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Scheme to measure the positive P distribution

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A four-port arrangement is shown to yield a direct measurement of the positive P distribution of Drummond and Gardiner [J. Phys. A **13**, 2353 (1980)].

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Recently several proposals [1–4] have appeared for the study of the quasiprobabilities or the quantum phase space distributions, such as the Q function [2] and the Wigner functions [3,4]. One of these proposals [4] on the Wigner function has also been implemented experimentally[5]. This is rather remarkable, as the quantum properties of the phase-space distributions [6–9] arise from the noncommutativity of the relevant conjugate variables, and yet these can be measured. In this Rapid Communication we propose a scheme, involving four input ports and four output ports, which enables us to measure directly the positive P distribution of Drummond and Gardiner [10–12]. The idea of operationally measuring the positive P distribution first originated in the work of Braunstein, Caves, and Milburn [1]. The proposed scheme provides an all optical arrangement for an explicit measurement of the positive P distribution.

Drummond and Gardiner [10] showed that one can associate a positive distribution $P(\alpha,\beta)$ in two complex variables α and β such that its moments $\langle\beta^m\alpha^n\rangle$ are equal to the normally ordered moments, i.e.,

$$\mathrm{Tr}\{\rho a^{\dagger m}a^n\}=\int d^2\alpha\, d^2\beta\,\beta^m\alpha^n P(\alpha,\beta)\ , \tag{1}$$

$$=\int d^2\alpha\,\alpha^{*n}\alpha^n P(\alpha)\ , \tag{2}$$

where $P(\alpha)$ represents the function appearing in the diagonal coherent-state representation of ρ. The function $P(\alpha)$ need not always exist and can be singular for many quantum states of the radiation field. In contrast, Drummond and Gardiner show that it is possible to choose $P(\alpha,\beta)$ such that it possesses all the properties of a classical probability distribution. A useful choice is [12]

$$P(\alpha,\beta)=\frac{1}{4\pi}\exp\{-\tfrac{1}{4}|\alpha-\beta|^2\}Q(\tfrac{1}{2}(\alpha+\beta),\tfrac{1}{2}(\alpha^*+\beta^*))\ , \tag{3}$$

where the Q function is defined by [7]

$$Q(\alpha,\alpha^*)\rightarrow Q(\alpha)=\frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle\ . \tag{4}$$

We now discuss the kind of experimental arrangement which will be suitable for a direct measurement of $P(\alpha,\beta)$. Our scheme is based on the expression (3) and what can be achieved by a beam splitter. Consider the arrangement shown in Fig. 1. Let the field whose positive P distribution to be measured be represented by the annihilation and creation operators $\hat a$ and $\hat a^\dagger$. Let the field $\hat b$ be in the vacuum state. We first calculate the Q distribution associated with the two output ports $\hat c$ and $\hat d$. The beam splitter is represented by a SU(2) transformation [13]. For a 50%-50% beam splitter we write

$$\begin{bmatrix}\hat c\\ \hat d\end{bmatrix}=\frac{1}{\sqrt2}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}\hat a\\ \hat b\end{bmatrix}\ . \tag{5}$$

Consider next the generating function $\chi_{cd}(z_1,z_2)$ for the antinormally ordered moments

$$\chi_{cd}(z_1,z_2)=\langle\exp(-z_1^*\hat c-z_2^*\hat d)\exp(z_1\hat c^\dagger+z_2\hat d^\dagger)\rangle\ . \tag{6}$$

The Q function associated with the output ports is

$$Q_{cd}(\gamma,\delta)\equiv\frac{1}{\pi^2}\int d^2z_1d^2z_2\exp\{z_1^*\gamma+z_2^*\delta-z_1\gamma^*-z_2\delta^*\}\times\chi_{cd}(z_1,z_2)\ . \tag{7}$$

The transformation (5) enables us to calculate Q_{cd} in terms of the quantum properties of the input beams $\hat a$ and $\hat b$. We substitute (5) in (6) and simplify the resulting expression by noting that $\hat a$ and $\hat b$ are independent, to obtain

$$\chi_{cd}(z_1,z_2)=\left\langle\exp\left\{-\frac{\hat a}{\sqrt2}(z_1^*+z_2^*)\right\}\exp\left\{-\frac{\hat b}{\sqrt2}(z_2^*-z_1^*)\right\}\exp\left\{\frac{\hat b^\dagger}{\sqrt2}(z_2-z_1)\right\}\exp\left\{\frac{\hat a^\dagger}{\sqrt2}(z_1+z_2)\right\}\right\rangle\ . \tag{8}$$

On using the fact that the mode $\hat b$ is in vacuum, this reduces to

$$\chi_{cd}(z_1,z_2)=\exp\{-\tfrac{1}{2}|z_2-z_1|^2\}\chi_a\left[\frac{z_1+z_2}{\sqrt2}\right]\ , \tag{9}$$

where χ_a is the generating function for the a mode,

$$\chi_a(z)=\langle\exp(-\hat a z^*)\exp(\hat a^\dagger z)\rangle\ . \tag{10}$$

On substituting (9) in (7) and on changing the variables of integration, we arrive at the result

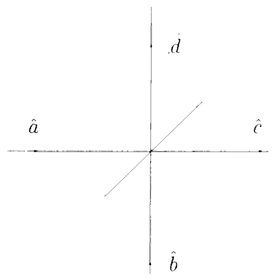

FIG. 1. Arrangement to show that the positive P distribution of the $\hat{a}$ mode can be measured in terms of the Q function associated with the modes $\hat{c}$ and $\hat{d}$.

$$Q_{cd}(\gamma,\delta)=\frac{1}{\pi}\exp\{-\tfrac{1}{2}|\delta-\gamma|^2\}Q_a\left[\frac{\gamma+\delta}{\sqrt{2}}\right] . \tag{11}$$

On comparison of (11) and (3) we get an important relation

$$4P_a(\sqrt{2}\gamma,\sqrt{2}\delta)\equiv Q_{cd}(\gamma,\delta) . \tag{12}$$

Hence the arrangement of Fig. 1 can be used to measure the positive P distribution of the input field $\hat{a}$, provided we can measure the joint Q distribution of the fields at the two output ports $\hat{c}$ and $\hat{d}$. We will now show how an argument originally due to Leonhardt and Paul [2] can be *generalized* to measure $Q_{cd}(\gamma,\delta)$. Consider now the scheme shown in Fig. 2. Here $\hat{a}'$ and $\hat{b}'$ are the vacuum fields at the unused ports. The fields at the four output ports are related to the input fields by

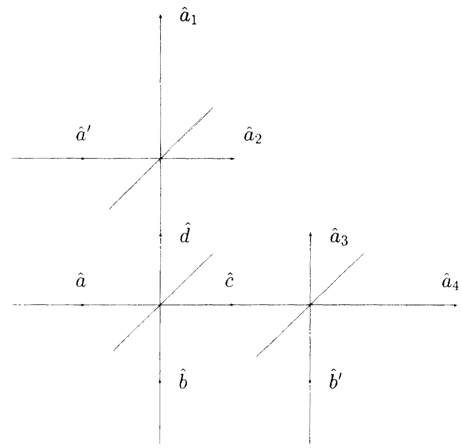

FIG. 2. Scheme to measure the joint Q function of the modes $\hat{c}$ and $\hat{d}$.

$$\hat{a}_1=\frac{\hat{a}'+\hat{d}}{\sqrt{2}},\quad \hat{a}_2=\frac{\hat{a}'-\hat{d}}{\sqrt{2}},\quad \hat{a}_3=\frac{\hat{c}+\hat{b}'}{\sqrt{2}},\quad \hat{a}_4=\frac{\hat{c}-\hat{b}'}{\sqrt{2}} . \tag{13}$$

Let us introduce the quadratures of the fields $\hat{x}$ and $\hat{p}$ by

$$\hat{x}_1=\frac{\hat{a}_1+\hat{a}_1^\dagger}{\sqrt{2}},\quad \hat{p}_1=\frac{\hat{a}_1-\hat{a}_1^\dagger}{\sqrt{2}i} . \tag{14}$$

Let $P(x_1,p_2,p_3,x_4)$ be the probability distribution for measuring the four commuting quadratures at the four output ports:

$$P(x_1,p_2,p_3,x_4)=\langle\delta(x_1-\hat{x}_1)\delta(p_2-\hat{p}_2)\delta(p_3-\hat{p}_3)\delta(x_4-\hat{x}_4)\rangle$$
$$\equiv\left[\frac{1}{2\pi}\right]^4\int d^4(k)\exp\{-ik_1x_1-ik_2p_2-ik_3p_3-ik_4x_4\}C(\{k_i\}) , \tag{15}$$

$$C(\{k_i\})=\langle\exp\{ik_1\hat{x}_1+ik_2\hat{p}_2+ik_3\hat{p}_3+ik_4\hat{x}_4\}\rangle . \tag{16}$$

On substituting (13) and (14) in (15) and in using the fact that the fields $\hat{a}'$ and $\hat{b}'$ are in vacuum, we get

$$C(\{k_i\})=\left\langle\exp\left\{i\left[\frac{k_1+ik_2}{2}\right]\hat{d}+i\left[\frac{k_4-ik_3}{2}\right]\hat{c}\right\}\times\exp\left\{i\left[\frac{k_1-ik_2}{2}\right]\hat{d}^\dagger+i\left[\frac{k_4+ik_3}{2}\right]\hat{c}^\dagger\right\}\right\rangle . \tag{17}$$

In deriving (17) we have used (i) the Baker-Hausdorff identity, (ii) normal ordering for the vacuum ports $\hat{a}'$ and $\hat{b}'$ and (iii) antinormal ordering for the fields $\hat{c}$ and $\hat{d}$. The choice of these two orderings enables us to cancel the extra factors coming from the Baker-Hausdorff formula. It should be borne in mind that (17) is the characteristic function associated with the Q function. We next use (17) in (15) and change the integration variables. A simple calculation then shows that

$$P(x_1,p_2,p_3,x_4)\equiv Q_{cd}(x_1-ip_2,x_4+ip_3) . \tag{18}$$

On combining (18) and (19) we obtain the main result of this Rapid Communication,

$$P_a(\gamma,\delta)\equiv\tfrac{1}{4}P\left[\frac{x_1}{\sqrt{2}},\frac{p_2}{\sqrt{2}},\frac{p_3}{\sqrt{2}},\frac{x_4}{\sqrt{2}}\right] ,$$
$$\gamma=x_1-ip_2,\quad \delta=x_4+ip_3 . \tag{19}$$

We have thus proved that the four-port arrangement of Fig. 2 can be used to obtain a direct measurement of the positive P distribution. The measurement scheme involves the measurement of the four quadratures (commuting variables) at the four output ports

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