Orbital Susceptibility of an Interaction Electron Gas

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The wave-vector and frequency-dependent orbital susceptibility (χ^{orb}) of an interacting electron gas is expressed in terms of suitable vertex functions. This makes the theory parallel to that of the spin susceptibility (χ^{sp}) of this system. The equation for the vertex function is solved in the statically screened exchange approximation by a variational method introduced earlier by one of the authors. The static long-wavelength limit of χ^{orb} is shown to be related to the difference between the f- and p-wave decomposition of the effective interaction, whereas χ^{sp} is related to the difference between the corresponding p- and s-wave parts in the same limit. The classic result that χ^{orb} is minus one-third of χ^{sp} for the noninteracting system is modified when the interactions are included. Explicit results are given for a model Yukawa interaction. From these, it follows that, for very short-range interactions, χ^{sp} reduces to the Stoner-enhanced form while χ^{orb} is *unaffected*. The momentum dependence of the interaction is thus more important for the determination of χ^{orb} than for χ^{sp}. In the unscreened Coulomb limit as well as for small screening, our results reduce to those obtained earlier by Kanazawa and Matsudaira. Several errors in the existing expressions for $\chi^{orb}(\vec{q}, q_0)$ are corrected in this work.

I. INTRODUCTION

It is well known that, for a system of noninteracting electrons, the steady orbital susceptibility is minus one-third of the spin susceptibility.[1] Much attention has been paid to the calculation of the wave-vector- and frequency-dependent spin susceptibility $\chi^{sp}(q)$ of an interacting electron system, whereas the same cannot be said of the corresponding orbital susceptibility $\chi^{orb}(q)$, as may be inferred from a recent review by Hebborn and March.[2] The spin susceptibility is enhanced by the interactions in the static long-wavelength limit. The corresponding orbital susceptibility has only been calculated in perturbation theory by Kanazawa and Matsudaira,[3] Stephen,[4] and, very recently, by Ishihara and Wadati,[5] who found a different form of the enhancement.

A unified treatment of the susceptibilities may be given by expressing them in terms of current-current correlation functions. These correlation functions have been formulated in terms of suitable vertex functions[6,7] Γ, and this description is used here to calculate the steady orbital susceptibility. A variational method of solution of the equation for Γ^{sp}, appropriate for computing χ^{sp} of an interacting system, was developed recently.[7] This technique will be used here to derive the analogous result for χ^{orb}. The equation for the relevant vertex function Γ^{orb} is set up in the statically screened exchange approximation. The variational method gives directly the real part of the susceptibility; the imaginary part may be inferred by an appropriate Kramers-Kronig relation.

The self-energy and vertex contribution played a crucial role in the earlier calculation of the long-wavelength spin-wave dispersion in an interacting ferromagnetic electron gas[7]; these play an even more interesting role in the computation of the χ^{orb}. For instance, the screening of the Coulomb interaction is essential to arrive at a nondivergent expression for the long-wavelength static χ^{orb}, whereas the χ^{sp} is finite even for an unscreened Coulomb gas for certain electron densities. More interesting is the result that in the perfectly screened limit—the Stoner model—the interaction contributes little to χ^{orb}, whereas χ^{sp} is the well-known Stoner-enhanced Pauli susceptibility. This aspect of the role of the momentum dependence of the electron interactions in determining χ^{orb} is an important conclusion of this work. The present approach puts the computation of χ^{orb} on the same footing as that of χ^{sp}, a feature absent in the previous perturbative treatments.[2]

In Sec. II we set up the relevant equations and solve them by a variational method. The spin susceptibility is also given here for the sake of completeness. We express the static long-wavelength limit of χ^{sp} and χ^{orb} in terms of the spherical wave decomposition of the screened Coulomb potential. In Sec. III the explicit expressions for the orbital and spin susceptibilities are given for a model Yukawa interaction. Section IV discusses the results and implications of this calculation to the Knight shift[8] in simple metals. Stated in terms of the language of Fermi liquids, it appears that the Knight-shift measurements in fact yield information about the f-wave part of the interaction, whereas

the spin-wave and other experiments yield only the s-, p- and sometimes d-wave parts. The general expression for $\chi^{\rm orb}(\vec{q}, q_0)$ contained in the literature[2] is found to be in error and the correct expression, due to Lindhard[9] (and Kadanoff and Martin[10]) is derived in the Appendix.

II. FORMULATION AND SOLUTION OF PROBLEM

The general formulation in coordinate space is given in Ref. 6 and we shall adopt it here without derivation. We quote here the expressions for a paramagnetic electron system with a nondegenerate spherical band. We have here excluded electron-phonon interaction from consideration even though its inclusion is straightforward as shown, for example, in Ref. 11. We choose the wave vector $\vec{q}$ to be $(0, q, 0)$ and the external magnetic field along the z axis[2] so that we have only to compute (for a derivation see the Appendix)

$$\frac{\chi_{zz}^{\rm orb}(q)}{1+\chi_{zz}^{\rm orb}(q)} = \frac{e^2}{m^2c^2q^2}\left[\frac{\tilde{K}_{xx}(q)}{1-(e^2/m^2)[\tilde{K}_{xx}(q)/(q_0^2-c^2q^2)]} - \left(\frac{mq_0}{q}\right)^2 \tilde{K}_{\rho\rho}\right], \quad (2.1a)$$

with

$$\tilde{K}_{xx}(q) = \left(-2\int k_x\, \mathfrak{F}(k;q)\, \tilde{\Gamma}_0(k;q)\, \frac{d^3k}{(2\pi)^3} - nm\right), \quad (2.1b)$$

$$\tilde{K}_{\rho\rho}(q) = 2\int \mathfrak{F}(k;q)\, \tilde{\Gamma}_\rho(k;q)\, \frac{d^3k}{(2\pi)^3}, \quad (2.1c)$$

$$\chi_{zz}^{\rm sp}(q) = -\frac{e^2}{2m^2c^2}\left(\int \mathfrak{F}(k;q)\, \tilde{\Gamma}_s(k;q)\, \frac{d^3k}{(2\pi)^3}\right), \quad (2.2)$$

where $\mathfrak{F}(k;q) = f_0(k+q) - f_0(k)$, $f_0(k)$ is the usual Fermi function, $\tilde{\Gamma}$ is an appropriate vertex function which, in the statically screened exchange approximation, obeys the equation (sometimes called ladder-bubble scheme)

$$[q_0 + i\eta - \Omega_k(q)]\, \tilde{\Gamma}_A(k;q) = \gamma_A(k;q) + \int V_s(k-k_1)\, \mathfrak{F}(k_1;q) \times [\tilde{\Gamma}_A(k_1;q) - \tilde{\Gamma}_A(k;q)]\, \frac{d^3k_1}{(2\pi)^3}, \quad (2.3a)$$

with

$$\gamma_A(k;q) = \begin{cases} -1, & A = s \\ +1, & A = \rho \\ -k_x, & A = 0. \end{cases} \quad (2.3b)$$

We have $\eta \to 0^+$ at the end of the calculation, where $\Omega_k(q) = \epsilon(k+q) - \epsilon(k)$ and $\epsilon(k) = k^2/2m$. In (2.3) we have isolated the self-energy contribution to $\Omega_k(q)$; it is the last term in the right-hand side of (2.3). The first term stems from the driving external field and the second term corresponds to the statically screened exchange contribution to the vertex. The bubble term is identically zero for the paramagnetic system where $A = s, \rho$ and hence is absent in (2.3). For $A = \rho$, Γ_A is the irreducible vertex function. Because of the statically screened exchange approximation it follows that the frequency dependence associated with k in (2.1), (2.2), and (2.3) is absent. We have $q = (\vec{q}, q_0)$ in all the above expressions. We shall not specify the structure of $V_s(k-k_1)$ so as to keep the formalism quite general as in Refs. 7 and 12.

The formula (2.1a) should be used to calculate $\chi_{zz}^{\rm orb}(\vec{q}, q_0)$ rather than that given in Ref. 2; the latter is correct only for $\chi_{zz}^{\rm orb}(\vec{q}, 0)$ to which our expression reduces in the appropriate limit. Our formula, besides incorporating a correct definition of $\chi_{zz}^{\rm orb}(\vec{q}, q_0)$, includes the contribution of the longitudinal polarization (ϵ_L) of the system to the magnetization $\vec{M}$. Also included is the proper screening of the transverse polarization (ϵ_T) which is often neglected as being of order $(v/c)^2$. As pointed out by Martin,[10] $(v/c)^2$ effects must be taken into account in computing $\chi^{\rm orb}(\vec{q}, q_0)$, since magnetic effects are always of this order. The static long-wavelength limit of our $\chi^{\rm orb}(\vec{q}, q_0)$ reduces to the Landau result without approximation.

The generic equation (2.3) can be solved by a variational method when $\tilde{\Gamma}_A$ is real, which is true in a limited region of $(\vec{q}, q_0)$. One then obtains the real part of $\chi^{\rm orb/sp}(q)$ (see Ref. 7 and, for a more recent discussion, see Ref. 13). The imaginary part can in principle be deduced from a dispersion relation (Kramers-Kronig). The variational principle proceeds by constructing a Lagrangian $J(\tilde{\Gamma}_A)$ whose first variation is Eq. (2.3). The extremum of $J(\tilde{\Gamma}_A)$ is then seen to be directly related to $\mathrm{Re}\chi^{\rm orb/sp}(q)$. Choosing a trial solution

$$\tilde{\Gamma}_A^{\rm trial}(k;q) = +\lambda_A \gamma_A(k;q)/[q_0 - \Omega_k(q)] \quad (2.4)$$

and determining the constant λ_A variationally, one arrives at the following expressions:

$$\mathrm{Re}\chi_{zz}^{\rm sp}(q) = \frac{e^2}{2m^2c^2}\left(\frac{I_s^2(q)}{I_s(q) - J_s(q)}\right), \quad (2.5a)$$

$$\mathrm{Re}\tilde{K}_{xx}(q) = \left(2\,\frac{I_0^2(q)}{I_0(q) - J_0(q)} - nm\right), \quad (2.5b)$$

$$\mathrm{Re}K_{\rho\rho}(q) = 2I_\rho^2(q)/[I_\rho(q) - J_\rho(q)], \quad (2.5c)$$

with

$$I_A(q) = \int \frac{\mathfrak{F}(k;q)}{q_0 - \Omega_k(q)}\, \gamma_A^2(k;q)\, \frac{d^3k}{(2\pi)^3}, \quad (2.6a)$$

$$J_A(q) = \iint \frac{\mathfrak{F}(k;q)}{q_0 - \Omega_k(q)}\, V_s(k-k_1) \left(\frac{\mathfrak{F}(k_1;q)}{q_0 - \Omega_{k_1}(q)}\right.$$

$$\times\gamma_A(k;q)\,\gamma_A(k_1;q)-\gamma_A^2(k;q)\,\frac{q_0-\Omega_{k_1}(q)}{q_0-\Omega_k(q)}\Bigg)\frac{d^3k\,d^3k_1}{(2\pi)^6}\ . \tag{2.6b}$$

These integrals can all be computed, at least in principle. The method of integration given here is for the static limit, but seems amenable to extension at least for some small region of $(\vec{q}, q_0)$. In fact, $I_A(q)$ has been evaluated for general q in the literature.[2] We will here derive expressions for $J_A(q)$ to leading order in $\vec{q}$ but with $q_0=0$. Before this, a few remarks concerning the methods employed by other authors may be made here. Kanazawa and Matsudaira's procedure is to evaluate the orbital susceptibility by expressing it in terms of a suitable two-particle Green's function; they then compute this two-particle Green's function to leading order in V_s. This procedure is equivalent to an iterative solution of (2.3) for the vertex function. Stephen[4] and Ishihara and Wadati[5] compute the free energy of the system in the presence of the magnetic field and derive from it the susceptibility. This procedure is perturbation theoretic in its structure. Our procedure presented here thus differs from all these. The expression similar to (2.5a) for the paramagnetic spin susceptibility was given by Iwamoto and Sawada[14] and generalized by one of the present authors[7] in the present form for the ferromagnetic case. It must be stressed that in the $q\to 0$ limit, the vertex equation can be solved exactly[11,12] and this coincides with the variational answers.

We now proceed to calculate the various integrals in (2.5a)–(2.5c) in the static long-wavelength limit at zero temperature. The calculation of (2.6a) is straightforward and the result is

$$I_s(q\to 0)\simeq\int\left(k^2\,\frac{dk}{d\epsilon(k)}\right)\delta(\epsilon(k)-E_F)\,\frac{d\epsilon(k)}{2\pi^2}$$

$$=\rho_0(E_F)=I_\rho(q\to 0), \tag{2.7a}$$

$$I_0(q\to 0)\simeq\tfrac{1}{2}mn-\rho_0(E_F)\,q^2/12\ , \tag{2.7b}$$

where $\rho_0(E_F)\simeq mk_F/2\pi^2$ is the density of states of the noninteracting system at the Fermi surface. It should be pointed out that this is a consequence of the separation of the self-energy effects from the noninteracting part in our vertex equation; a procedure which does not employ this separation is given, for example, in Appendix C of Ref. 11, yielding the same final answers. It is found that the renormalization of the density of states and the vertex corrections appear in exactly the way they have occured in (2.5a)–(2.5c).

To compute $J_A(q)$, we proceed as in Ref. 7. We expand $V_s(\vec{k}\text{-}\vec{k}_1)$ in terms of spherical harmonics:

$$V_s(\vec{k}\text{-}\vec{k}_1)=\sum_{lm}\frac{4\pi}{(2l+1)}\,Y_{lm}(<kq)$$

$$\times\,Y_{lm}^*(<kq)\;V_s^{(l)}(k;k_1), \tag{2.8}$$

with $<kq$ denoting the angle between $\vec{k}$ and $\vec{q}$ and

$$V_s^{(l)}(k;k_1)=\tfrac{1}{2}(2l+1)\int_{-1}^{+1}V_s(|\vec{k}-\vec{k}_1|)$$

$$\times P_l(\mu_{kk_1})\,d\mu_{kk_1}. \tag{2.9}$$

One then immediately obtains $J_s(q)$ for $q\to 0$:

$$J_s(q)=J_\rho(q)\simeq-\rho_0^2(E_F)\left[\tfrac{1}{3}V_s^{(1)}(k_F;k_F)-V_s^{(0)}(k_F;k_F)\right]$$

$$\equiv+\rho_0(E_F)\,J_s^{(0)}. \tag{2.10}$$

Thus we have

$$\chi_{zz}^{sp}(q)\simeq\chi_p^{sp}/[1-J_s^{(0)}], \tag{2.11}$$

with

$$\chi_p^{sp}=e^2k_F/4\pi^2mc^2.$$

The computation of $J_0(q)$ is a little more involved. We first shift $\vec{k}, \vec{k}_1$ to $\vec{k}-\frac{1}{2}\vec{q}$, $\vec{k}_1-\frac{1}{2}\vec{q}$, respectively, in (2.6b) [recall that $\vec{q}=(0,q,0)$] and then use the expansion

$$\frac{\mathcal{F}(k;q)}{-\Omega_k(\rho)}\cong\delta(\epsilon(k)-E_F)+\frac{1}{8}\,\frac{q^2}{m}\,\left[\delta'(\epsilon(k)-E_F)\right.$$

$$\left.+\tfrac{2}{3}\,\epsilon(k)\cos^2\theta_{<kq}\,\delta''(\epsilon(k)-E_F)\right]+O(q^4), \tag{2.12}$$

where the primes denote differentiation with respect to energy $\epsilon(k)$. Writing $x=\epsilon(k)$, $y=\epsilon(k_1)$ and $\theta_{<kq}=\theta_k$, $\theta_{<k_1q}=\theta_{k_1}$, etc., we obtain

$$J_0(q)\simeq(m^4/\pi^4)\int_0^\infty x\,dx\int_0^\infty y\,dy\;V_s(k-k_1)\,[d\Omega_k\,d\Omega_{k_1}/(4\pi)^2]$$

$$\times[\sin\theta_k\cos\phi_k\sin\theta_{k_1}\cos\phi_{k_1}-\sin^2\theta_k\cos^2\phi_k\;(\cos\theta_{k_1}/\cos\theta_k)]$$

$$\times\{[\delta(x-E_F)+(q^2/8m)(\delta'(x-E_F)+\tfrac{2}{3}x\cos^2\theta_k\delta''(x-E_F))](\delta(y-E_F)+(q^2/2m)\,\delta'(y-E_F))$$

$$+(q^2/12m)\,y\;\delta(x-E_F)\;\delta''(y-E_F)\cos^2\theta_{k_1}+O(q^4)\}\ . \tag{2.13}$$

We now perform the Ω_{k_1} integration after using the various well-known relationships between the $\sin\theta_k\cos\phi_k$, $\cos\theta_k$, etc., with the spherical harmonics and then perform the summation over lm. This makes the entire contribution from the terms grouped as $[\cdots]\,(\cdots)$ in $\{\cdots\}$ zero, and the second one becomes simply

$$J_0(q)\simeq\frac{\rho_0(E_F)q^2m}{360\pi^2}\,(2mE_F)^{1/2}\,\left[\left(\frac{d^2}{dx^2}-\frac{d}{dx}\right)\right.$$

$$\times \{x^4[\tfrac{1}{7} V_s^{(3)}(k_F; k_F x) - \tfrac{1}{3} V_s^{(1)}(k_F; k_F x)]\}\Big]$$

$$\equiv -\tfrac{1}{12} \rho_0(E_F) q^2 J_0^{(2)} \ . \qquad (2.14)$$

Using (2.7b), (2.14), and (2.5b) in (2.1a) we obtain

$$\chi_{zz}^{\text{orb}}(q \to 0) = -\tfrac{1}{3} \chi_p^{\text{sp}}[1 + J_0^{(2)}] \ . \qquad (2.15)$$

From (2.11) and (2.15) it is at once clear that the interactions play different roles in their contribution to the spin and orbital susceptibility for $q \to 0$ even though for finite q they seem to have similar structure.

It may also be stated that the present variational method reduces to known results in all the problems where it has been applied.

III. EXPLICIT RESULTS FOR YUKAWA INTERACTION

To make the above results quantitative we now evaluate $J_s^{(0)}$, $J_0^{(2)}$ for the Yukawa potential

$$V_s(k) = 4\pi e^2/(k^2 + \xi^2 k_F^2). \qquad (3.1)$$

ξ here is dimensionless and considered as a parameter, as in Refs. 7 and 12. Then

$$\frac{1}{(2l+1)} V_s^{(l)}(k; k_1) = \frac{2\pi e^2}{k k_1} Q_l\left(\frac{k^2 + k_1^2 + \xi^2 k_F^2}{2k k_1}\right), \qquad (3.2)$$

where $Q_l(z)$ is the associated Legendre function. After some algebra one then obtains

$$J_s^{(0)} = (\alpha r_s/\pi)\,[1 - \tfrac{1}{4}\xi^2 \ln(1 + 4/\xi^2)], \qquad (3.3a)$$

$$J_0^{(2)} = (\alpha r_s/\pi)\,[\tfrac{2}{3} - \tfrac{1}{8}\xi^4 + \tfrac{1}{24}\xi^2 (8 + 12\xi^2 + 3\xi^4)/(4 + \xi^2) - (\tfrac{1}{6} + \tfrac{1}{4}\xi^2) \ln(1 + 4/\xi^2)]. \qquad (3.3b)$$

Here r_s is the usual parameter $me^2/k_F = \alpha r_s/\pi$; $\alpha = (4/9\pi)^{1/3} = 0.521$. In the unscreened Coulomb case $\xi \to 0$ and we note that

$$J_0^{(0)} \to \alpha r_s, \quad J_0^{(2)} \to -\infty. \qquad (3.4)$$

On the other hand, when the screening is nonzero but small, we have

$$J_s^{(0)} \to (\alpha r_s/\pi)(1 + \tfrac{1}{4}\xi^2 \ln \tfrac{1}{4}\xi^2),$$
$$J_0^{(2)} \to (\alpha r_s/\pi)(\tfrac{2}{3} + \tfrac{1}{6} \ln \tfrac{1}{4}\xi^2). \qquad (3.5)$$

These features were noticed by Kanazawa and Matsudaira.[3] When the screening is large, $\xi \to \infty$, one has a short-range scheme as in the Stoner model[7,12], and it is then useful to introduce the dimensionless Stoner parameter[7,12]

$$K\theta'/E_F = \tfrac{4}{3}(\alpha r_s/\pi\xi^2), \qquad (3.6)$$

and we then obtain

$$J_s^{(0)} \to \tfrac{3}{2}(K\theta'/E_F),$$
$$J_0^{(2)} \to (1/\xi^2)(K\theta'/E_F) \to 0. \qquad (3.7)$$

These results coincide with the exact answers which can be obtained by directly solving (2.3) in this limit. This calculation shows that in this model, while the paramagnetic susceptibility is enhanced by short-range interactions, the orbital susceptibility is not at all affected by it.

IV. DISCUSSION OF RESULTS

The main results of the paper are contained in expressions (2.1a), (2.5), and (2.15). Our calculation succintly brings to light several aspects of the role of interactions in these problems. The "enhancement factors" appearing in the spin-paramagnetic and orbital susceptibilities are *not* of the same form. The precise forms for the Yukawa interaction model are given in expressions (3.3a) and (3.3b). This may be expected on physical grounds since the spin-density fluctuation depends only on the difference in spin densities, while the current-density fluctuations depend on the rate of change of the densities. That these two are widely different in an interacting system is brought to light by our calculations.

Our calculation may be of importance in the estimation of the interaction effects on the Knight shift. The inclusion of electron-phonon interactions in our calculation is straightforward as for the paramagnetic case.[11] It is found that V_s has an extra contribution from it. In the Jellium model it is known (see, for instance, Ref. 11) that the paramagnetic susceptibility is unaffected by the electron-phonon interaction in the static long-wavelength limit and is essentially a result following from the use of Migdal's theorem. A similar conclusion may be reached for the orbital susceptibility in agreement with the earlier perturbation calculation of Tani.[15] All these approaches treat the electron-phonon interaction in the weak coupling limit where the dynamical aspects of this interaction do not appear. When these are included, the simplification that $\tilde{\Gamma}_A(k; q)$ is independent of k_0 does not hold and we have a more complicated situation. In fact, the incipient Cooper instability in the normal state lies buried in $\tilde{\Gamma}_A(k; q)$ and, as one would expect, the spin and orbital susceptibilities also exhibit this via $\tilde{\Gamma}_A$. It may be of interest to note that a Knight-shift experiment could give a measurement of the $l = 3$ part of $V_s^{(l)}$, a new feature of the present formulation.

We have only calculated the steady part of the orbital susceptibility. The inclusion of the de Haas–van Alphen oscillations can in principle be incorporated in our formalism by employing a suitable starting Green's function given in Ref. 6. The corresponding perturbation-theoretic results may be found in Refs. 4 and 16.

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APPENDIX

We give here a derivation of Eq. (2.1a). The essential steps are contained in the works of Lindhard,[9] and Kadanoff and Martin,[10] and they are presented here for the sake of completeness. Following Kadanoff and Martin, if $(\vec{A}_{ext}, \phi_{ext})$ are the external gauge potentials, one computes the linear response functions with respect to these as perturbations and then, using the Maxwell equations, relates them to the physical response functions. We use the notation of Ref. 2 and take $\vec{q} = (0, q, 0)$, $\vec{A}_{ext} = (A^x_{ext}, 0, 0)$ so that the x and z directions correspond to the transverse (T) and the y direction to the longitudinal (L) polarizations of any vector. The only nonzero linear response functions (in a homogeneous uniform gas) then are

$$\begin{aligned}\langle J^T(q)\rangle_{ind} &\equiv \langle J_x(q)\rangle_{ind} = K_{xx}(q)\, A^x_{ext}(q),\\ \langle J^L(q)\rangle_{ind} &\equiv \langle J_y(q)\rangle_{ind} = -K_{y0}(q)\,\phi^{ext}(q)\\ &\equiv \langle \rho(q)\rangle_{ind} = -K_{00}(q)\,\phi^{ext}(q).\end{aligned} \tag{A1}$$

The equation of continuity requires

$$q\langle J_y\rangle_{ind} = q_0\langle\rho\rangle_{ind}, \tag{A2}$$

so that

$$\langle J_y\rangle_{ind} = -(q_0/q)\, K_{00}(q)\,\phi^{ext}. \tag{A3}$$

We define quite generally the following:

$$\begin{aligned}\langle \vec{J}\rangle_{ind} &\equiv -iq_0(\underline{\epsilon}' - \underline{1})\,\vec{E}^{ext}\\ &\equiv -iq_0(\underline{\epsilon} - \underline{1})\,\vec{E}^{tot},\end{aligned} \tag{A4}$$

$$\vec{E}^{tot} = \langle \vec{E}^{int}\rangle + E^{ext}. \tag{A5}$$

Then the Maxwell equation combined with (A2) gives us

$$iq\langle E^{int}\rangle^L = \langle\rho\rangle_{ind} = (q/q_0)\,\langle J_y\rangle_{ind}.$$

Using (A5) we arrive at

$$\begin{aligned}\langle J_y\rangle_{ind} &= iq_0\, E_y^{tot} - iq_0 E_y^{ext}\\ &= -iq_0\left(1 - \frac{\epsilon'_L - 1}{\epsilon_L - 1}\right) E_y^{ext}\end{aligned} \tag{A6}$$

after employing (A4) as a relation between $\vec{E}^{tot}$ and $\vec{E}^{ext}$. Thus, we obtain

$$\epsilon_L - 1 = (\epsilon'_L - 1)/[1 - (\epsilon'_L - 1)]. \tag{A7}$$

In terms of K_{00} defined by (A1) we have

$$\epsilon'_L - 1 = (1/q^2)\,K_{00}. \tag{A8}$$

In a similar way, employing the Maxwell equations

$$\begin{aligned}(q_0^2 - c^2q^2)\, E_T^{tot} &= -iq_0\, J_T^{tot},\\ (q_0^2 - c^2q^2)\, E_T^{ext} &= -i\,q_0\, J_T^{ext}\end{aligned} \tag{A9}$$

and using (A4) for the transverse components we arrive at

$$E_T^{tot} = \left(1 - \frac{q_0^2}{q_0^2 - c^2q^2}\,(\epsilon'_T - 1)\right) E_T^{ext}, \tag{A10}$$

or equivalently

$$\epsilon_T - 1 = \frac{(\epsilon'_T - 1)}{1 - [q_0^2/(q_0^2 - c^2q^2)]\,(\epsilon'_T - 1)}\ . \tag{A11}$$

In terms of the linear response function (A1), we then have

$$\epsilon_T - 1 = \frac{(c/q_0^2)\, K_{xx}}{1 - cK_{xx}/(q_0^2 - c^2q^2)}\ . \tag{A12}$$

Now by definition, $\vec{E} = \vec{E}_L + \vec{E}_T$ and

$$\begin{aligned}\vec{J}^{tot} &= -iq_0(\epsilon_T - 1)\,\vec{E}_T^{tot} - iq_0(\epsilon_L - 1)\,\vec{E}_L^{tot}\\ &= -iq_0(\epsilon_L - 1)\,\vec{E}^{tot} - iq_0(\epsilon_T - \epsilon_L)\,\vec{E}_L^{tot}.\end{aligned} \tag{A13}$$

Using the other Maxwell equation,

$$\vec{E}_T^{tot} = (iq_0/q^2)\,(\nabla\times\vec{B}) \tag{A14}$$

we have

$$\begin{aligned}\vec{J}^{tot} &= -iq_0(\epsilon_L - 1)\,\vec{E}^{tot} + (q_0^2/q^2)\,(\epsilon_T - \epsilon_L)\,(\nabla\vec{B})_T\\ &\equiv -iq_0\,(\epsilon_L - 1)\,\vec{E}^{tot} + c(\nabla\times\vec{M})_T.\end{aligned}$$

We now employ the definitions

$$\vec{B} = \mu\vec{H},\ \vec{M} = \chi^{orb}\vec{H} \tag{A15}$$

to arrive at

$$\chi^{orb}/(1 + \chi^{orb}) = (1 - \mu^{-1}) = (q_0^2/cq^2)\,(\epsilon_T - \epsilon_L). \tag{A16}$$

This result was originally derived by Lindhard[9] who, however, did not use the correct relation (A12), having neglected the transverse screening effect. Martin[10] stresses the importance of this in his work and we find it to be crucial in deriving the proper extension of $\chi^{orb}(q, 0)$ to finite q_0. Previous work in the literature[2] contains three errors: the definition of χ^{orb}, calculation of ϵ_T, and omission of ϵ_L in (A16). In the static limit, however, these errors do not affect the results.

In terms of an irreducible vertex function, K_{00} can be expressed as

$$\begin{aligned}K_{00}(q, q_0) &= 2e^2\int \mathfrak{F}(k; q)\,\tilde{\Gamma}_\rho(k; q)\,\frac{d^3k}{(2\pi)^3}\Big/\epsilon_L(q)\\ &\equiv e^2\,\tilde{K}_{\rho\rho}(q)/\epsilon_L(q),\end{aligned} \tag{A17}$$

with

$$\epsilon_L(q) = 1 + 2V(q)\int \mathfrak{F}(k; q)\,\tilde{\Gamma}_\rho(k; q)\,\frac{d^3k}{(2\pi)^3}.$$

Similarly, we have

$$\begin{aligned}cK_{xx}(q, q_0) &= \frac{e^2}{m^2}\left(-2\int k_x\mathfrak{F}(k; q)\,\tilde{\Gamma}_0(k; q)\,\frac{d^3k}{(2\pi)^3} - nm\right)\\ &\equiv \frac{e^2}{m^2}\,\tilde{K}_{xx}(q).\end{aligned} \tag{A18}$$

Using these expressions in (A7) and (A11), in (A16) we arrive at the result (2.1a) quoted in the

text. We may stress that the choice of the gauge used above was only a calculational device and that the final results are indeed gauge independent. Another point to be mentioned is that when the spins are included for a paramagnetic system, the orbital and spin parts of the susceptibility decouple and thus can be calculated separately. Moreover, the definition (A15) is always employed in deriving χ^{sp}.

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Perturbation Theory Based on Currents for Interacting Bosons at Absolute Zero

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As a continuation of a recent paper, using density and current density as coordinates for interacting bosons at absolute zero, this work deals with the perturbation theory. The zeroth-order Hamiltonian is the one shown to yield the Bogoliubov energy spectrum. First- or second-order perturbation calculations have been made of the three-phonon and the four-phonon vertices for the energy corrections of the low-lying excited states. The results in the long-wavelength limit differ, although only slightly, from the quantum hydrodynamic calculations, and indicate that the perturbed excitation energy $\epsilon_{\vec{k}}$ is an odd-power series in k if the Fourier coefficients of the two-body potential concerned can be approximated by a constant or an even-power series in k. However, whether this remains so in all higher orders of various perturbations is an open question. Two methods were used in the calculation. The first involves functional representations of the coordinates in the $\rho_{\vec{k}}$ representation recently reported. The second method is new, and employs "occupation-number" representations of the current algebra by introducing creation and annihilation operators for "phonons" from density fluctuations. These latter representations enable us to benefit from the advantages of the canonical field theory.

I. INTRODUCTION

In a previous paper, [1] a new method of dealing with a pairwise interacting boson system at absolute zero was developed in which we employed the density and the current density as coordinates and the functional representations of their equal-time commutation algebra in the $\rho_{\vec{k}}$ representa-