vec{\nabla}n_p'$ makes Eq. (11) more general than any of the forms given by Khalatnikov.[30]

In order to solve Eqs. (6)–(9) and (11), we will need $J[n_p']$ and ϵ_p. For our purposes we will usually take $\epsilon_p=cp$, thereby neglecting the effects of phonon dispersion. Then $\vec{v}_p\equiv\partial\epsilon_p/\partial\vec{p}=c\hat{p}$. However, we will occasionally discuss the effects of dispersion. We will discuss the form of $J[n_p']$ in Sec. III, where we calculate the transport coefficients.

Before proceeding to our calculations, we review the procedure for calculating transport coefficients, and clarify the relationship between kinetic theory and hydrodynamics. To lowest order, nondissipative hydrodynamics may be recovered from the kinetic theory by neglecting the right-hand side of Eqs. (6)–(9). This will give the hydrodynamic velocities, but it neglects attenuation, which depends upon $\{n_p'\}$. To obtain the attenuation one goes to a higher order, and employs, on the left-hand side of Eq. (11), the information obtained from the lowest-order solution of Eqs. (6)–(9), which defines the state of local equilibrium. On the right-hand side of Eq. (11) one keeps only the collision integral. Solving Eq. (11), one finds n_p'. Substitution into the right-hand side of Eqs. (6)–(9) gives the dissipative terms. From these one can deduce the transport coefficients by comparison with the equations of superfluid hydrodynamics including dissipation. One may then compute the attenuation by the formulas of superfluid hydrodynamics.[30]

To obtain a solution valid at all frequencies, one must exactly solve Eq. (11) for n_p', and substitute into the right-hand side of Eqs. (6)–(9). Solving these equations one finds $\omega(k)$. The primary difficulty, however, lies in solving for n_p'. This first requires a detailed knowledge of $J[n_p]$, and then the extraction of n_p' from Eq. (11) once $J[n_p]$ is known. So far there has been no solution of this problem, except when $J[n_p]$ has been grossly simplified.

We should not neglect to mention that Eqs. (6)–(9) and (11) are phenomenological; they have not been derived from a microscopic theory. Certainly one may question their validity when applied to situations involving atomic distances $a\approx 4\times10^{-8}$ cm and frequencies of $O(2\pi c/a\approx 5\times10^{12}\ \mathrm{sec}^{-1})$.

Indeed, their range of validity may not even extend to such short distances or high frequencies. However, these equations are expected to describe the frequency regime discussed in this paper.

III. TRANSPORT COEFFICIENTS AND SUPERFLUID HYDRODYNAMICS

We will now study the transport coefficients using the kinetic theory and a collision-time model. Our principal results are that, when phonon dispersion is neglected: (i) The second viscosity coefficient $\zeta_2(\geq 0)$ is proportional to $\tau_l-\tau_t$, so that $\tau_l\geq\tau_t$, and (ii) the attenuation of hydrodynamic first sound is about a 30th of the value predicted by Refs. 3 and 4. In addition we point out certain difficulties with the collision-time approximation when phonon dispersion is included, although we perform

no explicit calculations.

Let us consider disturbances with an $e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ spatial and temporal variation. We may take $\vec{k}$ along the z axis. There are two possibilities now. All vector quantities may be parallel (longitudinal disturbances) or perpendicular (transverse disturbances) to the z axis. He II is known to support three longitudinal modes and one transverse mode.[31] Using the present formulation it should in principle be possible to study all of these modes, for all frequencies ω.[1,2]

For such disturbances, $\partial/\partial t \to -i\omega$ and $\vec{\nabla} \to ik\hat{z}$. With primes denoting the deviations of thermohydrodynamic quantities from their equilibrium values, multiplying each ω or k will be a primed thermohydrodynamic quantity or a moment of n'. Using Eqs. (6)–(9) and the thermodynamic relations

$$P' = \frac{\partial P}{\partial \rho}\bigg|_T \rho' + \frac{\partial P}{\partial T}\bigg|_\rho T' \ ,$$

$$S' = \frac{\partial S}{\partial \rho}\bigg|_T \rho' + \frac{\partial P}{\partial T}\bigg|_\rho T' \ ,$$

we may eliminate all thermohydrodynamic variables from Eq. (11), which then will become an equation linear in n' and its moments $\langle n'\vec{p}\rangle$, $\langle n'\,\partial\epsilon/\partial\rho\rangle$, $\langle n'\vec{p}\,\vec{v}\rangle$, $\langle n'\epsilon\rangle$, and $\langle n'\epsilon v\rangle$. By calculating each of these moments of n' from this resultant equation one will have five equations in these five moments. Setting their determinant to zero determines the various $\omega(k)$, and from this, for each $\omega(k)$ one may study the ratios of P', T', $\vec{v}_n$, $\vec{v}_s'$, $\langle n'\vec{p}\rangle$, $\langle n'\,\partial\epsilon/\partial\rho\rangle$, $\langle n'\vec{p}\,\vec{v}\rangle$, $\langle n'\epsilon\rangle$, and $\langle n'\epsilon\vec{v}\rangle$.

In this paper we do not intend to study the longitudinal mode of Kronig and Thellung, which does not have a simple ω vs k relation.[31,1] We wish to study, for the hydrodynamic regime, first and second sound and the transverse mode. Let us consider the latter first, since it is the simplest to treat. Let all transport vectors ($\langle n'\epsilon\vec{v}\rangle$, $\langle n'\vec{p}\rangle$, $\vec{v}_n'$, $\vec{v}_s'$) point along the x axis. Then the mass equation [Eq. (7)] says that $\rho' = 0$; the density does not change, as expected. Further, the x component of the superfluid equation [Eq. (8)] says that $(\vec{v}_s')_x = 0$; the superfluid does not take part in the motion, as expected.

Additional information may be gathered. The z component of the momentum equation [Eq. (6)] relates T' (since P' depends only on T' when $\rho' = 0$) to $\langle n'\,\partial\epsilon/\partial\rho\rangle$ and $\langle n'p_z v_z\rangle$; the z component of the superfluid equation [Eq. (8)] relates T' to $\langle n'\,\partial\epsilon/\partial\rho\rangle$; the energy equation [Eq. (9)] relates T' to $\langle n'\epsilon\rangle$. From these three equations we can relate $\langle n'\epsilon\rangle$ and $\langle n'p_z v_z\rangle$. This provides an additional restriction on $\omega(k)$. It can be trivially satisfied if these quantities are zero (by the symmetry of the problem). Therefore we will take $T' = \langle n'\epsilon\rangle = \langle n'\,\partial\epsilon/\partial\rho\rangle = \langle n'p_z v_z\rangle = 0$ and show this to yield a consistent solution. In that case we have only to solve

$$-i\omega\rho_n v_n = i\omega\langle n'p_x\rangle - ik\langle n'p_x v_z\rangle \tag{12}$$

and

$$\frac{\partial n_p}{\partial \epsilon_p}\left(-ikv_n p_x v_z + i\omega p_x v_n\right) = J[n'] \ , \tag{13}$$

where we have neglected $\partial n_p'/\partial t$ and $\vec{v}_p\cdot\vec{\nabla}n_p'$ in comparison with $J[n']$. Since

$$\rho_n \frac{\partial v_n}{\partial t} = \eta_n \frac{\partial^2}{\partial z^2} v_n \tag{14}$$

in the hydrodynamic regime, we see that $k^2 = i\omega\rho_n/\eta_n$. Then $k \sim \omega^{1/2}$ and $kc \gg \omega$, so Eqs. (12) and (13) become

$$-i\omega\rho_n v_n \approx -i\kappa\langle n'p_x v_z\rangle \tag{12'}$$

and

$$-ikv_n p_x v_z \frac{\partial n_p}{\partial \epsilon_p} \approx J[n_p'] \ . \tag{13'}$$

In general we have

$$J[n_p'] = \langle f_{pp'} n_{p'}'\rangle \ , \tag{15}$$

where $f_{pp'}$ is some function describing how nonequilibrium of the $\vec{p}'$ phonon causes collisions with the $\vec{p}$ phonon. In principle we may invert Eq. (15) to obtain

$$n_p' = \langle (f^{-1})_{pp'} J[n_{p'}']\rangle \ . \tag{16}$$

Then Eq. (12′) becomes

$$-i\omega\rho_n v_n = -k^2 v_n \left\langle\!\!\left\langle (f^{-1})_{pp'} p_x' v_z' \frac{\partial n_{p'}}{\partial \epsilon_{p'}}\right\rangle\!\!\right\rangle p_x v_z \ . \tag{17}$$

The $\partial n_{p'}/\partial\epsilon_{p'}$ factor causes the operator $(f^{-1})_{pp'}$ to be dominated by thermal phonons in the p' index. The collision-time model uses (for this transverse mode)

$$J[n_p'] = -n_p'/\tau_t \ , \tag{18}$$

so

$$f_{pp'} = -\frac{1}{\tau_t}\delta_{pp'} \quad \text{and} \quad (f^{-1})_{pp'} = -\tau_t\,\delta_{pp'} \ .$$

Using the results of Eq. (18) it is clear that

$$-i\omega\rho_n v_n = \tau_t k^2 v_n \left\langle p_x^2 v_z^2 \frac{\partial n}{\partial \epsilon}\right\rangle = -\tfrac{1}{5}\tau_t (ck)^2 \rho_n v_n \ . \tag{19}$$

[We have used $\rho_n = -\frac{1}{3}\langle(\partial n/\partial\epsilon)p^2\rangle$ and $\langle p_x^2 v_z^2\,\partial n/\partial\epsilon\rangle = \frac{1}{15}c^2\langle(\partial n/\partial\epsilon)p^2\rangle$]. Hence, by comparison of Eqs. (14) and (19),

$$\eta_n = \tfrac{1}{5}\rho_n c^2 \tau_t \ . \tag{20}$$

This agrees with Khalatnikov and Chernikova, with our $\tau_t \to$ their τ_{pp}. Note that by symmetry $\langle n'\epsilon\rangle$, $\langle n'\,\partial\epsilon/\partial\rho\rangle$, $\langle n'p_z v_z\rangle$, and (hence) T' are

all zero.

Let us now compute the transport coefficients associated with longitudinal oscillations. First consider κ, which is related to $\langle n'\epsilon\vec{v}\rangle$. From Eq. (11), in the hydrodynamic regime (neglecting $\partial n_p'/\partial t$ and $\vec{v}_p\cdot\vec{\nabla}n_p'$) we have, with $(f^{-1})_{pp'} = -\tau_l\delta_{pp'}$,

$$\langle n'\epsilon v_z\rangle = -\tfrac{1}{3}\tau_l\left\langle\frac{\partial n}{\partial\epsilon}\epsilon^2\right\rangle\left(\frac{\partial}{\partial t}(v_s'-v_n')-\frac{c^2}{T}\frac{\partial T'}{\partial z}\right). \tag{21}$$

In the above we used $\langle\cos\theta\rangle = \langle\cos^3\theta\rangle = 0$, and $\langle\cos^2\theta\rangle = \frac{1}{3}$. From Eqs. (6)–(8), when evaluated in the hydrodynamic limit (i.e., neglecting the nonequilibrium terms) we have

$$\langle n'\epsilon v_z\rangle = -\tfrac{1}{3}\tau_l\left\langle\frac{\partial n}{\partial\epsilon}\epsilon^2\right\rangle\left(\frac{S}{\rho_n}-\frac{c^2}{T}\right)\frac{\partial T'}{\partial z}. \tag{22}$$

For dispersionless phonons, $\rho_n c^2 = ST$, so $\langle n'\epsilon v_z\rangle = 0$. Hence the thermal conductivity κ must be zero. This is a well-known result.[30] We also note that for dispersionless phonons

$$\langle n' p_z\rangle = (1/c^2)\langle n'\epsilon v_z\rangle = 0. \tag{23}$$

The viscosity coefficients ζ_3 and ζ_4 are contained in $\langle n'\,\partial\epsilon/\partial\rho\rangle$, so let us study this quantity:

$$\begin{aligned}\left\langle n'\frac{\partial\epsilon}{\partial\rho}\right\rangle &= -\tau_l\left\langle\frac{\partial n}{\partial\epsilon}p\frac{\partial c}{\partial\rho}\left(\frac{\partial\epsilon_p}{\partial t}-\frac{\epsilon_p}{T}\frac{\partial T'}{\partial t}-p_z v_z\vec{\nabla}\cdot\vec{v}_n'\right)\right\rangle\\ &= -\tau_l\frac{\partial c}{\partial\rho}\left\langle\frac{\partial n}{\partial\epsilon}p^2\right\rangle\left(\frac{\partial c}{\partial\rho}\frac{\partial\rho'}{\partial t}-\frac{c}{T}\frac{\partial T'}{\partial t}-\frac{c}{3}\vec{\nabla}\cdot\vec{v}_n'\right)\\ &= -\tau_l c\frac{\partial c}{\partial\rho}\left\langle\frac{\partial n}{\partial\epsilon}p^2\right\rangle\left[\frac{u}{\rho}\frac{\partial\rho'}{\partial t}-\frac{1}{T}\frac{\partial T}{\partial S}\bigg|_\rho\left(\frac{\partial S'}{\partial t}-\frac{\partial S}{\partial\rho}\bigg|_T\frac{\partial\rho'}{\partial t}\right)-\frac{1}{3}\vec{\nabla}\cdot\vec{v}_n'\right]\\ &= -\tau_l\frac{uc}{\rho}\left\langle\frac{\partial n}{\partial\epsilon}p^2\right\rangle\left\{\frac{1}{\rho}\vec{\nabla}\cdot\vec{j}'\left[u+\frac{1}{C_v}\left(S-\frac{\partial P}{\partial T}\bigg|_\rho\right)\right]+\left(\frac{1}{3}-\frac{S}{C_v}\right)\vec{\nabla}\cdot\vec{v}_n'\right\}.\end{aligned} \tag{24}$$

In the above we have used $\vec{j}' = \rho_n\vec{v}_n' + \rho_s\vec{v}_s'$, the thermodynamic identity $\partial S/\partial\rho|_T = (1/\rho)(S-\partial P/\partial T|_\rho)$, and $u \equiv (\rho/c)\partial c/\partial\rho$. For dispersionless phonons it is well known that $C_v = 3S$, so the coefficient of $\vec{\nabla}\cdot\vec{v}_n'$ is zero. In addition,

$$u+\frac{1}{C_v}\left(S-\frac{\partial P}{\partial T}\bigg|_\rho\right) = 0$$

for dispersionless phonons. Hence the coefficient of $\vec{\nabla}\cdot\vec{j}'$ is also zero. We therefore have

$$\left\langle n'\frac{\partial\epsilon}{\partial\rho}\right\rangle = \zeta_3 = \zeta_4 = 0. \tag{25}$$

Another property of dispersionless phonons is that

$$\langle n'\epsilon\rangle = \frac{\rho}{u}\left\langle n'\frac{\partial\epsilon}{\partial\rho}\right\rangle = 0. \tag{26}$$

Let us now compute $\langle n' p_z v_z\rangle$, which will give ζ_1 and ζ_2 (we expect that $\zeta_1 = 0$ since $\zeta_1 = \zeta_4$ by the Onsager symmetry principle for kinetic coefficients[29]):

$$\begin{aligned}\langle n' p_z v_z\rangle &= -\tau_l\left\langle\frac{\partial n}{\partial\epsilon}p_z v_z\left(\frac{\partial\epsilon_p}{\partial t}-\frac{\epsilon_p}{T}\frac{\partial T'}{\partial t}-p_z v_z\vec{\nabla}\cdot\vec{v}_n'\right)\right\rangle\\ &= -\tfrac{1}{3}\tau_l\left\langle\frac{\partial n}{\partial\epsilon}p^2\right\rangle c^2\left(\frac{u}{\rho}\frac{\partial\rho'}{\partial t}-\frac{1}{T}\frac{\partial T'}{\partial t}-\tfrac{3}{5}\vec{\nabla}\cdot\vec{v}_n'\right)\\ &= -\tfrac{1}{3}c^2\tau_l\left\langle\frac{\partial n}{\partial\epsilon}p^2\right\rangle\left(\frac{u}{\rho}\frac{\partial\rho'}{\partial t}-\frac{1}{T}\frac{\partial T'}{\partial t}-\tfrac{1}{3}\vec{\nabla}\cdot\vec{v}_n'-\tfrac{4}{15}\vec{\nabla}\cdot\vec{v}_n'\right)\\ &= \frac{1}{3}\frac{\rho}{u}\left\langle n'\frac{\partial\epsilon}{\partial\rho}\right\rangle-\tfrac{4}{15}\rho_n c^2\tau_l\vec{\nabla}\cdot\vec{v}_n'\\ &= -\tfrac{4}{15}\rho_n c^2\tau_l\vec{\nabla}\cdot\vec{v}_n'.\end{aligned} \tag{27}$$

In deriving the above equations we used $\langle\cos^4\theta\rangle = \frac{1}{5}$ and Eqs. (24) and (25). Putting Eq. (27) into Eq. (6) yields

$$\frac{\partial\vec{j}}{\partial t}+\vec{\nabla}P = \tfrac{4}{15}\rho_n c^2\tau_l\vec{\nabla}(\vec{\nabla}\cdot\vec{v}_n). \tag{28}$$

Superfluid hydrodynamics takes the form[30]

$$\frac{\partial\vec{j}}{\partial t}+\vec{\nabla}P = \tfrac{4}{3}\eta_n\vec{\nabla}(\vec{\nabla}\cdot\vec{v}_n)+(\zeta_2-\rho\zeta_1)\vec{\nabla}(\vec{\nabla}\cdot\vec{v}_n)+\zeta_1\vec{\nabla}(\vec{\nabla}\cdot\vec{j}). \tag{29}$$

Clearly $\zeta_1 = 0$, as expected. Also, since $\eta_n = \frac{1}{5}\rho_n c^2\times\tau_t$, we see that

$$\zeta_2 = \tfrac{4}{15}\rho_n c^2(\tau_l-\tau_t). \tag{30}$$

Because $\zeta_2 \geq 0$,[30] we must have $\tau_l \geq \tau_t$. This will be one of the restrictions on calculations of τ_l.[16] We repeat that τ_t is the same as the τ_{pp} calculated by Landau and Khalatnikov.[15]

We have shown that not all of the coefficients of second viscosity are zero, in disagreement with KC.[3(b)] They use only one collision time for both longitudinal and transverse processes, so it is clear how they could obtain $\zeta_2 = 0$.

Using hydrodynamics we may now compute the attenuation coefficients of first and second sound. They are determined only by the shear viscosity, and are given by[30]

$$\alpha_1 = \frac{\omega^2}{2\rho c^3}\left(\tfrac{4}{3}\eta + \zeta_2\right) = \tfrac{2}{15}\,\frac{\omega^2\tau_1}{c}\,\frac{\rho_n}{\rho} \tag{31}$$

and

$$\alpha_2 = \frac{\omega^2}{2\rho(c/\sqrt{3})^3}\,\frac{\rho_s}{\rho_n}\left(\tfrac{4}{3}\eta + \zeta_2\right) \approx \frac{2\sqrt{3}}{5}\,\frac{\omega^2\tau_1}{c}\ . \tag{32}$$

The result for α_2 is in agreement with Disatnik and with KC (there is a misprint in their equation for α_2, with $\sqrt{3}$ in the denominator). However, for α_1 we obtain a very different result than do Disatnik and KC, who are in agreement with one another. They obtain

$$\alpha_1' = \tfrac{3}{10}\,(u+1)^2\,\frac{\omega^2\tau}{c}\,\frac{\rho_n}{\rho}\ . \tag{33}$$

With $u = 2.8$, this expression is seen to be about 30 times the expression obtained using hydrodynamics.

One may also show that superfluid hydrodynamics yields[30]

$$v_2 = \frac{c}{\sqrt{3}}\left(1 - \frac{3}{4}\,\frac{\rho_n}{\rho}\,(1 + 2u + 3u^2)\right) \tag{34}$$

and

$$v_1 = c\left(1 + \frac{1}{4}\,\frac{\rho_n}{\rho}\,(1 + 6u + 3u^2 + \tfrac{3}{2}w)\right), \tag{35}$$

where $w = (\rho^2/c)\,\partial^2 c/\partial\rho^2$. These results are in agreement with KC and Disatnik.

It is instructive to study the effects of phonon dispersion on the transport coefficients, since it points out some of the difficulties of collision-time models. In Eq. (22) the coefficient of $\partial T/\partial z$ must be nonpositive ($\kappa \geq 0$), and in Eq. (24) the coefficient of $\vec{\nabla}\cdot(\vec{j} - \rho\vec{v}_n)$ must be non-negative ($\zeta_3, \zeta_4 \geq 0$). This is so that dissipative processes may increase the entropy of the system. However, $(TS - \rho_n c^2)$ and $[u + (1/C_v)(S - \partial P/\partial T|_\rho)]$ are proportional to the phonon dispersion parameter, which can be either positive or negative.[32] Therefore the collision times that must be used in calculating κ, ζ_3, and ζ_4 may be of negative sign. One must therefore be cautious in applying the collision-time model to the collision operator or its inverse. Clearly, the value of f_{pp}^{-1}, depends very much upon the operand. In a less extreme fashion, it is well known that the collision time used in the transport theory of Fermi liquids depends upon whether one is studying the thermal conductivity, the viscosity, or the spin diffusivity. One should not be surprised by similar occurrences in the transport theory of He II.

We note that inclusion of phonon dispersion affects the second-sound velocity, so that

$$\frac{\Delta V_2}{V_2} = -\frac{15\gamma}{7}\left(\frac{2\pi K_B T}{C}\right)^2 - \frac{945\gamma\gamma'}{8}\left(\frac{2\pi K_B T}{C}\right)^3, \tag{36}$$

where $\gamma = \zeta(7)/\pi^3\zeta(4) = 0.030047\ldots$, and $\epsilon_p = cp(1 - \gamma p^2 - \gamma' p^3)$ defines γ and γ'.[33] One usually refers to γ alone when referring to the dispersion parameter. At very low T this correction dominates those described in Eq. (35).

It is important to recognize, as was first done by Khalatnikov and Chernikova,[3] that $\kappa = \zeta_1 = \zeta_3 = 0$ for dispersionless phonons. Although it appeared that these results depended upon the collision-time approximation, they are of very general validity. They rest upon the Boltzmann equation, the restriction to the hydrodynamic regime, and certain thermodynamic identities satisfied by a dispersionless Bose gas at low temperatures. Only a microscopic calculation, however, can yield η or ζ_2.[15,16]

IV. ANALYSIS OF KC AND OF DISATNIK

We now study the work of KC and of Disatnik. It will be seen that neither work explicitly satisfies the requirements of energy and momentum conservation. In order for their problems to be soluble, they find two other conditions on their variables. These conditions are distinct from energy and momentum conservation. However, their expressions for the sound velocities are correct to $O(\rho_n/\rho)$, which indicates that their approximations to energy and momentum conservation are rather accurate. We will also show that Disatnik and (presumably) KC do not satisfy the requirement that the system relax to local equilibrium. This causes them to overestimate the attenuation of first sound by a factor of about 30.

The work of Khalatnikov and Chernikova has been divided into two papers. The first[3(a)] (which we shall call KCI) is devoted to the relaxation phenomena which take place in a range of temperatures up to about 1.6 K. The second[3(b)] (KCII) is devoted to the effects of these relaxation phenomena upon the attenuation of sound. We will here be concerned with those results which are applicable to $T \lesssim 0.6$ K, where only phonon-phonon scattering is of importance.

The relevant set of equations is given in KCII, Eqs. (4.1). The first two equations are simply mass conservation and the superfluid equation of motion. The second two equations are integrals of the Boltzmann equation. Their derivation is outlined in KCI. They start with the Boltzmann equation, written in the form [Eq. (2.8) of KCI]:

$$\left(1 - \frac{\vec{k}\cdot\vec{v}_p}{\omega}\right) n_p'' + \frac{\vec{k}\cdot\vec{v}_p}{\omega}\,\frac{\partial n_p}{\partial\epsilon_p}\left(\frac{\partial\epsilon_p}{\partial\rho}\,\rho' + \vec{p}\cdot\vec{v}_s\right) = -\frac{1}{i\omega}\,J[n_p]\ .$$

Here $n_p'' \equiv n_p^{(1e)} + n_p'$, and an $e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ dependence has been assumed. They also use (with $\vec{k}$, $\vec{v}_s$, and $\vec{v}_n$ all along the z axis) [Eq. (2.12) of KCI]

$$n_p'' = \frac{\partial n_p}{\partial \epsilon_p}\left(\frac{\partial \epsilon}{\partial \rho}\rho' + \epsilon\nu(\cos\theta)\right)$$

and $\nu(\cos\theta) = \nu_0 + \nu_1\cos\theta + \nu_2 P_2\cos\theta + \cdots$, where $\nu_0 = -T'/T$ and $\nu_1 = (v_n - v_s)/c$ (as shown in KCII). The collision integral, which involves only phonon-phonon scattering, is separated into $J'_{pp}[n]$ and $J_{pp}[n]$, for small- and large-angle phonon scattering, respectively.

Their first step in obtaining the integrals of motion from the Boltzmann equation involves integrating over $\epsilon_p p^2 dp$ (but not over $d\Omega_p$) and dividing by $\int(\partial n_p/\partial\epsilon_p)\epsilon_p^2 p^2 dp$. This gives [Eq. (2.16) of KCI]

$$(\tilde{\omega} - \cos\theta)\nu(\cos\theta) + \tilde{\omega}u\rho' + \cos^2\theta v_s = -\frac{\tilde{\omega}}{i\omega}\int J_{pp}[n]\epsilon_p p^2 dp \Big/ \int \frac{\partial n_p}{\partial \epsilon_p}\epsilon_p^2 p^2 dp \ ,$$

where $\tilde{\omega} = \omega/ck$. They set $\int J'_{pp}[n]\epsilon_p p^2 dp \approx 0$ by energy conservation for small-angle scattering. This step is not valid at high frequencies, where the mode is sharply peaked in momentum space; J'_{pp} is so large for such momenta that it dominates J_{pp}. This will be discussed in Sec. V.

Their second step involves treating $\int J_{pp}[n]\epsilon_p p^2 dp$, which they do by approximation [Eq. (2.26) of KCI]:

$$J_{pp}[n]\epsilon_p p^2 dp = -\frac{1}{\tau_{pp}}[\nu(\cos\theta) - \nu_0 - \nu_1\cos\theta] \times \int \frac{\partial n_p}{\partial \epsilon_p}\epsilon_p^2 p^2 dp \ .$$

Instead of next integrating on $d\Omega_p$ to obtain energy conservation, and then integrating on $\cos\theta\, d\Omega_p$ to obtain momentum conservation (thereby obtaining equations independent of τ_{pp}, since the collision terms integrate to zero) they rearrange (2.16) using (2.26) and obtain Eq. (2.32) of KCI:

$$-(z_{pp} - \cos\theta)\nu(\cos\theta) = \tilde{\omega}u\rho' + v_s\cos^2\theta + (\tilde{\omega} - z_{pp})(\nu_0 + \nu_1\cos\theta) \ .$$

Here $\tilde{z}_{pp} = [1 - i(\omega\tau_{pp})^{-1}]$. (Note that $v_s = j_r - (\rho_n/\rho)w_r$, $\nu_1 = -w_r$, $\tilde{z}_{pr} \to \tilde{z}_{pp}$, and $\beta \to 0$ account for the difference between the above form and that which actually appears in KCI.) They describe how they obtain their integrals of the Boltzmann equation: "We multiply the left- and right-hand sides of Eq. (2.32) first by unity and then by $(\tilde{z}_{pp} - \cos\theta)^{-1}$ and average in each case over $\cos\theta$." The first step, when done without rearrangement, does not produce momentum conservation. Both of the resulting equations depend on τ_{pp}, which should not be so. By rearranging, one loses track of the condition that the collision integral should not contribute to these integrals of motion.

In Disatnik's single-collision-time method [Eq. (2.3) of D], the analog to KCI (2.8) is

$$\left(\omega + \frac{i}{\tau} - kv_p\cos\theta\right)n_p'' + kv_p\cos\theta\,\frac{\partial n_p}{\partial\epsilon_p}\left(\frac{\partial\epsilon}{\partial\rho}\rho' + pv_s\cos\theta\right) = \frac{i}{\tau}\left[\left(\int n_p''\epsilon_p d\tau_p \Big/ \int \frac{\partial n_p}{\partial\epsilon_p}\epsilon_p^2 d\tau_p\right)\frac{\partial n_p}{\partial\epsilon_p}\epsilon_p + 3\left(\int n_p'' p\cos\theta\, d\tau_p \Big/ \int \frac{\partial n_p}{\partial\epsilon_p}p^2 d\tau_p\right)\frac{\partial n_p}{\partial\epsilon_p}p\cos\theta\right] \ .$$

His integrals of motion are obtained by multiplying both sides of D (23) first by $\epsilon_p/[c^2\rho(\omega - kv_p\cos\theta)]$ and next by $p\cos\theta/[c^2\rho(\omega - kv_p\cos\theta)]$ and then integrating both over momentum space. The first integral thus obtained is similar to the momentum integral of KC, but the second integral differs from that of KC. The resulting equations depend upon τ. As has been mentioned, this should not be the case.

Despite the fact that Disatnik and KC do not satisfy energy and momentum conservation exactly, they do obtain results for the sound velocities which are in agreement with superfluid hydrodynamics, to $O(\rho_n/\rho)$. This lends support to the use of collision-time models at low frequencies, where the works of Disatnik and KC differ. We see no reason to prefer one of these methods over the other. Neither method is exact, since it was necessary to simplify the equations in order to solve them.

It is important to note that only n_p' should appear in the collision terms of D(23), corresponding to relaxation to local equilibrium, rather than to static equilibrium. A proper discription of relaxation to local equilibrium is essential to a correct calculation of the transport coefficients, and to a correct kinetic theory. Disatnik's use of $n_p' + \delta n_p^{(1e)}$ in the collision term, rather than n_p', is responsible for his large first-sound attenuation. Most likely this is also the case with the theory of KC; however, their calculation is less readily analyzed than that of Disatnik. For a simple treatment indicating the importance of relaxation to local, as opposed to static equilibrium, the reader is referred to Ref. 34. This contains a calculation of the second viscosity for ordinary fluids.

V. COLLISION TIMES AND ANALYSES OF EXPERIMENTS

Many collision times have been calculated for helium, so it is understandable that some confusion may exist as to which of these times determines the regime for which hydrodynamics should cer-

tainly be valid. Since one must know the meaning of each collision time in order to use them properly in any collision-time kinetic theory, we present a discussion of some of these collision times.

For example, the three-phonon process involving the scattering of two nearly collinear thermal phonon (and the inverse process) has recently been calculated (for dispersionless phonons) by Jäckle to be[26]:

$$\tau_3^{-1} = 4.0 \times 10^9\, T^5 \text{ sec}^{-1}. \tag{37}$$

For $T = 0.5$ K this yields $\tau_3^{-1} \approx 10^8$ sec^{-1}, so that $\nu \lesssim 10^7$ Hz should be hydrodynamic if this is the proper collision time. In fact, the experiments[5-14] do not give an attenuation proportional to ω^2, as would occur were such frequencies to lie in the hydrodynamic regime.

Another collision time is that for small-angle phonon-phonon scattering. The original calculation of Landau and Khalatnikov[15] has recently been found to be in error.[35] With $c = 2.38 \times 10^4$ cm sec^{-1} and $u = 2.8 \times 10^{37}$ cgs, one can find from Eckstein's results that

$$\tau_4^{-1} \approx 9.3 \times 10^{44}\, T^7/\gamma \text{ sec}^{-1}, \tag{38}$$

where T is in deg. K, and γ is in cgs units. For $T = 0.5$ K and $\gamma \approx 10^{35} - 10^{37}$ cgs this yields $\tau_4^{-1} \approx 10^6 - 10^8$ sec^{-1}, so that $\nu \lesssim 10^5 - 10^7$ Hx should be hydrodynamic if this is the criterion. The above process causes energy equilibration along the scattering direction. It is a second-order process involving the three-phonon interaction.

Another process is the absorption of three phonons with the emission of two (and the inverse process), which involves only nearly collinear phonons. This process causes number equilibration in a given direction. It is discussed in Ref. 3(a).

The difficulty with the processes discussed above is that all of them involve nearly collinear phonons. To cause equilibrium with phonons moving perpendicularly, many such scattering processes must occur. For example, if $\Delta\theta$ is the angular scattering of thermal phonons involved in the three-phonon process, then on the order of $(\pi/2\Delta\theta)^2$ random three-phonon processes will cause equilibration of perpendicularly moving phonons. Since $\Delta\theta \ll 1$, the three-phonon equilibration time becomes on the order of $\tau_3(\pi/2\Delta\theta)^2 \gg \tau_3$, which shows how ineffective such a process can be at causing equilibration. Similar remarks hold for the other processes we have already discussed.

It should be apparent that equilibrium must be caused by wide-angle scattering. Such a collision process has been studied by Landau and Khalatnikov,[15] and is responsible for producing equilibrium in the phonon gas when subjected to a transverse disturbance. It is the result of two phonons scattering off one another (due to the same interaction that produces τ_4) at a large angle, and gives the collision time

$$\tau_{pp}^{-1} = \frac{9 \times 13!}{2^{13}} \frac{(K_B T)^9 (u+1)^4}{h^7 (\rho c^5)^2}$$

$$= 3.134 \times 10^7\, T^9 \text{ sec}^{-1}. \tag{39}$$

For $T = 0.5$ K, $\tau_{pp}^{-1} \approx 6 \times 10^4$ sec^{-1}. We will return to discuss some data which bear on τ_{pp}.

As the frequency increases, and one moves from the hydrodynamic regime (whose modes consist of the coherent amplitudes of many elementary excitations) to the collisionless regime (whose modes are dominated by a single elementary excitation), the mode "sharpens up" in momentum space. The small-angle scattering which would not cause attenuation from a broadened hydrodynamic mode has a dominant effect upon the collisionless mode. For this reason, in KCI (2.16) one should neglect J_{pp}, rather than J'_{pp}, at high frequencies. The τ for a mode in the collisionless regime is its lifetime, and is expected to be frequency dependent. Since the three-phonon process is generally the most rapid, τ is largely determined by this process. It has been discussed by many authors.[17-19,22-29]

The above comments have relevance to the attenuation and velocity measurements of the Argonne group.[10,11,13,14] In interpreting their data, they use $\tau = \tau_{pp}$ in the equation describing attenuation due to the three-phonon process:

$$\alpha(\omega, T) = \frac{\pi^2}{30} \frac{(u+1)^4}{\rho} \frac{K_B^4}{h^3 c^6} \times \omega T^4 [\tan^{-1}(2\omega\tau) - \tan^{-1}(3\gamma \bar{p}^2 \omega\tau)], \tag{40}$$

where $\bar{p} \equiv 3K_B T/c$. In this equation, which has been derived using many-body theory,[24,25] τ represents the lifetime of a thermal phonon. Hence one should take $\tau = \tau_3$ in the above equation. One should do likewise in the equation for the velocity shift as a function of frequency and temperature. With regard to the latter, the theory of Andreev and Khalatnikov[21] for the velocity of collisionless phonons is expected to be valid when $3\gamma\bar{p}^2\omega\tau \gg 1$. In that case the energy broadening of thermal phonons is insufficient to overcome the energy deficit for the three-phonon process to occur. With $\tau = \tau_{pp}$, it was believed that $3\gamma\bar{p}^2\omega\tau \gg 1$ had been satisfied in the velocity measurements at Argonne.[13] leading to discrepancies with the work of Andreev and Khalatnikov. However, with $\tau = \tau_3$, we find that $3\gamma\bar{p}^2\omega\tau \gg 1$ has not been satisfied in the Argonne experiments, so the theory for the collisionless regime has not yet been put to the test of experiment.

Returning to the question of attenuation, the use of $\tau = \tau_3$ has particular relevance to the measurements at high pressure, where the attenuation is suppressed. We note that the specific-heat mea-

surements of Phillips *et al.*[32] have indicated that γ goes from a negative value to a large positive value (−4.1 to 19.6, in units of 10^{37} cgs) as the helium goes from saturated vapor pressure to near the freezing pressure. Now the lifetime of thermal phonons is determined by the three-phonon process, which again depends upon the thermal-phonon lifetime. Clearly, there is a certain self-consistency involved. If γ increases, the lifetime increases, so the attenuation decreases, so the lifetime increases, so the attenuation decreases, etc. This will cause small changes in τ_2 if γ is not too large. For γ large enough a small increase in γ will cause large changes in τ_3, until the three-phonon process is suppressed in comparison with the four-phonon and other processes. Such a mechanism may be responsible for the suppression of phonon attenuation as the pressure (and therefore as γ) increases beyond a critical value (near 10 atm). As the temperature increases, phonon-roton scattering grows in importance, and the attenuation increases sharply. Related, but somewhat different comments have been made by Jäckle and Kehr, who have made a detailed study.[36] We note that τ_{pp} shows no such dramatic dependence on pressure.

It is noted the above attenuation formula has been derived by a kinetic-theory treatment for $\omega\tau \gg 1$.[37] The same criticisms can be made of this treatment as were made in the case of KC and Disatnik. However, collisions are not nearly so important for this frequency regime, so that it may well be that a more accurate collision-time kinetic-theory treatment will yield the same result in this region. One might think that the phenomenological collision time τ in such an equation should correspond to $\tau_3(\omega)$, the lifetime (due to the three-phonon process) of an excitation of frequency ω. However, it is clear from the many-body derivation that one must use $\tau_3(K_BT) \equiv \tau_3$. This puzzling result [that one must use τ_3 rather than $\tau_3(\omega)$] is probably due to the simplified nature of the collision-time approximation. Therefore collision-time models should be used with care. Accepting, however, that one must use τ_3 in a collision-time model for phonons with $\omega\tau_3 \gg 1$, this still leaves the difficult problem of determining the phenomenological collision time for $\tau_3^{-1} \leq \omega \leq \tau_{pp}^{-1}$. Perhaps one can find some physical argument to guide an interpolation between the hydrodynamic and collisionless regimes.

We now discuss some work relevant to τ_{pp}, since at first glance it would seem that Whitworth's heat-flow experiment[38] casts doubt upon the validity of the calculation. We wish to show that interpretation of Whitworth's data is a complicated affair, but we can provide an interpretation which supports the τ_{pp} calculation. We also present simple theoretical arguments to indicate that the temperature dependence of the τ_{pp} calculation is correct, but that its coefficient is incomplete.

That Whitworth was measuring a property of He II itself was manifested in the temperature-dependent mean-free path $\lambda(\sim T^{-4.3\pm0.3})$: Wall scattering would cause a temperature-independent mean free path.[39] Since τ_p varies as T^{-9}, and since $\tau_3(\sim T^{-5})$ is not associated with transport processes (and, moreover, is much too short), Whitworth's data have been a puzzle.

We find the following argument suggestive. It is based upon our study of heat flow in the He II from one wall to another (i.e., heat flow perpendicular to the walls, as opposed to Whitworth's parallel geometry.)[1] Temperature gradients are found to damp out exponentially in a length proportional to the geometric mean of the (nonlocal) transport coefficients $[\frac{4}{3}\eta_n + (\zeta_2 + \rho^2\zeta_3 - 2\rho\zeta_1)]$ and κ. Assuming that the nonlocality and the geometry only produce numerical factors, this means that the damping length is proportional to the geometric mean of a viscous mean free path ($\sim c\tau_{pp}$) and a thermal conductivity mean free path. This latter is very long at low temperatures, and in Whitworth's case it is determined by boundary scattering. The geometric mean then goes as $(cd\tau_{pp})^{0.5} \sim d^{0.5}T^{-4.5}$. Taking the damping length to be proportional to the length measured by Whitworth, the exponent is within the experimental error. Unfortunately, there are insufficient data to determine the dependence upon d, although the larger tubes gave longer mean free paths.[38]

Although the above argument is by no means conclusive, we present it as evidence that there should be no simple relationship between τ_{pp} and Whitworth's data. (It should be mentioned that a theoretical treatment of the parallel heat-flow problem is considerably more difficult than the perpendicular heat-flow problem. This is primarily because viscous drag along the walls can no longer be neglected.) We therefore believe that no experiment has yet contradicted the calculation of τ_{pp}. Indeed, our simple analysis of the data supports the predicted $\tau_{pp} \sim T^{-9}$ behavior.

Since the related calculation of τ_4 has been found to be in error,[35] doubts have been raised as to the validity of the calculation of τ_{pp}.[40] On purely dimensional grounds we believe that the temperature dependence predicted for τ_{pp} is correct, although we doubt the accuracy of the coefficient. Certainly, Ref. 15 does not include the effect of scattering via the four-phonon interaction, since nowhere does $w\,(\equiv(\rho^2/c)\partial^2 c/\partial\rho^2$, the four-phonon coupling constant, appear in τ_{pp} [Eq. (39)]. However, one can easily show that if only thermal phonon are involved the first-order four-phonon interaction and the second-order three-phonon interaction (for wide-angle scattering) both give contributions to the transition matrix element

V_{IF} proportional to T^2. The collision integral J involves multiplying $|V_{IF}|^2$ by energy and momentum δ functions and integrating over one initial and two final states. The momentum δ function reduces the problem to one of integration over two phase spaces, each of which give a factor T^3. In addition, the energy δ function gives a factor T^{-1}, so that the collision integral is proportional to T^9. Consequently, $\tau_{pp} \sim T^{-9}$, as obtained by Landau and Khalatnikov.

This concludes our discussion of collision times.

VI. EXPERIMENTAL PROSPECTS

From Sec. III we have seen that the velocities of first and second sound deviate from their zero temperature values by terms of order (ρ_n/ρ), which are quite small. Measurement of these velocities in the hydrodynamic regime may prove to be difficult, but such measurements have been made in the collisionless regime.[13] The primary interest in the second-sound velocity is that it provides a measure of γ, when the effect of phonon dispersion is included.

We now consider attenuation measurements. First sound will be weakly attenuated, with[26]

$$\alpha_1'(dB\ \mathrm{cm}^{-1}) = \frac{20}{\ln 10}\ \alpha_1(\mathrm{cm}^{-1}) = 8.69\ \alpha_1(\mathrm{cm}^{-1}).$$

For $\omega\tau_l \approx 1$, $\tau_l \approx \tau_{pp}$, and $T = 0.5$ K, one finds that $\alpha_1' \approx 10^{-5}\ dB\ \mathrm{cm}^{-1}$, $\alpha_2' \approx 10\ dB\ \mathrm{cm}^{-1}$. Hence it appears doubtful that the attenuation of first sound is measureable, although the attenuation of second sound appears much more promising. For example, Abraham *et al.*[13] measure attenuation to 10^{-2} dB cm^{-1}.

None of the above discussion will be of any help if it is not practical to generate hydrodynamic sound. There are spatial limitations of three kinds, which may be summarized as $d \gg \lambda \gg l$, where d is a typical experimental dimension, λ is the wavelength of the sound, and l is the phonon mean-free path. It is reasonable to take $l \approx c\tau_{pp}$ ($\gtrsim 0.4$ cm for $T \leq 0.5$ K). Consequently measurements by hydrodynamic second sound will require large chambers for propagation or for resonance. (We remark that viscous edge effects may complicate matters for second-sound attenuation, as discussed by Khalatnikov.[41] However, at low temperatures ρ_n is so small that this effect should be negligible.) The gravitational wave experiments planned jointly by Stanford and LSU, requiring the cooling of 3 tons of aluminum by immersion in helium, may provide large enough chambers for such measurements.[42]

Second-sound attenuation will provide τ_l, but only shear-wave measurements will determine τ_t. Webeler and Hammer have made low-temperature measurements of shear waves in He II, but their experiments were not conducted in the hydrodynamic regime.[43] Again, the solution is to have larger chambers and lower frequencies. Unlike the case of first and second sound, $\omega\tau_t \ll 1$ is not equivalent to $\mathrm{Re}(k)l \ll 1$. In the case of shear waves we have (with $\tau_t \approx \tau_{pp}$) $\mathrm{Re}(k)l = (10\omega\tau_+)^{1/2}$. Thus when frequency dispersion has grown important ($\omega\tau_t \approx \frac{1}{2}$), spatial dispersion already dominates [$\mathrm{Re}(k)l \approx 2$]. One needs somewhat larger chambers to study hydrodynamic shear waves than to study hydrodynamic longitudinal sound.

From the above discussion we conclude that low-temperature hydrodynamic measurements in helium will not be simple, most likely requiring a chamber with linear dimensions $\gtrsim 10$ cm. Nevertheless, such measurements are of great interest because they will provide experimental values for τ_l, τ_t, γ, and γ'. We note that so far contradictory experimental evidence exists as to the value of γ — the calorimetric evidence of Philips *et al.*[32] indicates that $\gamma \approx -4.1 \times 10^{37}$ cgs for pressures near the saturated vapor pressure, whereas the neutron scattering measurements of Woods and Cowley[44] (and the sum-rule analysis by Pines and Woo[45] give $\gamma \ll 10^{37}$ cgs. On the other hand, the theory of Eckstein and Varga[46] indicates that γ should be positive, as do the calculations of Sunakawa *et al.*[47]

VII. SUMMARY

We have shown how to construct a kinetic theory of He II at low temperatures which is in agreement with the predictions of superfluid hydrodynamics. It has been pointed out that one must ensure that the system relaxes to local equilibrium, rather than static equilibrium, in order to obtain the correct hydrodynamic transport coefficients.

To extend the theory to higher frequencies, one must not neglect $\partial n_{p'}/\partial_t$ and $\vec{v}_{p'} \cdot \vec{\nabla} n_{p'}'$ compared to $J[n_{p'}']$. Solution of the Boltzmann equation for $n_{p'}'$ now becomes more complicated, for one must now invert the operator $f_{pp'} + i(\omega - \vec{k} \cdot \vec{v}_p) n_p' \delta_{pp'}$. This formidable problem can be reduced considerably by making a collision-time approximation. The primary difficulty with such an approximation is that it provides us with no way to reliably estimate the errors thereby introduced. Because of its simplicity, however, it has been widely used. We have illustrated some of the difficulties of the method, which must be used with caution when applied to He II. We have pointed out, for example, the need for two different collision times τ_l and τ_t to describe longitudinal and transverse processes. We have also shown some of the difficulties of the collision-time model when phonon dispersion is included. We have not discussed how one calculates, from first principles, the relevant collision times, but we have shown the need for such a calculation of τ_l.[16] It is appropriate to mention that in applying the collision-time model to shear waves with

finite $\omega\tau$, we found that no solutions could be obtained for $\omega\tau \gtrsim 0.758$.[2] Therefore the off-diagonal elements of $f_{pp'}$ cannot be neglected in this calculation. Such effects may also manifest themselves in collision-time calculations for longitudinal waves, perhaps by affecting the choice of an appropriate collision time. The possibility should also be considered that the effects of neglecting these off-diagonal elements cannot be repaired, to the accuracy of the present experiments, by a judicious choice of the collision time.

In Sec. V we presented a short review of collision times for He II at low temperatures and discussed when certain of them were appropriate. We gave a qualitative argument for the use of a wide-angle collision time to define the hydrodynamic regime and a small-angle collision time to define the collisionless regime. On this basis we analyzed some experimental results. We argued that certain disagreements between theory and experiment can be explained, with the existing collision times, by reinterpretation of the data.

In Sec. VI we discussed the possibility of generating and measuring the properties of hydrodynamic modes in He II at low temperatures ($T \lesssim 0.6$ K). We determined that relatively large chambers would be needed, with linear dimensions $\gtrsim 10$ cm. Once generated, the attenuation of second sound should be measurable, yielding values for τ_l. The temperature dependence of the velocity of second sound may yield values for the dispersion parameters γ and γ'. In addition, hydrodynamic shear waves will yield values for τ_t. We conclude that much work remains to be done in this area, both experimentally and theoretically.

ACKNOWLEDGMENTS

I would like to thank Pierre Hohenberg for helpful discussions, and Harvey Gould for reading and commenting upon the first draft. I am indebted to I. M. Khalatnikov for pointing out an error in the original manuscript.

---

*Work supported by the National Science Foundation.

†Present address: Texas A & M University, College Station, Tex. 77843.

[1]W. M. Saslow, Phys. Letters 35A, 241 (1971).

[2]W. M. Saslow and J. Stocco (report of work prior to publication).

[3](a) I. M. Khalatnikov and D. M. Chernikova, Zh. Eksperim. i Teor. Fiz. 49, 1957 (1965) [Sov. Phys. JETP 22, 1336 (1966)]; (b) I. M. Khalatnikov and D. M. Chernikova, *ibid.* 50, 411 (1966) [*ibid.* 23, 274 (1966)].

[4]Y. Disatnik, Phys. Rev. 158, 162 (1967).

[5]C. E. Chase and M. A. Herlin, Phys. Rev. 97, 1447 (1955).

[6]W. M. Whitney, Phys. Rev. 105, 38 (1965).

[7]J. A. Newell and J. Wilks, Phil. Mag. 1, 588 (1956).

[8]K. Dransfeld, J. A. Newell, and J. Wilks, Proc. Roy. Soc. (London), A243, 500 (1958).

[9]W. A. Jeffers and W. M. Whitney, Phys. Rev. 159, 1082 (1965).

[10]B. M. Abraham, Y. Eckstein, J. B. Ketterson, and J. H. Vignos, Phys. Rev. Letters 16, 1039 (1966).

[11]B. M. Abraham, Y. Eckstein, J. B. Ketterson, and M. Kuchnir, Phys. Rev. Letters 19, 690 (1967).

[12]G. A. Waters, D. J. Watmough, and J. Wilks, Phys. Letters 26A, 15 (1967).

[13]B. M. Abraham, Y. Eckstein, J. B. Ketterson, M. Kuchnir, and J. Vignos, Phys. Rev. 181, 347 (1969).

[14]P. R. Roach, J. B. Ketterson, and M. Kuchnir, Phys. Rev. Letters 25, 1002 (1970).

[15]*Collected Papers of Lev D. Landau*, edited by D. T. Haar (Gordon and Breach, New York, 1965), pp. 494 and 511.

[16]S. Ma, J. Math. Phys. 12, 2157 (1971).

[17]K. Kawasaki, Progr. Theoret. Phys. (Kyoto) 26, 795 (1961).

[18]K. Dransfeld, Phys. Rev. 127, 16 (1962).

[19]T. O. Woodruff, Phys. Rev. 127, 682 (1962).

[20]I. M. Khalatnikov, Zh. Eksperim. i Teor. Fiz. 44, 769 (1963) [Sov. Phys. JETP 17, 519 (1963)].

[21]A. Andreev and I. M. Khalatnikov, Zh. Eksperim. i Teor. Fiz. 44, 2058 (1963) [Sov. Phys. JETP 17, 1384 (1963)]; J. Low Temp. Phys. 2, 173 (1970).

[22]S. Simons, Proc. Phys. Soc. (London) 82, 401 (1963).

[23]A. J. Legget and D. ter Haar, Physica 32, 1905 (1966).

[24]P. C. Kwok, P. C. Martin, and P. B. Miller, Solid State Commun. 3, 181 (1965).

[25]C. J. Pethick and D. ter Haar, Physica 32, 1905 (1966).

[26]J. Jäckle, Z. Physik 231, 362 (1970).

[27]C. Carroll, Phys. Rev. A 2, 497 (1970).

[28]R. Wehner and R. Klein, Phys. Rev. Letters 23, 1372 (1969).

[29]T. H. Cheung, Phys. Letters 32A, 211 (1970).

[30]I. M. Khalatnikov, *An Introduction to the Theory of Superfluidity* (Benjamin, New York, 1965).

[31]R. Kronig and A. Thellung, Physica 16, 678 (1950).

[32]N. E. Phillips, C. G. Waterfield, and J. K. Hoffer, Phys. Rev. Letters 25, 1260 (1970).

[33]W. M. Saslow (unpublished). See E. Feenberg [Phys. Rev. Letters 26, 301 (1971)] for a discussion of the $\gamma' pq$ term in ϵ_p.

[34]L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon, London, 1959), p. 304.

[35]S. G. Eckstein (unpublished).

[36]J. Jäckle and K. Kehr, Phys. Rev. Letters 27, 654 (1971).

[37]S. G. Eckstein, Y. Eckstein, J. B. Ketterson, and J. H. Vignos, in *Physical Acoustics*, edited by W. P. Mason and R. N. Thurston (Academic, New York, 1970), Vol. 6.

[38]R. W. Whitworth, Proc. Roy. Soc. (London) 246, 390 (1958).

[39]H. A. Fairbank and J. Wilks, Proc. Roy. Soc. (London) A231, 545 (1955).

[40]J. B. Ketterson (private communication).

[41]I. M. Khalatnikov, Zh. Eksperim. i Teor. Fiz. 22, 687 (1952); Usp. Fiz. Nauk 60, 69 (1956).

[42]W. O. Hamilton (private communication).

[43]R. W. H. Webeler and D. C. Hammer, Phys. Letters

19, 533 (1965).
[44]A. D. B. Woods and R. A. Cowley, Phys. Rev. Letters 24, 646 (1970).
[45]D. Pines and C. W. Woo, Phys. Rev. Letters 24, 1044 (1970).
[46]S. Eckstein and B. B. Varga, Phys. Rev. Letters 21, 1311 (1968).
[47]S. Sunakawa, S. Yamasaki, and T. Kebukawa, Progr. Theoret. Phys. (Kyoto) 41, 919 (1969); T. Kebukawa, S. Yamasaki, and S. Sunakawa, *ibid.* 44, 565 (1970).

PHYSICAL REVIEW A VOLUME 5, NUMBER 3 MARCH 1972

Spectroscopic Study of the Early Afterglow of a Recombining Helium Plasma

J. Stevefelt*
AB Atomenergi, Studsvik, Fack, S-61101 *Nyköping, Sweden*

and

F. Robben†
Department of Mechanical Engineering, University of California, Berkeley, California 94720

(Received 14 June 1971)

The afterglow of a 1.3-A 1.5-μsec-duration discharge in helium at 11 Torr was studied in some detail during the times 15 to 35 μsec after the discharge pulse. Spectroscopic measurements were used to obtain the number densities of excited atomic and molecular states, the conductance of the plasma column was determined from simultaneous electric field and current measurements, and a 10-μsec current pulse was used to selectively heat the electrons, thus disturbing some of the afterglow processes. The atomic and molecular ion densities and the electron temperature, obtained from the spectroscopic measurements, were in good agreement with the plasma conductance and field strengths. The inferred recombination rate favors the recent calculations of Mansbach and Keck over the Bates, Kingston, and McWhirter calculations of the collisional radiative-recombination rate. The rate of conversion of atomic into molecular ions was dominated by associative ionization of excited atomic states, and good agreement was obtained by including this process with other known processes.

I. INTRODUCTION

The decaying afterglow of electrical discharges in helium has been the subject of a very large number of experimental and theoretical investigations. In the last ten years we have collected almost 100 published papers dealing either directly with the helium afterglow or with processes which are of particular importance. Although the level of understanding has certainly increased as a result of all this labor and speculation, it is disconcerting to realize how much is still unknown.

For low-pressure afterglows ($p < 0.1$ Torr) the so-called collisional radiative-recombination model[1–3] is now generally considered to be a correct representation,[3–17] and only the rate constants in this model, and the finer details of the fate of the principal-quantum-number-two atomic states, intimately connected with the electron energy balance, need further clarification. For medium-pressure afterglows ($0.1 < p < 100$ Torr), where the molecular helium ion becomes important, an appropriate model which completely encompasses the observed phenomena has yet to be found.[8,18–28] However, a number of the experimental observations of afterglows in this pressure range have been explained reasonably satisfactorily. For high-pressure afterglows ($p > 100$ Torr) there have so far been only a few experimental measurements,[29–31] which indicate a very complex situation that is only beginning to be understood. Some preliminary measurements on pulsed and steady-state low-current arcs at near atmospheric pressure, where the molecular ion concentration is small, also showed unexplained phenomena.[32]

One concludes that useful experiments are very difficult to perform, and that as a corollary there are numerous errors in the published papers. A good review of the situation would be invaluable as an aid in determining what future work should be the most useful, and with what degree of care measurements must be performed. Such a review would be a quite difficult undertaking which is not attempted in the present paper; we have only cited some representative references to accompany the preceding brief comments.

The work presented in this and the following paper was initiated as a result of a proposed type of magnetohydrodynamic generator which would use an auxiliary ionization source to partially ionize the helium working gas, and would depend on a relatively slow recombination rate to maintain a reasonable electrical conductivity in the generator channel.[33,34] Thus a study of the recombination mechanisms and