Exact Results for a Quantum Many-Body Problem in One Dimension. II *

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We continue our previous investigation of the properties of a system of either fermions or bosons interacting in one dimension by a two-body potential $V(r)=g/r^2$ with periodic boundary conditions. A method is introduced which provides a physically intuitive description of the excited states. The energies are given explicitly for finite and infinite systems, including all corrections due to quasiparticle interaction. The method is shown to apply to other one-dimensional systems as well. In particular, the results are applied to a classical one-dimensional N-body model of Dyson—the Coulomb gas in a Brownian medium.

I. INTRODUCTION

This paper is a continuation of previous work, to be called I, [1] II, [2] and III, [3] respectively, on an interacting quantum many-body problem. This problem, the g/r^2 potential, has proven to be realistic, apart from the limitation of one dimension and the peculiarity of being exactly soluble. It is explicitly soluble in astonishing detail. To appreciate the extent to which it opens up, recall that the other nontrivial quantum many-body problem to be solved is the δ-function potential in one dimension. But in this case, the solution yields, so far, only the energy levels; missing are such interesting properties as correlation functions and momentum distributions. All of these are now available for the g/r^2 potential.

Such an explicit model allows one to use the results much as an experiment, to test our theoretical understanding of quantum fluids. One may test the various theoretical approximation procedures in common use, all of which are so involved as to make impossible exact determination of their convergence properties. We reserve such an investigation for later, however, first exploring the model to the fullest.

II. SOLUTION

We seek eigenfunctions Ψ with energy E of the Hamiltonian

$$H=-\sum_j \frac{\partial^2}{\partial X_j^2}+g\frac{\pi^2}{L^2}\sum_{j>l}\left(\sin\frac{\pi}{L}(X_j-X_l)\right)^{-2} . \tag{1}$$

We will write Ψ in the form

$$\Psi=\Psi_0\Phi , \tag{2}$$

where Ψ_0 is the ground-state energy E_0, as determined in Paper III. Considering Φ as a function of the variables

$$\theta_j=2\pi X_j/L , \tag{3}$$

it satisfies the equation

$$H'\Phi=(H_1+H_2)\Phi=\epsilon\Phi \tag{4}$$

with

$$H_1=-\textstyle\sum_j \partial^2/\partial\theta_j^2 , \tag{5}$$

$$H_2=-i\lambda\sum_{j>l}\left(\frac{e^{i\theta_j}+e^{i\theta_l}}{e^{i\theta_j}-e^{i\theta_l}}\right)\left(\frac{\partial}{\partial\theta_j}-\frac{\partial}{\partial\theta_l}\right) , \tag{6}$$

$$\epsilon=(L/2\pi)^2(E-E_0) . \tag{7}$$

Note that H' will not be Hermitian in general.

A. H' in a Basis of Free-Boson Eigenstates

We seek solutions to the eigenvalue equation (4) in a basis of free-boson eigenstates

$$\Phi_B\{n\}=\textstyle\sum_P\prod_j e^{in_j\theta_{Pj}} . \tag{8}$$

The quantum numbers are $\{n\}=\{n_1,n_2,\ldots,n_N\}$, which we order $n_1\le n_2\le\cdots\le n_N$. Then

$$H_1\Phi_B\{n\}=\epsilon_1\Phi_B\{n\} \tag{9}$$

with

$$\epsilon_1=\textstyle\sum n_j^2 . \tag{10}$$

Allow H_2 to act on $\Phi_B\{n\}$, and concentrate on the following term, assuming $k=n-m\ge 0$:

$$-i\lambda e^{i\Sigma'\theta_j n_{Pj}}\left(\frac{e^{i\theta}+e^{i\phi}}{e^{i\theta}-e^{i\phi}}\right)\left(\frac{\partial}{\partial\theta}-\frac{\partial}{\partial\phi}\right)\left[e^{i(n\theta+m\phi)}+e^{i(m\theta+n\phi)}\right]$$

$$=\lambda(n-m)e^{i\Sigma'\theta_j n_{Pj}}e^{im(\theta+\phi)}\left(\frac{e^{i\theta}+e^{i\phi}}{e^{i\theta}-e^{i\phi}}\right)(e^{ik\theta}-e^{ik\phi}) , \tag{11}$$

where the prime in the summation means the θ and ϕ terms are to be omitted. Using the fundamental identity derived in the Sec. II B, we write this term of Eq. (11) as

$$\lambda(n-m)\,e^{i\sum'\theta_j n_{Pj}}\{e^{i(n\theta+m\phi)}+2e^{i[(n-1)\theta+(m+1)\phi]} + \cdots + e^{i(m\theta+n\phi)}\} \quad . \tag{12}$$

Therefore, we have the result that $H_2\Phi_B\{n\}$ gives a diagonal element

$$\epsilon_2=\lambda\sum_{l>j}(n_l-n_j)=-\lambda\sum_l n_l(N+1-2l) \quad . \tag{13}$$

B. Fundamental Identity

Consider the quantity

$$(e^{i\theta}+e^{i\phi})\left(\frac{e^{ik\theta}-e^{ik\phi}}{e^{i\theta}-e^{i\phi}}\right) \quad , \quad k\geq 0 \quad . \tag{14}$$

Using the expression for the sum of a finite geometric series, this becomes

$$(e^{i\theta}+e^{i\phi})\{e^{i(k-1)\theta}+e^{i[(k-2)\theta+\phi]}+\cdots+e^{i(k-1)\phi}\} = e^{ik\theta}+e^{ik\phi}+2\sum_{l=1}^{k=1}e^{i[(k-l)\theta+l\phi]} \quad . \tag{15}$$

This is the fundamental identity used in Sec. II A to derive Eq. (12).

C. Matrix Elements of H_2

The operator H_2 acting on $\Phi_B\{n\}$ gives a contribution $\langle n'|H_2|n\rangle\,\Phi_B\{n'\}$ only if the set $\{n'\}$ can be produced from $\{n\}$ by squeezing together a pair $n_j<n_k$ into a pair $n_j'\leq n_k'$, where $n_j\to n_j'=n_j+l$, $n_k\to n_{k'}-l$, l an integer $0<l\leq\frac{1}{2}(n_k-n_j)$. The other n's are unchanged. (Note that the $\{n'\}$ obtained by this prescription may not be in proper order.) There is then a corresponding contribution $2\lambda\times(n_k-n_j)>0$ to $\langle n'|H_2|n\rangle$.

A free-boson state may also be specified by the occupancy $\nu(n)$ of the n wave numbers. We have $\nu(n)\geq 0$, $\sum\nu(n)=N$. Then the total matrix element $\langle n'|H_2|n\rangle$ for states connected as above will be

$$2\lambda\nu(n_k)\nu(n_j)(n_k-n_j)>0 \quad . \tag{16}$$

D. Ordering the Basis and Determining ϵ

It is clear that if $\langle n'|H_2|n\rangle\neq 0$, then $\langle n|H_2|n'\rangle=0$. Thus there is a partial ordering of the states $\{n\}$; a state $\{n\}$ will be said to lie below a state $\{n'\}$ in our basis if $\langle n'|H_2|n\rangle\neq 0$. Writing out H' in this ordered basis, we see that all elements below the diagonal vanish. We then immediately find the eigenvalues of H' to be the diagonal elements $\epsilon=\epsilon_1+\epsilon_2$, given by

$$\epsilon=\sum_j n_j^2+\lambda\sum_{l>j}(n_l-n_j) = \sum_j[n_j^2-\lambda n_j(N+1-2j)] \quad . \tag{17}$$

In terms of the occupancy $\nu(n)$, this becomes

$$\epsilon=\sum_n n^2\nu(n)+\frac{\lambda}{2}\sum_{n,m}\nu(n)\nu(m)|n-m| \quad . \tag{18}$$

Let us verify that the present result is the same as the expression in Ref. 2:

$$\epsilon=\sum k_j^2-\sum(k_j^0)^2 \quad , \tag{19}$$

where

$$k_j=m_j+\frac{\gamma}{2}\sum_l \mathrm{sgn}(k_j-k_l) \quad , \tag{20}$$

m_j being the free-fermion quantum numbers (m_j^0 being the ground state), i.e., unequal integers, ordered so $m_1<m_2<\cdots<m_N$. Thus the last term in Eq. (20) is simply $2j-N-1$, which is twice the corresponding quantum number m_j^0 for the ground state. We have

$$k_j=m_j+\gamma m_j^0 \quad , \quad k_j^0=\lambda m_j^0 \quad . \tag{21}$$

If we identify the fermion state $\{m\}$ with a boson state $\{n\}$ by

$$n_j=m_j-m_j^0 \quad , \tag{22}$$

then

$$k_j=n_j+\lambda m_j^0 \quad , \tag{23}$$

and

$$\epsilon=\sum_j[(n_j+\lambda m_j^0)^2-(\lambda m_j^0)^2] = \sum_j[n_j^2-\lambda n_j(N+1-2j)] \quad . \tag{24}$$

This agrees exactly with Eq. (17).

The question of which description, fermion or boson, is best or most useful remains. Certainly, for $\lambda=0$, the boson is best, while for $\lambda=1$, the fermion is best. (Remember that for $\lambda=1$, the eigenstates are just free-fermion wave functions divided by the ground-state free-fermion wave function.) Otherwise each description seems to have its usefulness. To summarize, the states are labeled by either free-boson quantum numbers $\{n\}$, or free-fermion quantum numbers $\{m\}$; the two are related by Eq. (22).

E. Algorithm for the Eigenvectors

Let us write H' as a matrix as follows. Choose a free-boson eigenvector $\Phi_B\{n\}$, and add all those, and only those, Φ_B' which are produced by successive application of H' on $\Phi_B\{n\}$. Label them $|j\rangle$, ordered as before so $j>l$ if $\langle l|H'|j\rangle\neq 0$. Thus, the highest index $|L\rangle$ is $\Phi_B\{n\}$ itself. In this basis, H' is a matrix,

$$\langle l|H'|j\rangle=\delta_{lj}\epsilon_l+h_{lj} \quad , \tag{25}$$

$$h_{ij} \begin{cases} = 0, & j \le l \\ \ge 0, & \text{otherwise.} \end{cases} \tag{26}$$

Then the eigenvector $\Phi\{n\}$ is that eigenvector of the above $L \times L$ matrix, none of whose components vanish.

Let us define the matrix M_{jl}:

$$M_{LL} = 1 \ , \quad M_{jl} = \frac{h_{jl}}{\epsilon_L - \epsilon_j} \quad \text{otherwise.} \tag{27}$$

We shall prove in Sec. II F that $\epsilon_j > \epsilon_l$ if $\langle l | H' | j \rangle \neq 0$, and hence $M_{jl} \ge 0$, and finite.

If ϕ_j are the components of $\Phi\{n\}$, then

$$\sum_l M_{jl} \phi_l = \phi_j \ . \tag{28}$$

Choose $\phi_L = 1$, then

$$\begin{aligned} \phi_{L-1} &= \sum_j M_{L-1,j} \phi_j = M_{L-1,L} \ , \\ \phi_{L-2} &= \sum_j M_{L-2,j} \phi_j = (M^2)_{L-2,L} \ , \end{aligned} \tag{29}$$

etc., or, to summarize,

$$\phi_j = (M^{L-j-1})_{jL} = (M^{L-1})_{jL} > 0 \ . \tag{30}$$

Clearly, in this way, we have given a procedure for determining $\Phi\{n\}$.

F. Theorem

We wish to prove that $\epsilon_j > \epsilon_k$ if $\langle k | H' | j \rangle \neq 0$. Let k be constructed from j by moving $n \to n - l$, $m \to m + l$, $n > m$, $\frac{1}{2}(n-m) \ge l > 0$.

Then

$$\begin{aligned} \epsilon_j - \epsilon_k &= n^2 + m^2 - (n-l)^2 - (m+l)^2 \\ &\quad + \lambda \sum_k{}' [\, |n-k| + |m-k| - |n-l-k| - |m+l-k| \,] \\ &\qquad \times \nu(k) + \lambda(n-m) - \lambda(n-m-2l) \\ &= 2l[n-m-l] + 2\lambda l + \lambda \sum_k{}' \nu(k) f(k) \ . \end{aligned} \tag{31}$$

This will be positive if the final sum is positive. But $2l \ge f(k) \ge 0$. Thus we have

$$\epsilon_j - \epsilon_k > 0. \tag{32}$$

G. Example

As an example, let us consider $N = 4$, $\{n\} = \{-2, 0, 0, 1\}$. Then $\langle n' | H' | n \rangle = 0$, except

$$\begin{aligned} \{n'\} &= \{n\} \equiv |3\rangle, & \langle 3 | H' | 3 \rangle &= 5 + 9\lambda \\ \{n'\} &= \{-1, -1, 0, 1\} \equiv |2\rangle, & \langle 2 | H' | 3 \rangle &= 8\lambda \\ \{n'\} &= \{-1, 0, 0, 0\} \equiv |1\rangle, & \langle 1 | H' | 3 \rangle &= 6\lambda \ . \end{aligned}$$

Then H' in this basis is the matrix

$$H' = \begin{vmatrix} 1+3\lambda & 8\lambda & 6\lambda \\ 0 & 3+7\lambda & 8\lambda \\ 0 & 0 & 5+9\lambda \end{vmatrix} \ .$$

The partial ordering is

$$\begin{array}{ccc} 3 & \to & 2 \\ & \searrow \swarrow & \\ & 1 & \end{array} \ .$$

The matrix M is

$$M = \begin{vmatrix} 0 & 4\lambda/(2+3\lambda) & 3\lambda/(2+3\lambda) \\ 0 & 0 & 4\lambda/(1+\lambda) \\ 0 & 0 & 1 \end{vmatrix} \ ,$$

$$M^2 = \begin{vmatrix} 0 & 0 & [\lambda(19\lambda+3)/(1+\lambda)(2+3\lambda)] \\ 0 & 0 & 4\lambda/(1+\lambda) \\ 0 & 0 & 1 \end{vmatrix} \ .$$

The last column of M^2 gives the components of $\Phi\{-2, 0, 0, 1\}$ according to Eq. (30).

H. Interpretation

The matrix form of H'—the fact that it is triangular with positive elements—invites one to view the action of H' as a relaxation process. However, since H' will not conserve probability, the two processes are not identical. Any state with momentum $2\pi K/L = 2\pi \sum n_j / L$ eventually relaxes, under a finite number of successive perturbations of H', each a two-body collision conserving momentum but not energy, to the state $\nu(0) = N - |K|$, $\nu(K/|K|) = |K|$, $\nu(n) = 0$, all other n. (We assume $|K| \le N$.) This state is an eigenstate of H', as well as of H_1, with

$$\epsilon = |K|\,[1 + \lambda(N - |K|)] \ . \tag{33}$$

The wave function is

$$\Phi = \sum_p e^{i[\theta_{P1} + \theta_{P2} + \cdots + \theta_{P|K|}]} \ . \tag{34}$$

The corresponding free-fermion state is a one hole state. By the theorem on the ordering of the energy levels, this state is the lowest-energy state for given momentum.

The energy levels outside the above momentum range are given by the following theorem: The spectrum of a one-dimensional system at density d is periodic in the momentum with period $2\pi d$.

III. OTHER PROBLEMS

We wish to briefly discuss how the methods introduced in Sec. II may be adapted to solve other related problems.

A. Scattering States

If one considers the particles interacting by a g/r^2 potential without periodic boundary conditions, but instead unconfined, i.e., the scattering states, then we may likewise analyze the problem. We seek eigenstates of the form

$$\Psi = \Psi_0 \Phi \ , \tag{35}$$

where

$$\Psi_0 = \prod_{i>j} |X_i - X_j|^\lambda, \quad X_i > X_j \text{ for } i > j \ . \tag{36}$$

Then Φ satisfies the equation

$$H'\Phi = E\Phi = \left[-\sum_j \frac{\partial^2}{\partial X_j^2} - 2\lambda \sum_{i>j} \frac{1}{X_i - X_j} \left(\frac{\partial}{\partial X_i} - \frac{\partial}{\partial X_j} \right) \right] \Phi \ . \tag{37}$$

We again seek Φ as a sum of free-boson eigenstates,

$$\Phi_B\{k\} = \sum_p \prod_j e^{ik_j X_{Pj}} \ , \tag{38}$$

and use instead of Eq. (15) the following fundamental identity:

$$\frac{\sin kr}{r} = \int_0^k dk' \cos k' r \ . \tag{39}$$

Then ordering our basis properly, H' will again have the triangular form of Sec. II. We find

$$E = \sum k_j^2 \ . \tag{40}$$

B. Harmonic Well

Sutherland originally considered the g/r^2 problem in a harmonic well $\omega^2 X^2$. The eigenstates are of the form

$$\Psi = \Psi_0 \Phi$$

with

$$\Psi_0 = \prod_{i>j} |X_i - X_j|^\lambda \prod_j e^{-1/2\,\omega X_j^2} \ . \tag{41}$$

Φ satisfies the equation

$$H'\Phi = \epsilon\Phi = (H_1 + H_2)\Phi \ , \tag{42}$$

where

$$\epsilon = E - E_0 = E - \omega N[1 + \lambda(N-1)] \ ,$$

$$H_1 = -\sum \frac{\partial^2}{\partial X_j^2} + 2\omega \sum X_j \frac{\partial}{\partial X_j} \ , \tag{43}$$

$$H_2 = -2\lambda \sum_{i>j} \frac{1}{X_i - X_j} \left(\frac{\partial}{\partial X_i} - \frac{\partial}{\partial X_j} \right) \ .$$

We now seek Φ as a sum of the free-boson eigenstates,

$$\Phi_B\{n\} = \sum_p \prod_i H_{n_i}(X_{Pi}\sqrt{\omega}) \ . \tag{44}$$

With $H_n(z)$ a Hermite polynomial, they are eigenstates of H_1.

Then the fundamental identity becomes

$$\frac{1}{X-Y} \left(\frac{\partial}{\partial X} - \frac{\partial}{\partial Y} \right) [H_n(X\sqrt{\omega}) H_m(Y\sqrt{\omega}) + H_n(Y\sqrt{\omega}) H_m(X\sqrt{\omega})]$$

$$= \sum_{l=0}^{n-1} \sum_{k=0}^{m-1} \alpha_{lk} [H_l(X\sqrt{\omega}) H_k(Y\sqrt{\omega}) + H_l(Y\sqrt{\omega}) H_k(X\sqrt{\omega})] \ . \tag{45}$$

This is true, although explicit formulas for α_{lk} are complicated. Since H_2 has no diagonal elements, ϵ is identical to the value for free bosons,

$$\epsilon = 2\omega \sum n_j \ . \tag{46}$$

C. Hard Core

One might like to make the correlation functions look more like those of real substances by introducing, in addition to the g/r^2 potential, a hard core of diameter a. This can be done, but at the cost of making the two-body potentials depend on the ordering of the particles.

The potential one uses is

$$V = \frac{g\pi^2}{(L-Na)^2} \sum_{i>j} \left\{ \sin\left(\frac{\pi[X_i - X_j - a(i-j)]}{L - Na} \right) \right\}^{-2} , \tag{47}$$

where

$$X_i > X_j \text{ for } i > j \ .$$

One can again investigate the problem in detail. For instance, the ground state will be

$$\Psi_0 = \prod_{i>j} \left\{ \sin\left(\frac{\pi[X_i - X_j - a(i-j)]}{L - Na} \right) \right\}^\lambda \tag{48}$$

with

$$\lambda = \tfrac{1}{2}[1 + (1 + 2g)^{1/2}] \ . \tag{49}$$

We save a complete discussion for a subsequent publication.

IV. DYSON'S BROWNIAN MOTION MODEL

A. Problem

In Dyson's original investigation[4] of the properties of random matrices, the square of our normalized ground-state wave function Ψ_0^2/D was identified as the equilibrium distribution for N classical point charges moving on a circular wire of circumference L. The charges repel each other according to the two-dimensional Coulomb law, so the potential energy is

$$W = -q^2 \sum_{i>j} \ln \left| \sin\left[\frac{\pi}{L}(X_i - X_j) \right] \right| \ . \tag{50}$$

The temperature is given by $\beta = 1/kT$. Thus D is the partition function Z for this system.

The dynamics of the system are introduced by interpreting the N particles as in Brownian motion,[5] in addition to the mutual interaction. Thus, if $P(x_1, \ldots, X_N; t)$ is the (unnormalized) probability density of the particles at time t, the time evolution is governed by the Fokker-Planck or Smoluchowski equation

$$f\frac{\partial P}{\partial t} = \sum_j \left\{ kT\frac{\partial^2 P}{\partial X_j^2} + \frac{\partial}{\partial X_j}\left[\frac{\partial W}{\partial X_j} P\right]\right\} , \tag{51}$$

where f is the friction coefficient. The unique time-independent solution of this equation is the equilibrium density

$$P_0 = e^{-\beta W} . \tag{52}$$

B. Eigenfunctions and Eigenvalues

We shall determine the eigenfunctions $P(\alpha)$ and eigenvalues α of Eq. (51), writing them in the form

$$P(\alpha) = e^{-\alpha t} P_0 \Phi(\alpha) . \tag{53}$$

α are inverse relaxation times for the normal modes of Eq. (51). Φ then satisfies the equation

$$\alpha\, \beta f \Phi = \left(-\sum_j \frac{\partial^2}{\partial X_j^2} + \beta\sum_j \frac{\partial W}{\partial X_j}\frac{\partial}{\partial X_j}\right)\Phi . \tag{54}$$

Using the expression, Eq. (50), for W, we find that this is identical to Eq. (4) if we make the identification

$$\lambda = \beta q^2/2 = q^2/2kT , \tag{55}$$

$$\alpha\beta f = (2\pi/L)\epsilon(\lambda) , \tag{56}$$

or

$$\alpha\{n\} = \frac{(2\pi)^2 kT}{fL^2}\sum n_j^2 + \frac{\pi^2 q^2}{fL^2}\sum_{j,l}|n_j - n_l| . \tag{57}$$

Then $\Phi\{n\}$ are eigenvectors of Eq. (54). The integers $\{n\} = \{n_1, n_2, \ldots, n_N\}$ label the eigenvectors and eigenvalues.

The first term in the expression for α is linear in kT, and would remain if there were no Coulomb interaction W. The second term is proportional to the charge squared, but independent of temperature. Thus as $T \to 0$ the second term, or interaction term, dominates the relaxation process.

C. Harmonic Brownian Problem

Dyson also considered the corresponding Coulomb gas on a line in a harmonic well.[5] The potential is then given by

$$W = \sum_j \omega^2 X_j^2 - q^2 \sum_{j>l} \ln|X_j - X_l| . \tag{58}$$

As noted in Ref. 1 the equilibrium density $p_0 = e^{-\beta W}$ is equal to Ψ_0^2 for a corresponding g/r^2 potential problem. Seeking eigenfunctions of the Fokker-Planck equation again of the form of Eq. (53), we find that Φ is the same as given by Eq. (42) of Sec. III. Thus the relaxation times are

$$\alpha = \frac{\omega}{f}\sum_j n_j , \tag{59}$$

n_j being integers ≥ 0. It is most surprising to find the spectrum of relaxation times to be unchanged by the mutual Coulomb interaction.

D. Eigenfunction Expansion

Note that in using these eigenfunctions $P(\alpha)$ to expand an arbitrary function of the particle coordinates, the proper scalar product to use is

$$(P(\alpha'), P(\alpha)) \equiv \int \cdots \int dx_1 \cdots dx_n P^*(\alpha') P(\alpha) P_0^{-1} , \tag{60}$$

for then they are orthogonal.

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