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Classical Laser*

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In this paper a completely classical model for laser action is discussed. An active medium consisting of classical anharmonic oscillators interacts with a classical electromagnetic field in a resonant cavity. Comparison with the case of a medium consisting of harmonic oscillators shows the significance of nonlinearities for producing self-sustained oscillations in the radiation field. The results for the classical model are found to be similar to those for a semiclassical model of the ammonia-beam maser. The conclusion is that laser action is not intrinsically a quantum-mechanical effect. The classical-laser theory as given in this paper can also be applied to the case of the electron-cyclotron maser.

I. INTRODUCTION

In recent years, there have been many advances in the theory of the laser. For a gas laser the active medium was treated as a quantum-mechanical ensemble of two-level atoms and the radiation as a classical electromagnetic field.[1] Scully and Lamb[2] have generalized this theory by treating both atoms and fields quantum mechanically. Other authors have given alternate formulations of this theory.[3] Results of these calculations have been in good agreement with experiments, and except for possible refinements, the understanding of laser theory appears to be satisfactory.

There is, however, a fundamental question still to be considered. Is the operation of the laser a result of quantum effects (an avalanche of photons caused by stimulated emission[4]) or can the laser be described completely in classical terms. (Maxwell's equations for the field and Newton's equations of motion for the medium?)

The laser is an example of a self-sustained oscillator. Such devices are well known in electronics. The first of these devices for which a theory was developed was the triode oscillator.[5] In that case, the energy required for sustaining oscillations was provided by a battery. The nonlinear characteristics of the triode-battery system served to provide a negative nonlinear resistance which could drive an L-C circuit into a state of sustained oscillations.

In this paper, a totally classical model of a laser is investigated. The possibility of such a system was first discussed by Gapanov.[6] The model here was independently suggested by one of the authors in a later publication.[7] It is shown that the essential features of laser action arise from nonlinearities in the active medium and not from quantum effects. The calculation closely parallels the semiclassical theory of Ref. 1.

II. MODEL FOR CALCULATION

The model to be used is similar to the one used by Helmer[8] and Lamb[9] to describe the ammonia-beam maser. An unpolarized beam of classical molecules passes through a resonant radiation cavity and interacts with the radiation field. The induced polarization of the beam of molecules is calculated from the dynamics on the interaction. It is required that this polarization be the source for the radiation field. The equations for this self-consistency requirement will be introduced in Sec. III.

The following simplifying assumptions will also be used: (i) The mechanical oscillators move with a single constant velocity through the cavity in a uniform one-dimensional beam perpendicular to the electric field. (ii) Only one cavity mode is considered and the spatial variation of its electric field along the beam will be neglected. Loss in the cavity is described by a phenomenological Q factor. (iii) The mechanical oscillators are represented by a particle of mass m and charge e vibrating in one dimension parallel to the electric field in the cavity. (iv) Internal damping of the mechanical oscillator is neglected. (v) The mechanical oscillators enter the cavity with a fixed internal energy but with random phase with respect to the electric field. The geometry of the model is shown in Fig. 1.

III. SELF-CONSISTENCY CONDITIONS

The following discussion, based on Maxwell's equations, can be found in Ref. 1 in greater detail. Only one mode of a high-Q electromagnetic resonantor is considered. Let its frequency be Ω in the absence of an active medium. The electric field is taken in the form

$$E(z,t)=A(t)\,U(z)\,, \tag{1}$$

where $U(z)$ satisfies the cavity-mode eigenvalue problem. Maxwell's equations can be combined to give

$$\ddot{A}+(\sigma/\epsilon_0)\dot{A}+\Omega^2 A=-(1/\epsilon_0)\ddot{P}\,; \tag{2}$$

P is the polarization of the medium, and σ is a

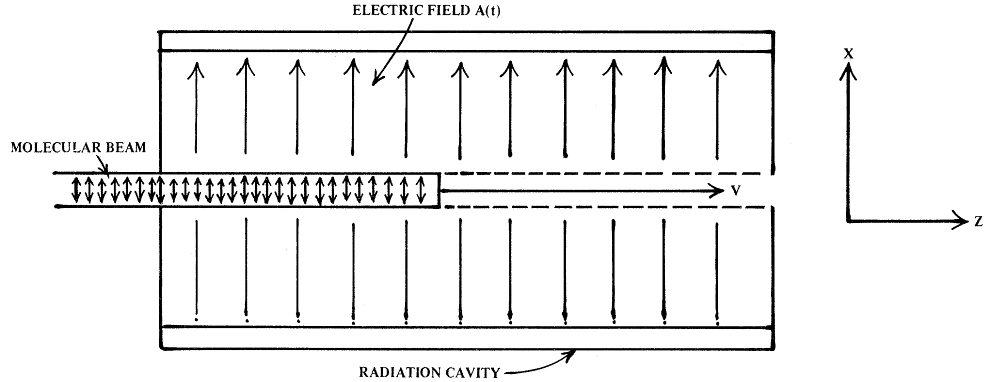

FIG. 1. Geometry of the classical laser. A one-dimensional beam of molecules moves through a resonant radiation cavity with velocity v. The direction of internal oscillation of each oscillator (x axis) is parallel to the electric field.

fictional conductivity adjusted to give the required damping of the radiation field in the cavity, i.e.,

$$\sigma = \epsilon_0 \nu / Q \,. \tag{3}$$

Further assume that the electric field and polarization can be taken in the slowly varying amplitude and phase approximation

$$A(t) = E(t)\cos[\nu t + \varphi(t)] \,, \tag{4a}$$

$$P(t) = C(t)\cos[\nu t + \varphi(t)] + S(t)\sin[\nu t + \varphi(t)] \,, \tag{4b}$$

where ν is a constant frequency yet to be determined. Inserting (4a) and (4b) into Eq. (2) and neglecting small terms in $\ddot{E}$, etc., the following self-consistency equations are obtained:

$$\dot{E} = -\tfrac{1}{2}(\nu/Q)E - \tfrac{1}{2}(\nu/\epsilon_0)S \,, \tag{5a}$$

$$(\nu + \dot{\varphi} - \Omega)E = -\tfrac{1}{2}(\nu/\epsilon_0)C \,. \tag{5b}$$

IV. POLARIZATION OF THE MEDIUM

Let the internal motion of the mechanical oscillator in the presence of the cavity electric field be $x(t_0, \theta_0; t)$. The oscillator entered the cavity at $z = 0$ at time t_0 with phase θ_0 with respect to the electric field $A(t)$. The oscillators move with a constant single velocity so that they are at $z = v \times (t - t_0)$ at time t. The dipole moment p of each oscillator is

$$p(t_0, \theta_0; t) = ex(t_0, \theta_0; t) \,. \tag{6}$$

The macroscopic polarization of the medium is obtained by summing up contributions of individual oscillators. For a collection of oscillators that entered the cavity with phases between θ_0 and $\theta_0 + d\theta_0$ around time t_0, the contribution to the macroscopic polarization $dP(\theta_0; z, t)$ (dipole moment/unit length) is

$$dP(\theta_0; z, t) = Np(t_0, \theta_0; t)d\theta_0/2\pi \,, \tag{7}$$

where N = the number of molecules/unit length in the cavity, and the entry time t_0 is replaced by

$$t_0 = t - (z/v) \,. \tag{8}$$

Summing the contributions from all initial phases θ_0 gives

$$P(z, t) = (N/2\pi)\int_0^{2\pi} d\theta_0\, p(t_0, \theta_0; t) \,. \tag{9}$$

The component of the polarization which is the source of the cavity radiation is found by projecting P on the uniform cavity mode. Thus,

$$P(t) = (1/L)\int_0^L dz\, P(z, t) \,, \tag{10}$$

where L is the length of the cavity.

V. LINEAR OSCILLATOR

The equation of motion of a linear oscillator of frequency ω in the presence of the assumed cavity field is

$$\ddot{x} + \omega^2 x = (eE/m)\cos(\nu t) \,. \tag{11}$$

The amplitude E and phase φ have been assumed to be constant in Eq. (11) since the time spent by the oscillators in the cavity is assumed to be short compared to the time required for appreciable change in E and φ. The phase φ has been set equal to zero. The phase of the oscillator is then measured relative to that of the cavity field.

The solution of Eq. (11) subject to the initial conditions $x(t_0) = \dot{x}(t_0) = 0$ for the driven part is

$$x(t) = A_0\cos[\omega(t - t_0) + \theta_0] + \frac{eE}{m}(\omega^2 - \nu^2)^{-1} \times \left(\cos(\nu t) - \frac{(\omega + \nu)}{2\omega}\cos[\omega(t - t_0) + \nu t_0] - \frac{(\omega - \nu)}{2\omega}\cos[\omega(t - t_0) - \nu t_0]\right) . \tag{12}$$

From Eqs. (7) and (8), the polarization of a collection of oscillators with initial phase θ_0 is

$$dP(\theta_0;z,t)=\frac{N}{2\pi}\,e\left[A_0\cos[(\omega-\nu)t-\omega t_0+\theta_0]+\frac{eE}{m}\times(\omega^2-\nu^2)^{-1}\left(1-\cos\frac{(\omega-\nu)z}{v}\right)\right]\cos(\nu t)$$
$$\times\frac{N}{2\pi}\,e\left[-A_0\sin[(\omega-\nu)t-\omega t_0+\theta_0]+\frac{eE}{m}\times(\omega^2-\nu^2)^{-1}\sin\frac{(\omega-\nu)z}{v}\right]\sin(\nu t),\quad(13)$$

where nonresonant terms have been neglected.

Equations (9) and (10) give

$$P(t)=\frac{Ne^2E}{m}(\omega^2-\nu^2)^{-1}\left(1-\frac{\sin[(\omega-\nu)T]}{T(\omega-\nu)}\right)\cos(\nu t)+\frac{2Ne^2E}{m}(\omega^2-\nu^2)^{-1}\left(\frac{\sin^2[\frac{1}{2}(\omega-\nu)T]}{T(\omega-\nu)}\right)\sin(\nu t),\quad(14)$$

where

$$T=L/v\quad(15)$$

is the time spent by a molecule in the cavity. Comparing (14) with (4b), and letting ν be close to resonance, the coefficients C and S can be determined:

$$C=\frac{Ne^2E}{2m\nu}(\omega-\nu)^{-2}T^{-1}\{T(\omega-\nu)-\sin[(\omega-\nu)T]\},\quad(16a)$$

$$S=\frac{Ne^2E}{m\nu}\,\frac{1/T}{(\omega-\nu)^2}\sin^2[\tfrac{1}{2}(\omega-\nu)T].\quad(16b)$$

The amplitude equation (5a) gives the following result for the cavity field:

$$\dot{E}(t)=-\left[\frac{\nu}{2Q}+\frac{Ne^2}{\epsilon_0 m}\,\frac{\sin^2[\frac{1}{2}(\omega-\nu)T]}{T(\omega-\nu)^2}\right]E(t).\quad(17)$$

Equation (17) shows that an injected stream of randomly phased harmonic oscillators will always increase the damping of the field in the cavity. Steady-state oscillations cannot be achieved with such a medium. The familiar result[10] that a randomly phased linear oscillator can only absorb energy from an electric field has been rederived.

If the phases of the oscillators before entering the cavity had been properly correlated to the electric field, S as calculated from (13) with a constant θ_0 could have been negative for suitable transit times T. That is equivalent to coupling a signal generator to the resonant cavity. The problem under consideration, however, is to construct a model for a generator.

In order to see more clearly why the harmonic oscillator will not sustain oscillations in the radiation field, evaluate (12) for $\dot{x}(t)$ at resonance ($\omega=\nu$):

$$\dot{x}(t)=A_0\cos[\omega(t-t_0)+\theta_0]+(eE/2m\omega)(t-t_0)\sin(\omega t).\quad(18)$$

The power absorbed by the oscillator is

$$\frac{dW(t)}{dt}=F(t)\dot{x}(t),\quad(19)$$

where $F(t)$ is the force on the oscillator. Using (11), (19) becomes

$$\frac{dW(t)}{dt}=(\tfrac{1}{2}\omega eEA_0)\sin(-\omega t_0+\theta_0)+\frac{(eE)^2}{4m}(t-t_0),\quad(20)$$

where high-frequency 2ω terms have been neglected. The oscillators that are initially phased to gain energy from the electric field will do so for all times. The others initially lose energy, but eventually gain. The average of (20) over the injection phase θ_0 is positive definite which corresponds to the result derived earlier for the entire ensemble.

VI. NONLINEAR OSCILLATOR

The frequency of oscillation in the case of a nonlinear or anharmonic oscillator is amplitude dependent. Consider the situation where such an oscillator is introduced into the resonant cavity at an amplitude corresponding to a frequency slightly lower than the cavity frequency (see Fig. 2). As in the case of the harmonic oscillator, upon entering the cavity some oscillators will gain energy

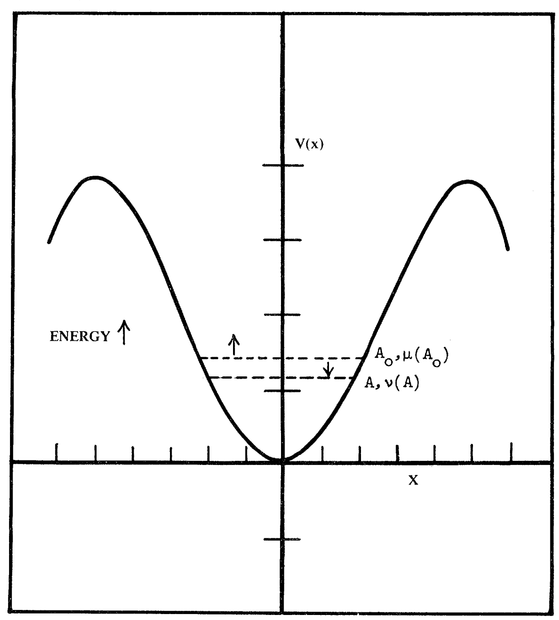

FIG. 2. Anharmonic-oscillator potential. The frequency of oscillation is amplitude dependent. The oscillators are injected into the cavity with amplitude A_0 and corresponding frequency $\mu=\mu(A_0)$. This gives an energy slightly higher than if they were oscillating at the cavity electric field frequency $\nu>\mu$. Depending on the initial phase, some oscillators gain energy from the field and move away from resonance while others lose energy and move toward resonance with the electric field.

from the field and some will lose, depending on the phase. As any oscillator gains energy it gradually goes out of resonance with the electric field in the cavity since the frequency is amplitude dependent. The energy absorption is thus severely limited in comparison with the linear oscillator.

Those oscillators that initially lose energy come closer to resonance with the driving field (and may even pass through resonance) and could lose a substantial amount of energy before rephasing or being removed from the cavity. Under certain conditions a net loss of energy to the cavity field is therefore possible.

This rough description gives some motivation for investigating a nonlinear oscillator as a medium for laser action.

The model for a classical nonlinear oscillator will be the familiar but nontrivial case of a simple pendulum of mass m, length a, and charge e. The equation of motion is

$$\ddot{x} + a\omega^2 \sin(x/a) = (eE/m)\cos(\nu t), \tag{21}$$

where ω denotes the small amplitude resonant frequency. Using the series expansion of $\sin(x/a)$ to third order, (21) becomes

$$\ddot{x} + \omega^2[x - \tfrac{1}{6}(x^3/a^2)] = (eE/m)\cos(\nu t), \tag{22}$$

which is known as Duffing's equation. There is extensive literature on the problem. There are subharmonic solutions, stable and unstable oscillations, and jump phenomena.[11] The following treatment corresponds most closely to that of Bogoliubov and Mitropolsky. Assume a solution with slowly varying amplitude and phase which can be expressed as a Fourier series in odd harmonics of the driving frequency ν. Let

$$x(t) = \sum_{n=0}^{\infty} B_{2n+1}(t)\cos[(2n+1)\nu t + \theta_{2n+1}(t)], \tag{23}$$

where the amplitudes B_{2n+1} and phases θ_{2n+1} are slowly varying in comparison with $\cos(\nu t)$. Only the component of the polarization varying at the fundamental frequency ν is of interest in this problem. Since numerical analysis has shown that the most significant elements of the motion vary at this frequency, (23) is replaced by

$$x(t) = B(t)\cos[\nu t + \theta(t)]. \tag{24}$$

Inserting (24) into (22), equating coefficients of $\cos(\nu t + \theta)$ and $\sin(\nu t + \theta)$, and neglecting terms in $\ddot{B}$, $\dot{B}^2$, $\ddot{\theta}$, $\dot{\theta}^2$, $\dot{B}\dot{\theta}$ (slowly varying amplitude and phase approximation) yields two coupled first-order differential equations for $B(t)$ and $\theta(t)$:

$$\dot{B} = -(eE/2m\omega)\sin\theta, \tag{25a}$$

$$\dot{\theta} = (\omega - \nu) - \frac{\omega B^2}{16a^2} - \frac{eE}{2m\omega} B^{-1}\cos\theta, \tag{25b}$$

where $|\omega - \nu| \ll \omega$. Equations (25) can be rewritten in terms of a dimensionless force parameter

$$G = (eE/2ma\omega^2), \tag{26a}$$

dimensionless variables

$$A = B/a \tag{26b}$$

for amplitude, and

$$\tau = \omega t \tag{26c}$$

for time as

$$\frac{dA}{d\tau} = -G\sin\theta, \tag{27a}$$

$$\frac{d\theta}{d\tau} = \Delta - \tfrac{1}{16}A^2 - \frac{G}{A}\cos\theta, \tag{27b}$$

where

$$\Delta = (\omega - \nu)/\omega. \tag{28}$$

When $G = 0$ the solutions of (27) are

$$A(\tau) = A_0 = B_0/a, \tag{28a}$$

$$\theta(\tau) = \Delta\tau - (\tfrac{1}{16}A_0^2)\tau + \theta_0, \tag{28b}$$

so that the motion of the oscillator is

$$x(t) = B_0\cos[\omega(1 - \tfrac{1}{16}A_0^2)t + \theta_0]. \tag{29}$$

The familiar $\frac{1}{16}A_0^2$ correction to the frequency of a simple pendulum is confirmed by this analysis.

Some of the properties of the solutions of Eqs. (27) can be found by investigating the stationary points. These occur when $dA/d\tau = 0$ and $d\theta/d\tau = 0$ giving stationary solutions

$$\theta = n\pi \quad \text{for } n = \pm 1, \pm 2, \pm 3, \ldots \tag{30a}$$

and A determined as a root of the cubic equation

$$A[\Delta - \tfrac{1}{16}A^2] - (-1)^n G = 0 \quad \text{for } n = \pm 1, \pm 2, \ldots. \tag{30b}$$

Without loss of generality, consider only the solutions with $A > 0$. Figures 3(a) and 3(b) show the solutions of (31b) for $\Delta > 0$ and $\Delta < 0$, respectively. For $\Delta > 0$, Fig. 3(a), there are three possible stationary points: a and b with $\theta = 0, 2\pi, \ldots$, and c with $\theta = \pi, 3\pi, \ldots$. By linearizing Eqs. (27) about these points, a and c are found to give stable solutions and b to give an unstable solution. When $G > (\frac{4}{3}\Delta)^{3/2}$ [G_2 in Fig. 3(a)], a and b disappear leaving c as the only possible stationary solution. When $\Delta < 0$, Fig. 3(b), only one stationary is found with the same stability as point c in Fig. 3(a).

Equations (27) have been solved numerically on an IBM 7094 computer using a predictor-corrector method.[12] Figures 4(a)–4(c) exhibit the solutions in a phase diagram where $\theta(t)$ is plotted as a function of $A(t)$. The relationship between the amplitude and phase of the oscillator can be used to determine some important qualitative aspects of the motion

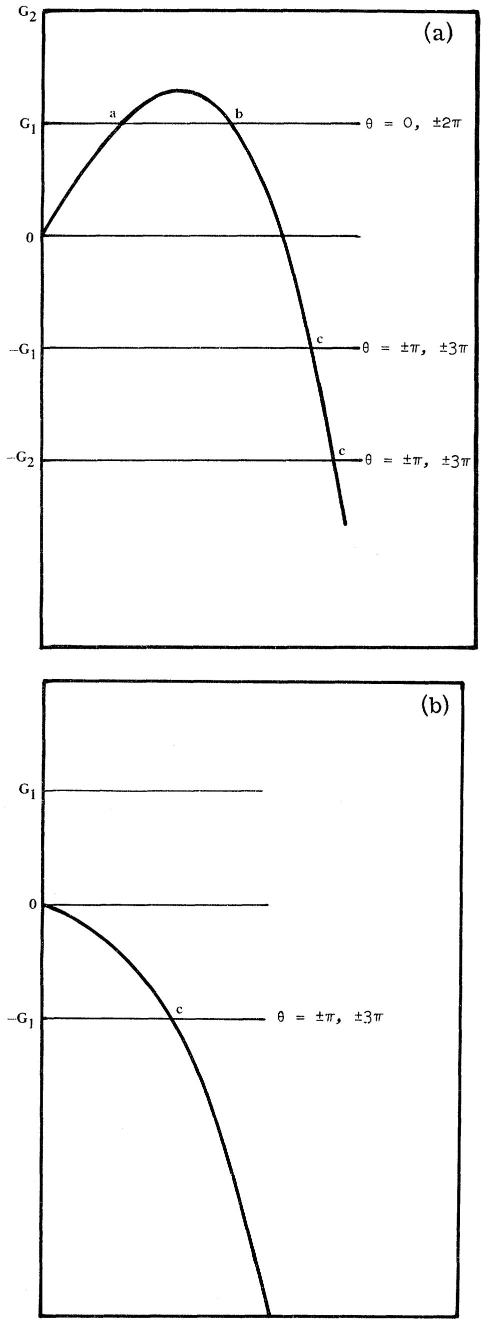

FIG. 3. (a) Plot of $A(\Delta-\frac{1}{16}A^2)$ for $\Delta>0$. The intersections of this curve with horizontal straight lines of ordinates $|G_1|<(\frac{4}{3}\Delta)^{3/2}$ give stationary points a, b, c. The intersection with $-G_2$, $|G_2|>(\frac{4}{3}\Delta)^{3/2}$ gives only stationary point c. (b) Plot of $A(\Delta-\frac{1}{16}A^2)$ for $\Delta<0$. The intersection with horizontal line of ordinate $-G_1$ gives only one stationary point c.

under the influence of a driving field.

In Fig. 4(a), $\Delta>0$ and $G<(\frac{4}{3}\Delta)^{3/2}$. The stability properties of the stationary points a, b, c are easily seen. Figure 4(b) corresponds to $\Delta>0$ and $G>(\frac{4}{3}\Delta)^{3/2}$, while in Fig. 4(c) $\Delta<0$. In each of the latter two cases only the one stable point c is found.

Figures 5 show the time evolution of a collection of Duffing oscillators which enter the radiation cavity at a fixed amplitude $A_0=0.32$ but with random phase. The amplitude A is an indicator of the energy of the oscillator [i.e., energy $=\omega^2 a^2(\frac{1}{2}A^2-\frac{1}{24}A^4)$]. The rough description of the nonlinear oscillator given at the beginning of this section can be made more explicit by examining Fig. 5. When the oscillators have been in the cavity for a time $\omega t=300$ the ones with initial phase greater than π are increasing in amplitude while those with initial phase less than π are decreasing in amplitude. By $\omega t=900$ [see Fig. 5(d)] most of the oscillators have lost energy. Those oscillators that initially gained energy have "rephased" so as to lose energy. Those oscillators that initially lost energy have not yet returned to their original amplitude. If the oscillators are removed from the cavity at such a time, a net transfer of energy to the cavity radiation field can be expected. Therefore, a beam of nonlinear molecules injected with a suitable energy and removed from the cavity at the proper time could produce laser action.

Section VII treats Eqs. (25) to first order in the driving field. That analysis will find a threshold condition for the onset of laser oscillations and frequency-pulling effects.

VII. WEAK-SIGNAL THEORY

To first order in the force parameter G, the amplitude and phase of the Duffing oscillator are

$$A=A^{(0)}+GA^{(1)}\,, \tag{31a}$$

$$\theta=\theta^{(0)}+G\theta^{(1)}\,. \tag{31b}$$

Using (31) in the differential equations (27) gives

$$\frac{dA^{(0)}}{d\tau}=0\,, \tag{32a}$$

$$\frac{dA^{(1)}}{d\tau}=-\sin\theta^{(0)}\,, \tag{32b}$$

$$\frac{d\theta^{(0)}}{d\tau}=\Delta-\tfrac{1}{16}A^{(0)2}, \tag{32c}$$

$$\frac{d\theta^{(1)}}{d\tau}=\tfrac{1}{8}A^{(0)}A^{(1)}-\frac{1}{A^{(0)}}\cos\theta^{(0)}\,, \tag{32d}$$

with solutions

$$A^{(0)}=A_0=\text{const}\,, \tag{33a}$$

$$\theta^{(0)}=(\mu-\nu)(t-t_0)+\theta_0\,, \tag{33b}$$

$$A^{(1)}=[\omega/(\mu-\nu)]\{\cos[(\mu-\nu)(t-t_0)+\theta_0]-\cos\theta_0\}\,, \tag{33c}$$

$$\theta^{(1)}=\frac{\omega^2A_0\cos\theta_0}{8(\mu-\nu)}(t-t_0)-\frac{\omega}{A_0(\mu-\nu)}\left(1+\frac{\omega A_0^2}{8(\mu-\nu)}\right)$$

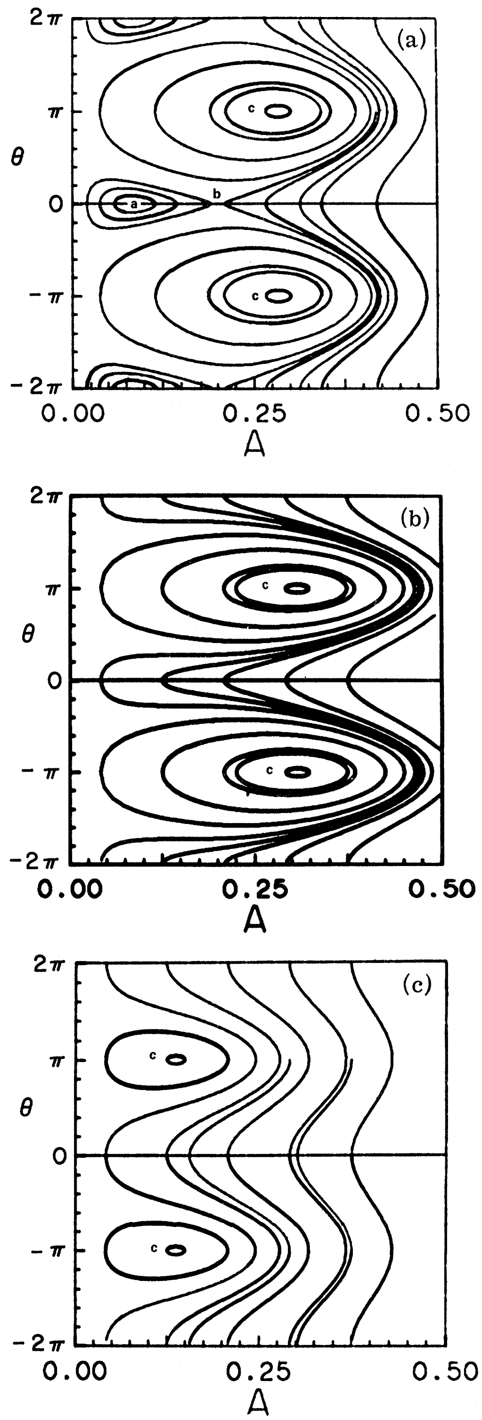

FIG. 4. (a) Duffing phase plot with $G=10^{-4}$ and $\Delta=4.625\times10^{-3}$. Solutions of Eqs. (27) where θ is plotted as a function of A with $\Delta>0$ and $G<(\frac{4}{3}\Delta)^{3/2}$. The stationary points a and c are stable while point b is unstable. (b) Duffing phase plot with $G=6.0\times10^{-4}$ and $\Delta=4.625\times10^{-3}$. Solution of Eqs. (27) where θ is plotted as a function of A for $\Delta>0$ and $G>(\frac{4}{3}\Delta)^{3/2}$. The only stationary point is c. (c) Duffing phase plot for $G=10^{-4}$ and $\Delta=-10^{-3}$. Solution of Eqs. (27) where θ is plotted as a function of A for $\Delta<0$. The only stationary point is c.

$$\times\{\sin[(\mu-\nu)(t-t_0)+\theta_0]-\sin\theta_0\}\ , \tag{33d}$$

where

$$\mu=\omega(1-\tfrac{1}{16}A_0^2) \tag{34}$$

is the free oscillation frequency of the injected oscillator.

Using Eq. (9) the polarization of a collection of oscillators with initial phase θ_0 is

$$dP(\theta_0;z,t)=(N/2\pi)\,ex(t-z/v,\ \theta_0;t)\,d\theta_0$$
$$=(Ne/2\pi)A[\cos\theta\cos(\nu t)-\sin\theta\,\sin(\nu t)]\,d\theta_0\ . \tag{35}$$

Identifying the coefficients of sin (νt) and (νt) gives

$$dC(\theta_0;z,t)=(Ne/2\pi)A\cos\theta\,d\theta_0\ , \tag{36a}$$

$$dS(\theta_0;z,t)=-(Ne/2\pi)A\sin\theta\,d\theta_0\ . \tag{36b}$$

The first-order contribution to C and S can now be found by using the solutions (33)

$$dC^{(1)}(\theta_0;z,t)=(Ne/2\pi)[A_0\cos\theta^{(0)}+G(A^{(1)}\cos\theta^{(0)}-A_0\theta^{(1)}\sin\theta^{(0)})]d\theta_0\ , \tag{37a}$$

$$dS^{(1)}(\theta_0;z,t)=-(Ne/2\pi)[A_0\sin\theta^{(0)}+G(A^{(1)}\sin\theta^{(0)}+A_0\theta^{(1)}\cos\theta^{(0)})]\,d\theta_0\ . \tag{37b}$$

Averaging Eqs. (37) according to the prescription of Eqs. (9) and (10) gives

$$C^{(1)}=\frac{\omega NeaG}{(\mu-\nu)^2}\left((\omega-\nu)-\frac{(\omega-\mu)+(\omega-\nu)}{T(\mu-\nu)}\sin[(\mu-\nu)T]+(\omega-\mu)\cos[(\mu-\nu)T]\right)\ , \tag{38a}$$

$$S^{(1)}=\frac{\omega NeaG}{(\mu-\nu)^2}\left((\omega-\mu)\sin[(\nu-\mu)T]-\frac{2[(\omega-\mu)+(\omega+\nu)]}{T(\nu-\mu)}\sin^2[(\nu-\mu)T/2]\right)\ . \tag{38b}$$

In the limit of a linear oscillator, $\omega=\mu$, and Eqs. (38) are identical to (16).

Using (38b) in the electric-field-amplitude equation (5a), the conditions necessary for the onset of laser oscillations can be determined. At steady state, $\dot{E}=0$ and

$$\frac{1}{Q}=\frac{Ne^2}{2m\omega\epsilon_0T(\nu-\mu)^2}\left(\frac{4d}{(\nu-\mu)}\sin^2[\tfrac{1}{2}(\nu-\mu)T]-2\sin^2[\tfrac{1}{2}(\nu-\mu)T]-Td\sin[(\nu-\mu)T]\right)\ , \tag{39}$$

where

$$d=(\omega-\mu)=\tfrac{1}{16}A_0^2 \tag{40}$$

is a measure of the initial excitation of the oscillator.

For a given cavity transit time $T=L/v$, the self-consistency condition (39) can be satisfied

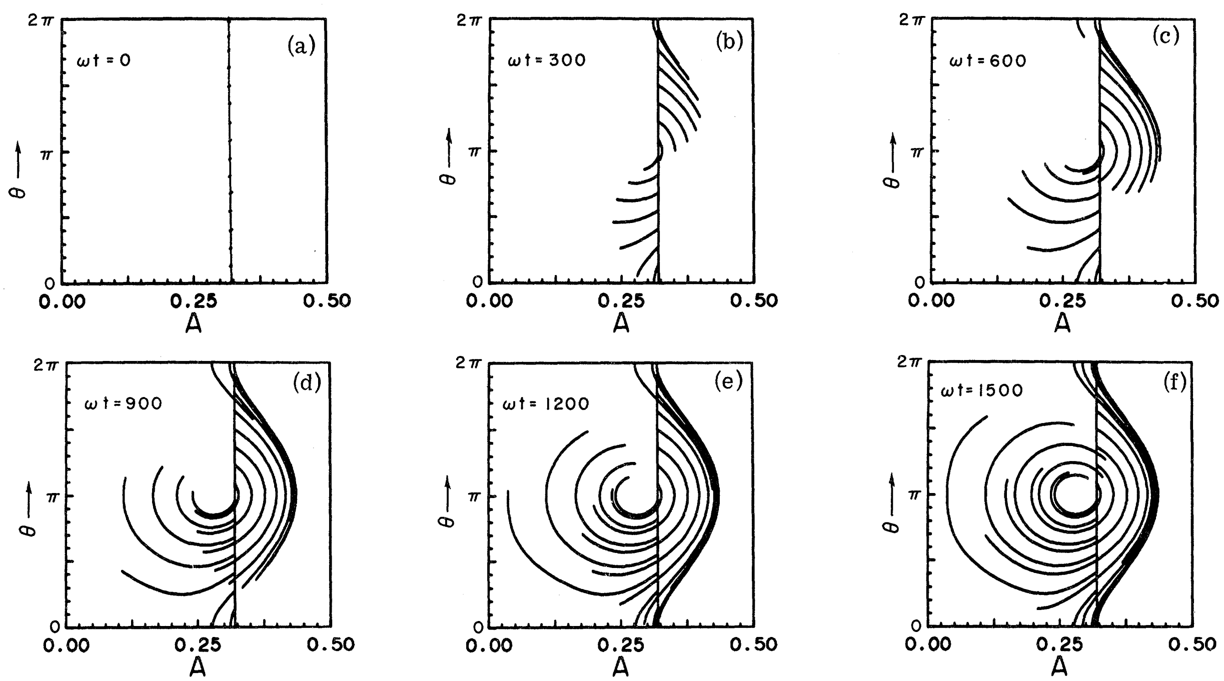

FIG. 5. Time evolution of Duffing oscillators. Solutions of Eqs. (27). θ (modulo 2π) is plotted as functions of A stopped at times $\omega t = 0$, 300, 600, 900, 1200, 1500. 15 oscillators start at equally spaced initial phases between 0 and 2π with amplitude $A_0 = 0.32$ and $G = 4.0 \times 10^{-4}$ and $\Delta = 3 \times 10^{-3}$.

within finite frequency bands. If the right-hand side of Eq. (39) is positive, then N and Q can be adjusted to give threshold. Figure 6 shows a plot of the large round bracketed expression on the right-hand side of (39) as a function of

$$\psi = \tfrac{1}{2}(\nu - \mu)T \tag{41}$$

for various values of the parameter Td. The domains where the right-hand side of (39) is positive occur when

$$n\pi \geq \psi \geq \psi_n^+ , \qquad -n\pi \geq \psi \geq \psi_n^- , \tag{42}$$

where $n = 1, 2, 3, \ldots$. The angles ψ_n^+ and ψ_n^- are solutions of the transcendental equation

$$\tan(\psi) = (Td)\psi/(Td - \psi) . \tag{43}$$

For the remainder of this paper, only the region $\pi \geq \psi \geq \psi_1^+$ will be considered. This corresponds to the widest frequency band which gives a self-consistent solution and is closest to the linear resonance $\nu = \omega$ (frequency for small amplitude oscillations). The frequency band to be considered is then

$$\nu_{\min} \leq \nu \leq (2\pi/T) + \mu , \tag{44}$$

where

$$\nu_{\min} = \mu + 2\psi_1^+/T .$$

Figure 7 shows a plot of the width of the above region as a function of transit time in the cavity. There is a linear variation for short transit times and $1/T$ dependence for long exposures. Figure 8 is a plot of $\nu_{\min}$ as a function of transit time. For this frequency band, $\nu_{\min}$ is always greater than μ. Shorter transit times require that the Duffing oscillators be sent through a cavity tuned to a higher frequency. As the transit time increases, the driving frequency must be proportionally decreased so that the oscillators do not begin to absorb energy.

Inserting (39a) into the "frequency-determining" equation (5b) at steady state gives

$$(\nu - \Omega) = -\frac{\nu}{2\epsilon_0}\,\frac{Ne^2}{2m(\nu-\mu)^2}\left((\omega - \nu) - \frac{d + (\omega - \nu)}{T(\nu - \mu)}\sin[(\nu-\mu)T] + d\cos[(\nu-\mu)T]\right) . \tag{45}$$

Using N at threshold in (45) gives

$$(\nu - \Omega) = \frac{(\nu T/2Q)\left((\omega-\nu) - \{[d + (\omega-\nu)]/T(\nu-\mu)\}\sin[(\nu-\mu)T] + d\cos[(\nu-\mu)T]\right)}{-2\{[d+(\omega-\nu)]/T(\nu-\mu)\}\sin^2[\frac{1}{2}(\nu-\mu)T] + Td\sin[(\nu-\mu)T]} . \tag{46}$$

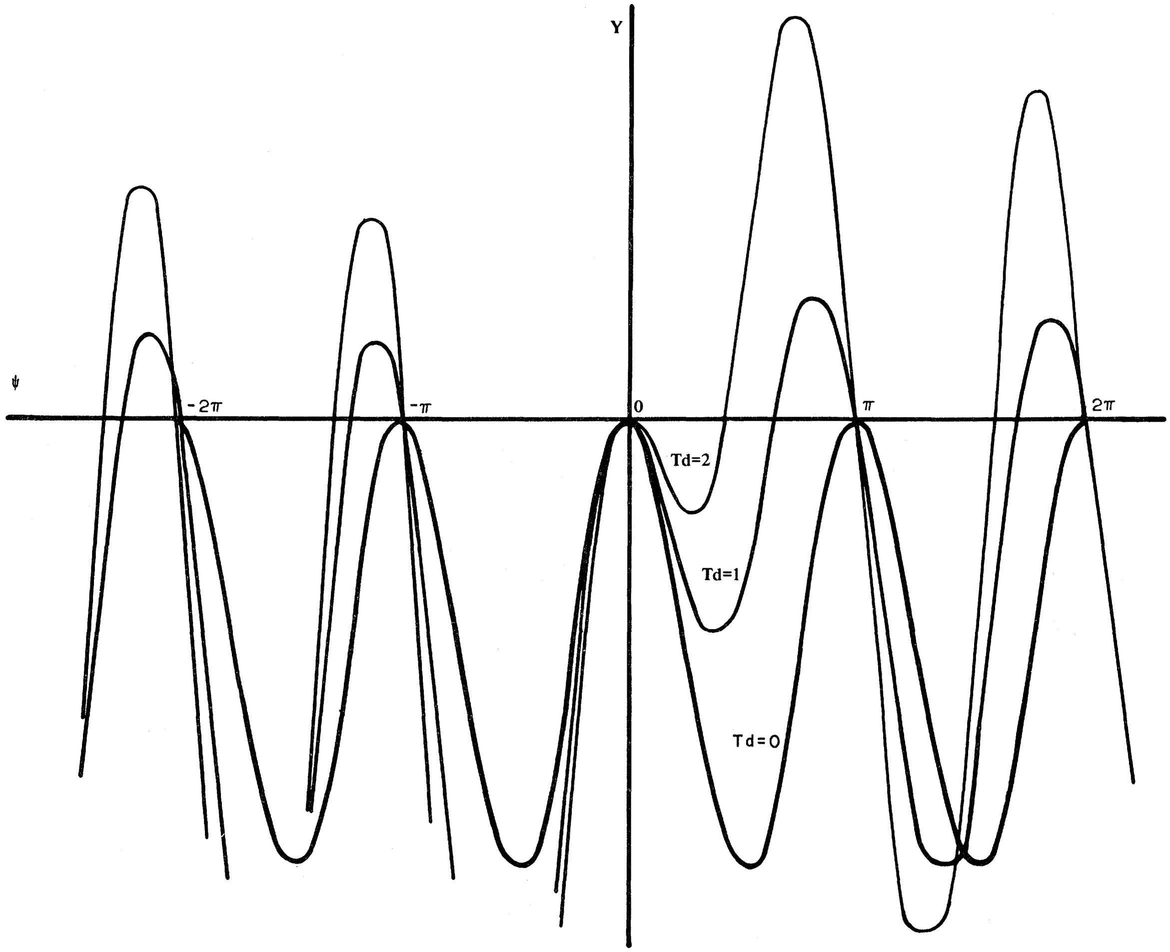

FIG. 6. Plots of $y=2[(Td-\psi)/\psi]\sin^2\psi - Td\sin 2\psi$, where $\psi=\frac{1}{2}(\nu-\mu)T$ for $Td=0$, 1, 2. The parameter $d=\omega-\mu$ of Eq. (40) is a measure of the injection energy of the oscillators. The ranges where y is positive give self-consistent solutions of the threshold-condition equation (39).

Figure 9(a) is a plot of the right-hand side of (46) as a function of $(\mu-\nu)$ for $\omega T=800$ while Fig. 9(b) has $\omega T=200$. For short transit times [e.g., Fig. 9(b)] the frequency is double valued. Thus it is possible to have two different types of oscillation under the single cavity mode. However, the analysis here is only of a single frequency. The equations of motion would have to be solved with a two-frequency driving force in order to determine whether they could coexist. Therefore, the analysis will be restricted to longer transit times such at $\omega T=800$ [Fig. 9(a)]. The frequency is then single valued and the pulling has a well-defined linear region. To examine linear pulling, expand Eq. (42) about the zero point $\nu=\nu_0$ giving

$$(\nu-\Omega)=-\mathcal{S}(\nu_0 T)(\nu-\nu_0)\ , \tag{47}$$

where

$$\mathcal{S}=(\nu_0 T/Q)F(\nu_0 T), \tag{48}$$

where F is a complicated dimensionless function. $\mathcal{S}$ is known as the stabilization factor which apart from F is the ratio of the cavity bandwidth (ν/Q) to the transit-time bandwidth $(1/T)$ of the molecules.

VIII. COMPARISONS WITH AMMONIA-BEAM MASER WEAK-SIGNAL THEORY

It is instructive to compare the classical theory with one in which a simple quantum-mechanical nonlinear oscillator is used. The results of the Helmer[8]-Lamb[7] small signal theory of an ammonia-beam maser will be used. The mechanical systems are two-level atoms with energy difference $\hbar\omega$ injected into the resonant cavity in their upper state. With the simplifying assumptions of Sec. II, Eq. (5a) at threshold gives

$$1/Q=\frac{N\wp^2}{\hbar\epsilon_0}\frac{2}{(\nu-\omega)^2 T}\sin^2[\tfrac{1}{2}(\nu-\omega)T]\ , \tag{49}$$

and the frequency equation (5b) becomes

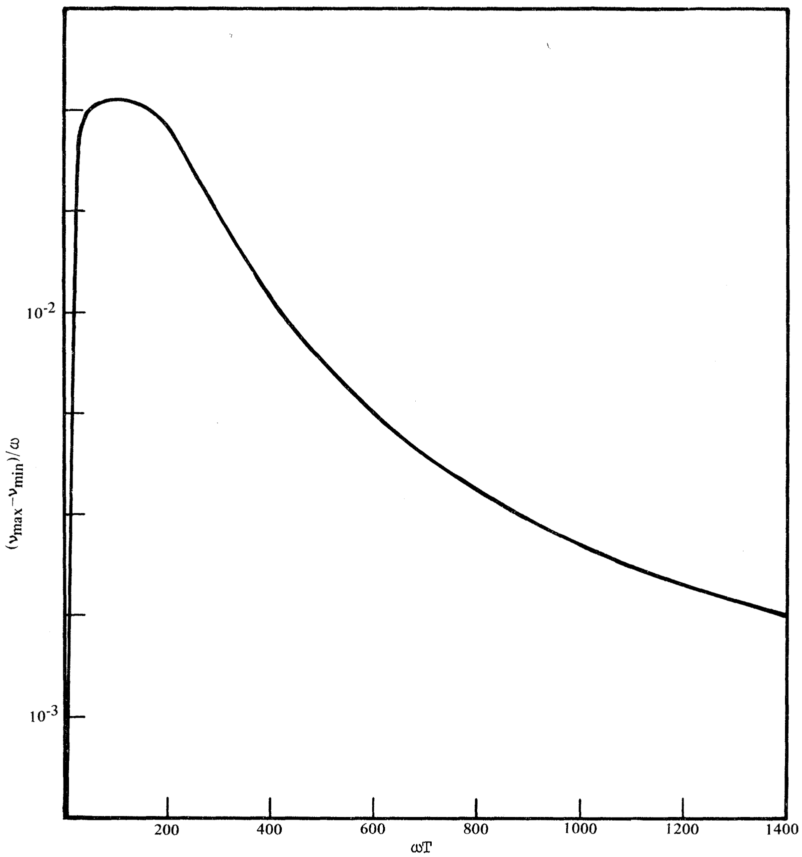

FIG. 7. The dimensionless width of the first frequency band of laser oscillations $(\nu_{max}-\nu_{min})/\omega$ is plotted as a function of the dimensionless transit time ωT in the cavity.

$$(\nu - \Omega) = -\frac{N\wp^2}{2\hbar e_0}\,\frac{\nu}{(\nu-\omega)^2}\left((\nu-\omega) - \frac{\sin[(\nu-\omega)T]}{T}\right), \tag{50}$$

where $\wp$ is the dipole matrix element for the radiative transition. The similarity between Eqs. (39) and (49) and (45) and (50) should be noted.[13] The second $\sin^2$ term in the expression for the Duffing oscillator (39) is always negative. That term is exactly the same as the total expression for the linear oscillator, Eq. (16). The other two terms in (39) combine to make the expression positive under certain conditions. They are both proportional to $d = (\omega - \mu)$ which is a measure of the nonlinearity of the oscillator.

IX. STRONG-SIGNAL THEORY

It has been seen that, at least for small signals, a completely classical system provides a reasonable model for laser action. An unpolarized beam of anharmonic oscillators of fairly high amplitude is injected into a radiation cavity, and the conditions for the buildup of laser oscillations are not very different from those of a simple quantum-mechanical model.

The nonlinearities of the system play an essential role in that they provide for a coupling between the amplitude and phase of the mechanical oscillator in the presence of an electric field. This coupling, not present in the linear oscillator, allows the phases to readjust giving the medium a net active polarization.

The next problem is to determine the intensity and frequency of the classical laser. Ideally, the perturbation expansion in the dimensionless parameter G could be continued to high orders. Unfortuantely, the amount of algebra involved is enormous. Using a computer, however, it was relatively easy to use numerical methods to calculate the polarization of an ensemble of Duffing oscillators.

The technique employed was to solve Eqs. (27) simultaneously with the same initial amplitude A_0 for a set of equally spaced initial phases between 0 and 2π. The phase averages, Eq. (9), of S and C were found using Simpson's rule at each time. Since the molecules move at uniform velocity, the mode projection of Eq. (10) is just the time average. In terms of the numerical procedure this time average is just the cumulative sum for previous times divided by the total elapsed time. For small amplitudes, these coarse phase and time averages are in excellent agreement with the first-order theory.

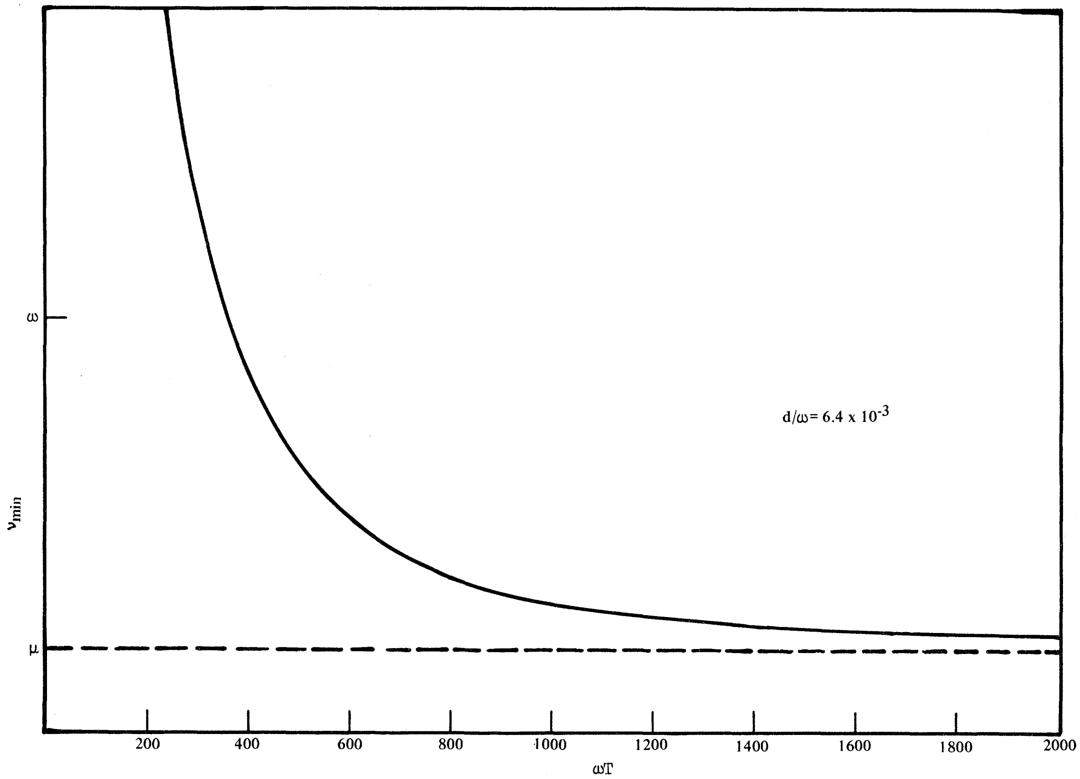

FIG. 8. The minimum frequency of laser oscillations ν_{min} is plotted as a function of the dimensionless transit time for the first band. For large T, $\nu_{min} \sim \mu$.

Figure 10 is a plot of $-S$ as a function of G for various values of $\Delta = (\omega - \nu)/\omega$ with $\omega T = 800$. Figure 11 shows a plot of $-C$ as a function of G. The amplitude equation at steady state ($\dot{E} = 0$) gives

$$E/Q = (G/Q)(2m\omega^2 a/e) = -(1/\epsilon_0)S(G). \quad (51)$$

The intersection of a straight line through the origin in Fig. 10 with slope $(1/Q)(2m\omega\nu a/e)$ with any one of the $-S$ curves will give an operating point of the laser. Figure 12 is a plot of a set of such intersection points showing E^2 as a function of $\nu - \mu$ for several values of Q. Thus, the theory has given the intensity as a function of $\nu - \mu$.

Figure 10 shows the characteristic behavior of saturation phenomena: $-S$ increases linearly for small amplitudes and then curves back downward for larger values of the electric field. The gain ($-S$) becomes negative at very high amplitudes.

From the values of the electric field obtained for the operating points and the numerical values of $C(E)$, the frequency of the laser can be determined. Figure (13) is a plot of $C(E)/E$ as a function of $(\nu - \mu)$ for different values of Q. The form $C^{(1)}(E)/E$ of Fig. 9(a) is also included to show the pulling is apparently linear.

Section X will show that the classical-laser theory can be applied to a physical problem of the electron-cyclotron maser.

X. ELECTRON-CYCLOTRON MASER

The electron-cyclotron maser[14] is an example of a real physical system for which the classical-laser theory is applicable. This oscillator uses a system of energetic free electrons in a dc magnetic field (H_z) which undergo radiative transitions in a microwave cavity. In quantum-mechanical terms, the transitions are induced between adjacent Landau levels w_n where

$$w_n = mc^2\{[1 + 2(n + \tfrac{1}{2})\hbar\omega/mc^2]^{1/2} - 1\} , \quad (52)$$

with

$$\omega = eH_z/mc \ \ \text{(cyclotron frequency)}. \quad (53)$$

For slightly relativistic electrons (~ 5 keV) and for typical laboratory magnetic fields ($H_z \sim 2000$ G) the relevant quantum numbers are of the order of 10^8 (i.e., $10^8\hbar\omega = 5$ keV). A classical treatment of this problem should be used for such high quantum numbers.

Consider electrons moving in a uniform magnetic field H_z in a rectangular microwave cavity. Assume a TE mode in the cavity with the dc magnetic field H_z perpendicular to the electric field. Neglect the

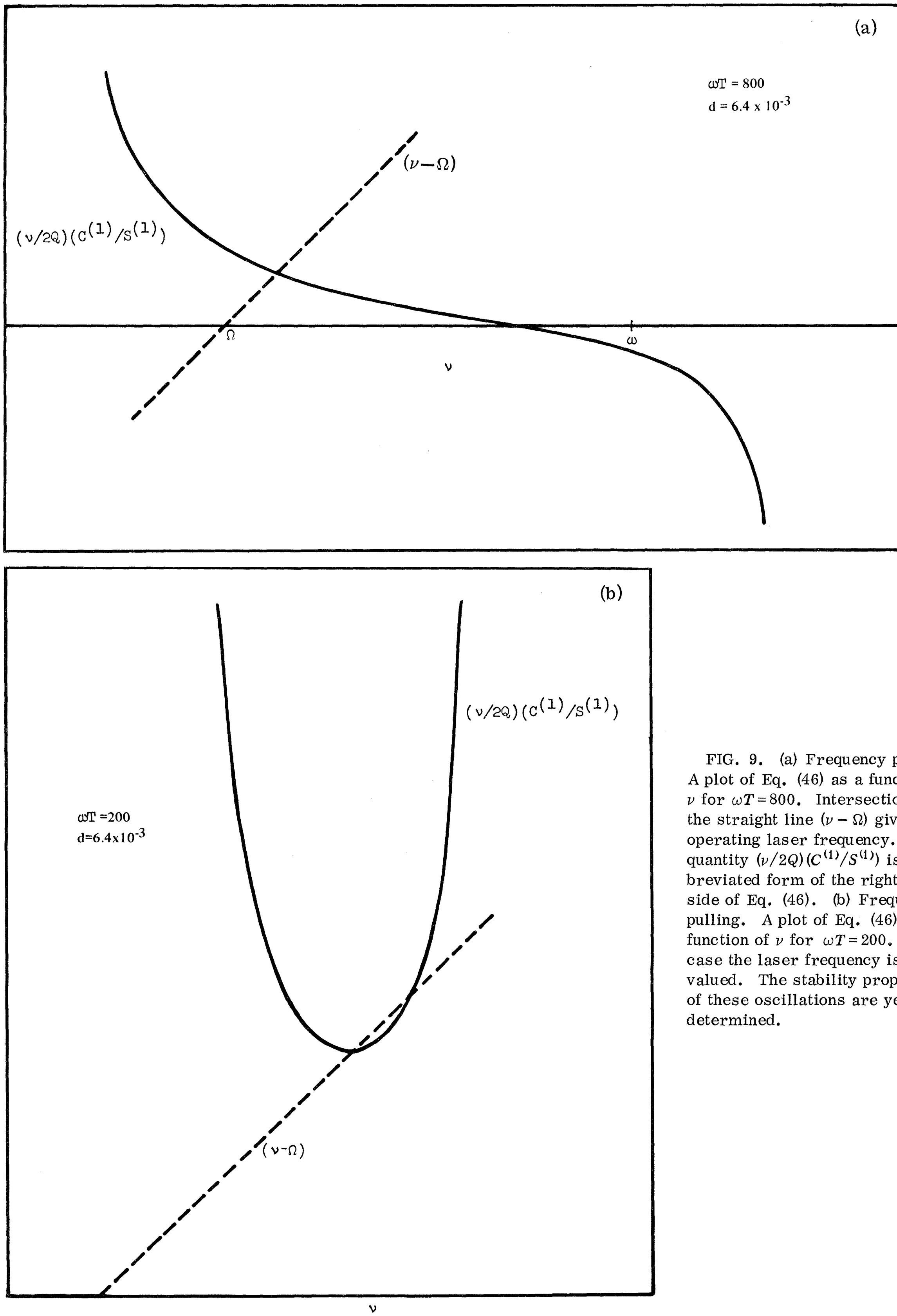

FIG. 9. (a) Frequency pulling. A plot of Eq. (46) as a function of ν for $\omega T = 800$. Intersection with the straight line $(\nu - \Omega)$ gives the operating laser frequency. The quantity $(\nu/2Q)(C^{(1)}/S^{(1)})$ is an abbreviated form of the right-hand side of Eq. (46). (b) Frequency pulling. A plot of Eq. (46) as a function of ν for $\omega T = 200$. In this case the laser frequency is double valued. The stability properties of these oscillations are yet to be determined.

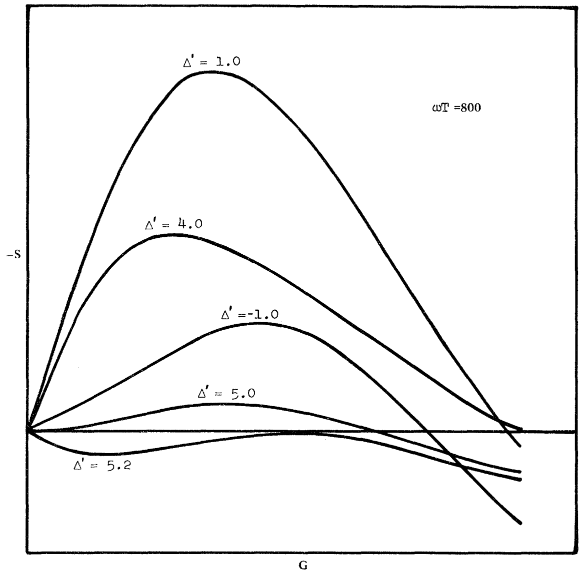

FIG. 10. $-S$ plotted as a function of G for strong signals for various values of $\Delta' = 10^3\Delta$ with $\omega T = 800$. Intersection of the curves with the straight line $-S = (2m\omega^2 a/e)(G/Q)$ will give the laser operating points as a function of Δ.

transverse spatial variations of the cavity mode and the rf magnetic fields. Also, assume that most of the electronic energy is in its transverse motion (i.e., $\dot{x}, \dot{y} \gg \dot{z}$).

The equations of motion of an electron with

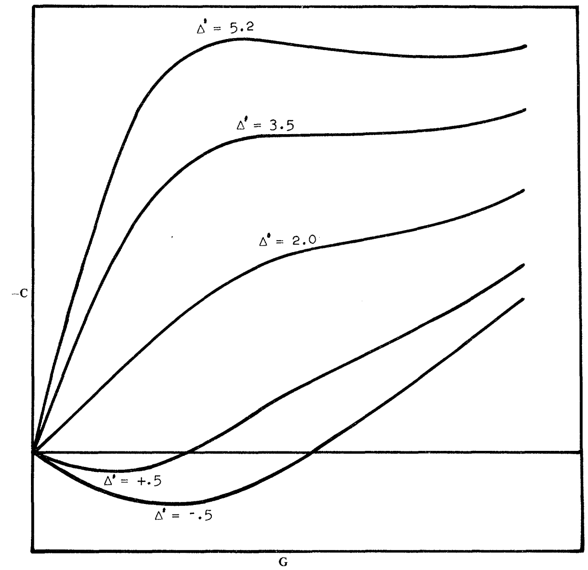

FIG. 11. $-C$ plotted as a function of G for strong signals for various values of $\Delta' = 10^3\Delta$ for $\omega T = 800$.

charge e and mass m injected into a cavity according to the above scheme are

$$\frac{d}{dt}\left(\gamma m \dot{x}\right) - \frac{eH_z}{c}\,\dot{y} = eE_x, \tag{54}$$

$$\frac{d}{dt}\left(\gamma m \dot{y}\right) + \frac{eH_z}{c}\,\dot{x} = eE_y, \tag{55}$$

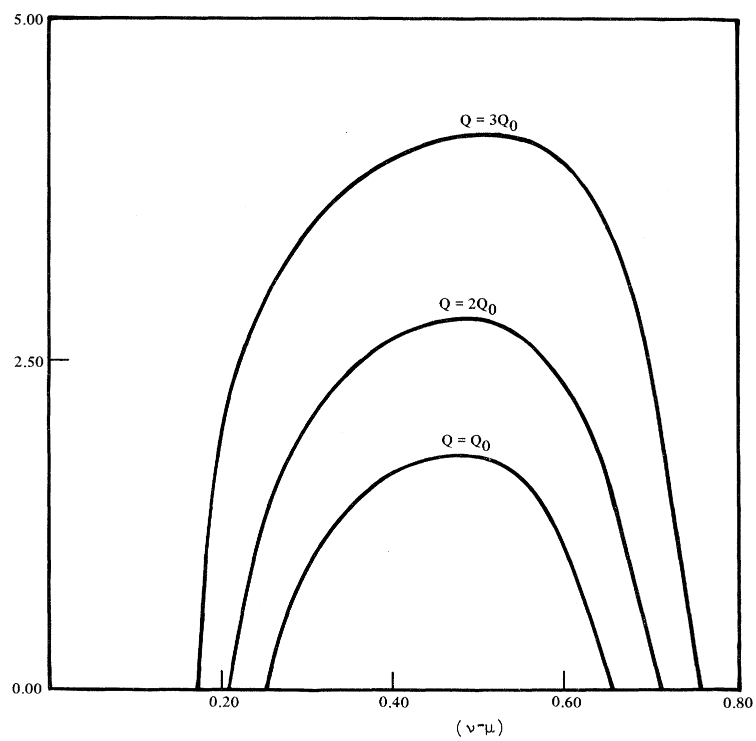

FIG. 12. Laser intensity I as a function of $(\nu - \mu)$ for various values of Q.

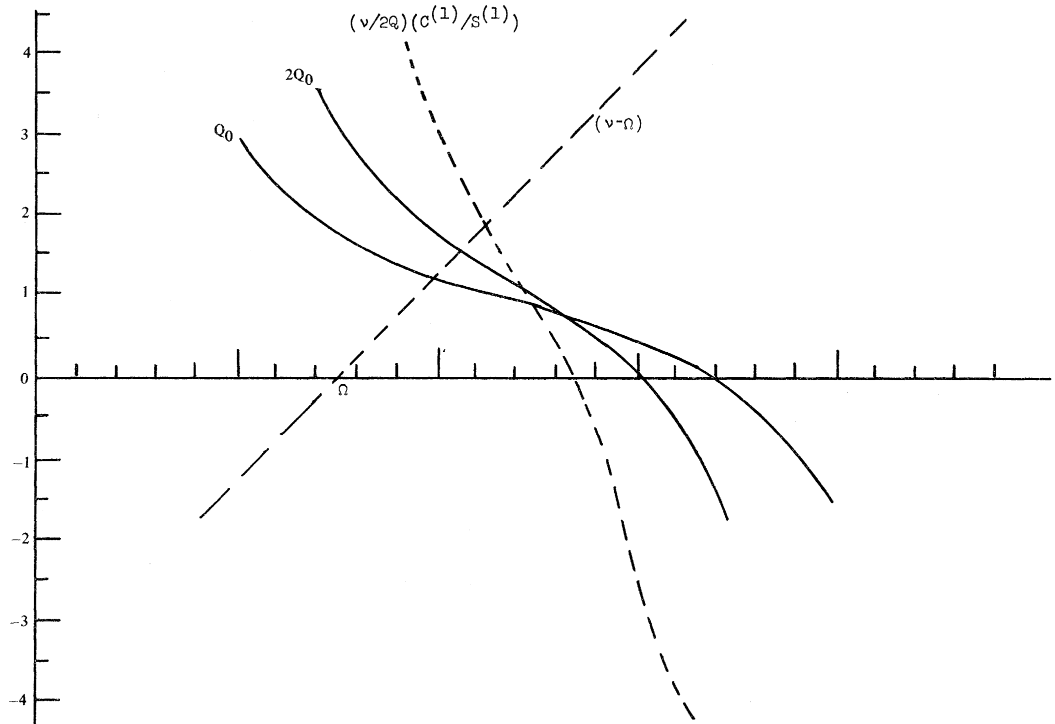

FIG. 13. Plots of $C(E)/E$ as a function of ν for strong signals and various values of Q. Intersection with the straight line $(\nu-\Omega)$ gives the laser frequency. The dashed curve shows $C(E)/E$ of Eq. (46) in the first-order theory for comparison.

where

$$\gamma=[1-(\dot{x}^2+\dot{y}^2)/c^2]^{-1/2}. \tag{56}$$

As in the case of the Duffing oscillator, let $E_x = E\cos(\nu t)$ and let $E_y=0$. Integrating (55) gives

$$\dot{y}=-eH_z x/\gamma mc. \tag{57}$$

Substituting (57) into (54) gives

$$\frac{d}{dt}\left(\gamma m\dot{x}\right)+\frac{e^2H_z^2x}{\gamma mc^2}=eE\cos(\nu t). \tag{58}$$

Assume the following solutions for $x(t)$ and $y(t)$ for single-mode operation:

$$x(t)=r(t)\cos[\nu t+\theta(t)],\quad y(t)=-r(t)\sin[\nu t+\theta(t)], \tag{59}$$

where $r(t)$ is the radius of the orbit of the electron, and $r(t)$ and $\theta(t)$ are taken in the slowly varying amplitude and phase approximation. Neglecting terms in $\dot{r}^2$, $\ddot{r}$, $\dot{r}\dot{\theta}$, $\ddot{\theta}$, and for slightly relativistic electrons we find that

$$\gamma\simeq 1+r^2\nu^2/2c^2, \tag{60}$$

$$\dot{\gamma}\cong\dot{r}\,r\nu^2/c^2. \tag{61}$$

Using (59)–(61) in (58) the following first-order differential equations for $r(t)$ and $\theta(t)$ are obtained:

$$\dot{r}=-G(1-r^2\omega^2/c^2)\sin\theta, \tag{62}$$

$$\dot{\theta}=\omega-\nu-r^2\omega^3/2c^2-Gr^{-1}\cos\theta, \tag{63}$$

where $G=(eE/2m\omega)$ and $|\omega-\nu|\ll\omega$. Since $r^2\omega^2/c^2$ is small compared to unity in (57), (62) and (63) are identical to the Eqs. (25) for A and θ in the Duffing problem:

$$\dot{r}=-G\sin\theta, \tag{64}$$

$$\dot{\theta}=\omega-\nu-r^2\omega^3/2c^2-Gr^{-1}\cos\theta. \tag{65}$$

The electron-cyclotron maser can therefore be treated using the theory given in Sec. IX.

XI. SUMMARY

A totally classical model of a laser has been treated, in which no mention has been made of photons or stimulated emission. A beam of randomly phased classical anharmonic oscillators passes through a resonant cavity and gives up energy to the radiation field. Nonlinearities in the medium are essential for producing self-sustained oscillations. A medium consisting of randomly phased harmonic oscillators (linear medium) can only absorb energy from the radiation field.

This model has been used to calculate the intensity and frequency of the resulting laser. The theory can be applied to Hirschfield's electron-cyclotron maser since extremely high quantum numbers are involved.

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†Work based on a thesis submitted by M. Borenstein to Yale University in partial fulfillment of the requirements for the Ph. D. degree.

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[12]Equations (27a) and (27b) were originally solved numerically by M. Sargent III and some of his results are shown in Ref. 7. We gladly acknowledge his assistance on the numerical part of this problem and his sharing of some of the difficulties.

[13]The coefficient on the right-hand side of Eq. (49) $N\wp^2/\hbar$ has the dimensions of number density × (dipole moment)^2/angular momentum. In Eq. (39) for the classical model, the coefficient on the right-hand side can be written in the form $N(ea)^2/2m(\omega a)a$ where "a" is the length of the pendulum, "ea" is the dipole moment, and $m(\omega a)a$ is the angular momentum.

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Effect of Velocity-Changing Collisions on the Output of a Gas Laser*

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A theoretical model for the pressure dependence of the intensity of a gas laser is presented in which only velocity-changing collisions with foreign-gas atoms are included. This is a special case where the phase shifts are the same for the two atomic-laser levels or are so small that deflections are the dominant effect of collisions. A collision model for hard-sphere repulsive interactions is derived and the collision parameters, persistence of velocity and collision frequency, are assumed to be independent of velocity. The collision theory is applied to a third-order expansion of the polarization in powers of the cavity electric field (weak-signal theory). The resulting expression for the intensity shows strong pressure dependence. The collisions reduce the amount of saturation and the laser intensity increases with pressure in a characteristic fashion. It is recommended that the best way to look for this effect is to make the measurements under conditions of constant relative excitation.

I. INTRODUCTION

The radiation emitted by an atomic system can be significantly affected by collisions with neighboring atoms. The parameters which determine the shape of a spectral line (atomic energy-level separation, decay rate, velocity) fluctuate due to random collisions during the radiative lifetime of the atomic system. There is an extensive literature on the effects of collisions on the shape of spectral lines covering about 70 years. A recent paper[1] gives a comprehensive list of references on this subject.

In a previous publication[2] (I) a model for a laser oscillator was presented in which the active atoms undergo collisions during their lifetimes. The result was a theoretical expression for the pressure dependence of the intensity of the laser in satisfac-