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Analogies between light and electrons: Density of states and Friedel's identity

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By explicitly making use of analogies between Schrödinger potential scattering and classical wave scattering, we generalize statements for electron-impurity scattering to similar statements for scattering of scalar classical waves from dielectric particles. We will focus upon density of states, spectral function, Friedel's identity for screening, the Wigner phase-delay time, and the Thouless criterion for localization.

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I. INTRODUCTION

Knowledge of classical wave propagation has undoubtedly facilitated the formulation of quantum-wave theory for material particles. Nevertheless, quantum theory has existed separately ever since, probably due to the wealth of nonclassical phenomena that emerged from wave-particle duality, nonclassical in the sense that these phenomena are not predicted by Newton's laws of mechanics.

During the last couple of decades the close analogy between the equations of motion for Schrödinger waves and classical waves has been appreciated and explored. Many concepts and ideas in quantum mechanics were reformulated for classical waves. Examples are Anderson localization [1,2], weak localization [3], conductance fluctuations [4,5], optical crystals [6], Berry phases [7], and, more recently, tunneling aspects [8] and quantized conductances [9].

In many respects classical waves are more convenient for studying interference effects in multiple scattering. In the electronic case one often deals with phase-destroying (inelastic) effects, making interference in multiple scattering measurable only at very low temperatures. Despite technical difficulties it has fortunately become possible to enter the so-called *mesoscopic* regime in condensed matter where inelastic processes are negligible. Classical wave scattering, on the other hand, exhibits macroscopic interference even at room temperature, since the role of absorption can be suppressed with relative ease, making the observation of weak localization (enhanced backscattering), and possibly the onset of strong localization (vanishing or scale dependence of diffusion) much more easy to achieve.

In studying classical waves, several fundamental differences with Schrödinger particles show up as well. Some of them are well known and hardly need to be mentioned again. Long-wavelength classical scattering vanishes as λ^{-4}, whereas in the same limit a finite (s-wave) cross section is obtained for Schrödinger potential scattering. In addition, the interpretation of the complex field $\psi(\mathbf{r},t)$ in quantum theory as a probability amplitude is fundamentally different from the one for classical waves. Here $\psi(\mathbf{r},t)$ describes a measurable quantity, say the electric-field vector. Furthermore, the impact of absorption on classical-wave localization was recently shown to be different from the one of phase-destroying mechanisms in electron localization [10,11].

We recently reported on a distinguished difference between classical and Schrödinger waves, showing up in multiple scattering [12–14]. The velocity v_E entering into the diffusion constant $D = \frac{1}{3}v_E\ell$ (ℓ being the transport mean free path) turned out to be considerably smaller than the phase velocity $v_p \sim E/k$ (E the frequency, k some average-medium wave number) in the regime of resonant scattering. In electron-impurity language, the slowing down of classical waves can be attributed to the occurrence of giant mass-enhancement corrections.

Mass-enhancement factors do not show up in the diffusion constant for multiple electron-impurity scattering [15]. Here v_E is equal to $\hbar k_F/m_e^*$, with m_e^* the effective mass of electrons in the underlying crystal and k_F the Fermi wave number. The cancellation of such factors is due to particle conservation. In optical language the electron-impurity transport velocity could be called the reciprocal phase velocity c_0^2/v_p. This can be seen by using the analogy $\hbar E \leftrightarrow m_e^* c_0^2$ and $k_F \leftrightarrow k$.

The different conservation law for classical wave scattering invalidates this typical electron-impurity result. This can best be illustrated by comparing the equations of motion, in particular the eigenvalue equations for $\psi(t) = \psi_E \exp(-iEt)$ (see Table I). The presence of an *energy-dependent* "potential" will affect dynamical properties in scattering, such as the transport speed for multiple scattering. Closely related is the fact that $|\psi|^2$ no longer denotes the conserved quantity.

On the other hand, the associated currents are very similar (Table I). Since the current is the probed physical quantity in a stationary (dc) experiment, we might anticipate that stationary properties of multiple scattering will, from a theoretical point of view, be very similar for both kinds of waves. In particular, transport quan-

TABLE I. Comparison of the equations of motion for Schrödinger potential scattering (describing spinless electrons) and for scalar dielectric scattering (describing spinless light).

Schrödinger waves	Quantity	Classical scalar waves
$\psi(\mathbf{r},t)$	wave function	$\psi(\mathbf{r},t)$
$\left[p^2+V(\mathbf{r})\right]\psi=i\partial_t\psi$	equation of motion	$\left[p^2+\varepsilon(\mathbf{r})\partial_t^2\right]\psi=0$
$\left[p^2+V(\mathbf{r})\right]\psi_E=E\psi_E$	eigenvalue equation	$\left[p^2+V_E(\mathbf{r})\right]\psi_E=E^2\psi_E$
$V(\mathbf{r})$	potential	$V_E(\mathbf{r})=[1-\varepsilon(\mathbf{r})]\,E^2$
E, energy	eigenvalue	E^2, square of frequency
$\lvert\psi\rvert^2$: probability density	conserved quantity	$\frac{1}{2}\varepsilon(\mathbf{r})\lvert\partial_t\psi\rvert^2+\frac{1}{2}\lvert\partial_\mathbf{r}\psi\rvert^2$: energy density
$\operatorname{Im}\psi^*\partial_\mathbf{r}\psi$	current	$-\operatorname{Re}\partial_t\psi^*\partial_\mathbf{r}\psi$
$G(z,p)=\left[z-p^2-V\right]^{-1}$	Green's-function amplitude	$G(z,p)=\left[\varepsilon z^2-p^2\right]^{-1}$

tities such as the dc conductivity should not be affected by the different equations of motion.

In this paper we will investigate what other concepts in electron-impurity scattering will be modified by the introduction of an energy-dependent potential. We shall focus on Friedel's identity [16], linking the change in density of states caused by impurities to the Wigner phase-delay time [18], and often quoted in the context of screening, Jauch's space-integral formulation [17] for the Wigner phase-delay time, density of states (DOS) and the associated spectral function, giving the distribution of the levels (E,p) in an unbounded random medium. We shall also formulate an Einstein relation for classical wave diffusion and discuss the consequences for the Thouless criterion for localization.

II. FRIEDEL'S IDENTITY AND JAUCH'S FORMULA FOR ELECTRONS

The formula of Jauch [17] is a rigorous identity which is best interpreted as a formula for the amount of screened charge surrounding some impurity potential. The quantity,

$$J(\mathbf{k})\equiv\int d^3\mathbf{r}\,\left[|\psi_\mathbf{k}(\mathbf{r})|^2-1\right] \qquad (1)$$

($\psi_\mathbf{k}$ being a normalized continuum eigenfunction at energy eigenvalue $E=k^2$ of the impurity Hamiltonian) is proportional the amount of screened charge ($e=1$) near one impurity in vacuum given an incoming stationary electron flux $\exp(i\mathbf{k}\cdot\mathbf{r})$ from direction $\mathbf{k}$. The Jauch formula relates $J(\mathbf{k})$ to the Wigner phase delay time [18]. This shows, not unexpectedly, that the accumulation of electrons near the impurity is proportional to the delay suffered by them due to the presence of the interaction.

In principle, by using the Born series for the eigenfunction,

$$|\psi_\mathbf{k}\rangle=\left[1+\frac{1}{k^2-p^2+i0}\,t(k)\right]|\mathbf{k}\rangle \qquad (2)$$

[$t(k)$ being the transition operator of one impurity at energy k^2] the right hand side of Eq. (1) can be worked out algebraically in terms of Green's functions and t matrices $t_{\mathbf{pp}'}(k)\equiv\langle\mathbf{p}|\,t(k)\,|\mathbf{p}'\rangle$, so to arrive at a t-matrix formulation for the phase-delay time in three dimensions. However, following Friedel *et al.* [16], such an elaborate procedure can be circumvented by making the *physical* observation that the presence of the impurity implies a local change in the number of states that the electrons will occupy. From Eq. (1) it follows that the expected number of extra electrons with wave number $\mathbf{k}_j$ surrounding one impurity in a large but finite volume V is given by the dimensionless quantity $J(\mathbf{k}_j)/V$. The total number is therefore

$$\delta N=\sum_j\frac{J(\mathbf{k}_j)}{V}\;\underset{V\to\infty}{\longrightarrow}\;\int\frac{d^3\mathbf{k}}{(2\pi)^3}\,J(\mathbf{k})\,.$$

For a *dilute* medium with n randomly positioned impurities per unit volume, the change in *differential* DOS per unit volume should, according to Friedel's argument, be

$$\delta\rho(k)\equiv n\,\frac{d\,\delta N}{dk}=n\,\frac{k^2}{(2\pi)^3}\int d^2\hat{\mathbf{k}}\,J(\mathbf{k})\equiv n\,J(k)\,. \qquad (3)$$

We shall refer to this result as Friedel's identity, and it is a well-known result of scattering theory [19]. We can now find $J(k)$ by using the conventional trace over the *spectral function* [20],

$$\rho(k)=-\frac{2k}{\pi}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\operatorname{Im}G(k,p)\,, \qquad (4)$$

with $G(k,p)=\left[k^2-p^2-\Sigma(k,p)\right]^{-1}$ the Dyson retarded Green's function. Inserting the usual "independent-scattering" approximation for the self-energy [21], $\Sigma(k,p)=nt_{\mathbf{pp}}(k)$, we arrive at

$$\begin{aligned}\delta\rho(k)\equiv{}&\rho(k)-\rho_0(k)\\ ={}&-\frac{2nk}{\pi}\operatorname{Im}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{t_{\mathbf{pp}}(k)}{(k^2-p^2+i0)^2}\\ ={}&-\frac{2nk}{\pi}\operatorname{Im}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{\partial G_0(k,p)}{\partial(p^2)}\,t_{\mathbf{pp}}(k)\,.\end{aligned} \qquad (5)$$

Using one integration by parts, this is readily shown to be equal to

$$\delta\rho(k) = \frac{nk}{(2\pi)^3}\left[-\int d^2\hat{\mathbf{p}}\left(\frac{\partial p \,\mathrm{Re}\, t_{\mathbf{pp}}(k)}{\partial(p^2)}\right)_{p=k} - \frac{2}{\pi}\int d^3\mathbf{p}\, \frac{\partial\, \mathrm{Re}\, G_0(k,p)}{\partial(p^2)}\, \mathrm{Im}\, t_{\mathbf{pp}}(k)\right]. \tag{6}$$

By invoking a Ward identity [15] (page 617) [22] for the off-shell t matrix $t \equiv |t|\exp(i\phi)$,

$$-\int d^2\hat{\mathbf{p}}\, \frac{\partial \mathrm{Re}\, t_{\mathbf{pp}}(k)}{\partial(k^2)} + \int d^2\hat{\mathbf{p}}\, d^2\hat{\mathbf{p}}'\, \frac{|t_{\mathbf{pp}'}(k)|^2}{(4\pi)^2}\, \frac{\partial\phi(\mathbf{p},\mathbf{p}',k)}{\partial k} - \frac{2}{\pi k}\int d^3\mathbf{p}\, \frac{\partial \mathrm{Re}\, G_0(k,p)}{\partial(k^2)}\, \mathrm{Im}\, t_{\mathbf{pp}}(k) = 0\,, \tag{7}$$

and using that $\partial\, \mathrm{Re}\, G_0/\partial p^2 = -\partial\, \mathrm{Re}\, G_0/\partial k^2$, Eq. (6) can be rewritten as [23]

$$\delta\rho(k) = n\, J(k) = \frac{nk^2}{(2\pi)^3}\left[-\int d^2\hat{\mathbf{k}}\, \frac{d\, k\, \mathrm{Re}\, t_{\mathbf{kk}}(k)}{k\, d(k^2)} + \int d^2\hat{\mathbf{k}}\, d^2\hat{\mathbf{k}}'\, \frac{|t_{\mathbf{kk}'}(k)|^2}{(4\pi)^2}\, \frac{d\phi(\mathbf{k},\mathbf{k}')}{dk}\right]. \tag{8}$$

This formula is the most important result of this section. Equation (8) gives $J(k)$ conveniently in terms of total derivatives of *on-shell* t matrices $t_{\mathbf{kk}'}(k)$ of the individual scatterers. The quantity $|t_{\mathbf{kk}'}(k)|^2/(4\pi)^2$ is usually called the differential cross section $d\sigma/d\Omega$. Given definition (1), the first equality in Eq. (8) is Friedel's identity, the second Jauch's formula. That the third term is indeed (proportional to) the Wigner phase-delay time in three dimensions can be illustrated explicitly for spherically symmetric potentials [13,14]. Then Eq. (8) reduces to

$$J(k) = \frac{1}{\pi}\sum_{\ell=0}^{\infty}(2\ell+1)\frac{d\phi_\ell}{dk}\,, \tag{9}$$

where $\phi_\ell(k)$ is the phase shift in the angular-momentum channel ℓ.

We stress that the Ward identity (7) states that all mass-enhancement factors cancel, as was mentioned earlier. Since these factors are (by our definition [24]) all energy derivatives, they will no longer cancel if the t matrix is obtained from an energy-dependent potential. This issue will be addressed in Sec. III. In this context we also mention that $J(k)$ is, contrary to what Eq. (9) might suggest, *not* a mass-enhancement correction. As is evident from Eq. (6), $J(k)$ is a genuine wave number derivative.

III. FRIEDEL'S ARGUMENT FOR CLASSICAL WAVES

Let us first modify the definition for $J(\mathbf{k})$ in Eq. (1) for classical scalar waves. Judging from Table I it seems logical to redefine $J(\mathbf{k})$ for this case (denoted by an asterisk) as

$$\begin{aligned} E^2 J^*(\mathbf{k}) &\equiv \int d^3\mathbf{r}\left[\frac{1}{2}\varepsilon(\mathbf{r})E^2|\psi_{\mathbf{k}}(\mathbf{r})|^2 + \frac{1}{2}|\partial_{\mathbf{r}}\psi_{\mathbf{k}}(\mathbf{r})|^2 - E^2\right] \\ &= E^2\int d^3\mathbf{r}\ \left[\varepsilon(\mathbf{r})|\psi_{\mathbf{k}}(\mathbf{r})|^2 - 1\right]\,, \end{aligned} \tag{10}$$

where $\psi_{\mathbf{k}}$ is the same eigenfunction as in Eq. (1), for a "potential" $V(\mathbf{r},E) = [1-\varepsilon(\mathbf{r})]E^2$. Furthermore,

$$J^*(E) \equiv \frac{E^2}{(2\pi)^3}\int d^2\hat{\mathbf{k}}\, J^*(\mathbf{k})\,. \tag{11}$$

The second equality in Eq. (10) is a natural consequence of "equipartition" that can easily be checked from the eigenvalue equation; $E = \pm k$ is now the frequency. By making explicit use of the analogy of an energy-dependent potential it was recently demonstrated by one of us [25] that Jauch's formula holds true for J^*, that is the second equality of Eq. (8) [and thus Eq. (9) as well] is valid for scalar waves as well provided one then uses $J^*(E)$. It was shown that the difference between J and J^*, that is the potential energy of the scalar field, is completely equivalent to the extra mass-enhancement contributions entering into Eq. (7) due to the energy-dependence of the potential.

Let us next find out whether Friedel's argument applies, linking $J^*(E)$ to the change in DOS, thus the first equality of Eq. (8). From Table I we are tempted to associate an "energy" $E^2 = k^2$ with the eigenfunction $\psi_{\mathbf{k}}$, since this is the eigenvalue of the "Hamiltonian." But Eq. (10) gives the total excess of classical-wave energy near the dielectric scatterer. The excess number of states near the dielectric is therefore $E^2 J^*(E)/E^2 = J^*(E)$. For a dilute sample of n dielectric particles per unit volume the excess DOS per unit volume thus becomes, according to the Friedel argument,

$$\delta\rho^*(E) = \tfrac{1}{2}\, n\, J^*(E)\,. \tag{12}$$

The extra factor 1/2 emerges since for classical waves there also exist counter-rotating states with $E < 0$, with $\rho^*(E) = \rho^*(-E)$.

Can this identity be derived directly from scattering theory? Following the analogies leads us to consider the scalar-wave Green's function

$$G(z,p,\mathbf{r}) = \frac{1}{\varepsilon(\mathbf{r})z^2 - p^2}\,. \tag{13}$$

After ensemble-averaging (denoted by $\langle\,\rangle$) the Dyson Green's function becomes translationally invariant and takes the familiar Dyson form $\langle G(z,p)\rangle = [z^2 - p^2 - \Sigma(z,p)]^{-1}$. As was done for electrons, we may construct a positive-definite distribution from its imaginary part,

$$S(E,p) = -\frac{E}{\pi}\, \mathrm{Im}\, \langle G(E+i0,p)\rangle\,. \tag{14}$$

[Note the extra factor 1/2 again when compared to Eq. (4)]. This distribution is not normalized to unity since it obeys the sumrule [13]

$$\int_{-\infty}^{\infty} dE\, S(E,p) = \langle \varepsilon^{-1} \rangle \neq 1 . \tag{15}$$

For this reason $S(E,p)$ cannot be the true averaged distribution of states (E,p) in the medium. The tempting analogy with electrons thus here leads to an erroneous result.

The spectral operator that is manifestly normalized to unity is

$$\begin{aligned} S^*(E) &= |E| \left\langle \delta \left(E^2 - \varepsilon(\mathbf{r})^{-1} p^2 \right) \right\rangle \\ &= -\frac{E}{\pi} \left\langle \varepsilon(\mathbf{r}) \operatorname{Im} G(E + i0, \mathbf{p}, \mathbf{r}) \right\rangle , \end{aligned} \tag{16}$$

which does not depend on $\mathbf{r}$ as a result of averaging. Using the complete normalized plane wave set $|\mathbf{p}_j\rangle / \sqrt{V}$, the spectral distribution $S^*(E, p_j)$ can be constructed as

$$S^*(E,p_j) \equiv \frac{1}{V} \langle \mathbf{p}_j | \, S^*(E) \, | \mathbf{p}_j \rangle \underset{V\to\infty}{\longrightarrow} \frac{p^2}{E^2} S(E,p) . \tag{17}$$

$S(E,p)$ was defined in Eq. (14); we have applied the identity $\langle \varepsilon G(E,p) \rangle = 1/E^2 + \left\langle (p^2/E^2) G(E,p) \right\rangle$. This treatment shows that $S^*(E,p)$ is the spectral function for scalar classical waves, and not $S(E,p)$. As a result, the total DOS per unit volume becomes

$$\rho^*(E) = \frac{1}{V} \sum_j S^*(E,p_j) \underset{V\to\infty}{\longrightarrow} \int \frac{d^3\mathbf{p}}{(2\pi)^3} S^*(E,p) , \tag{18}$$

which we will now show satisfies a Friedel identity. To this end we write

$$\begin{aligned} S^*(E,p) &= S(E,p) + \frac{p^2 - E^2}{E^2} S(E,p) \\ &= S(E,p) + \frac{n}{\pi E} \operatorname{Im} \frac{t_{\mathbf{pp}}(E)}{E^2 - p^2 + i0} + \mathcal{O}(n^2) . \end{aligned} \tag{19}$$

The first term is actually the spectral function in Eq. (4) for electron-impurity scattering and gives rise to a Friedel identity (3) with the extra factor of 1/2. We are left to show that the wave-number integral of the second extra term of the spectral function equals $\frac{1}{2} n \left[J^*(E) - J(E) \right]$. The identity

$$\begin{aligned} &\frac{1}{\pi E} \operatorname{Im} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{t_{\mathbf{pp}}(E)}{E^2 - p^2 + i0} \\ &\quad = \frac{1}{2} \frac{E^2}{(2\pi)^3} \int d^2\hat{\mathbf{k}} \, \langle \psi_{\mathbf{k}} | \, \varepsilon - 1 \, | \psi_{\mathbf{k}} \rangle \\ &\quad = \frac{1}{2} \left[J^*(E) - J(E) \right] \end{aligned} \tag{20}$$

follows straightforwardly from scattering theory [26], thereby using Eq. (2); the second equality is obtained by subtracting the definitions (10) and (1).

We can thus conclude that Friedel's argument to find the change in DOS using the Wigner phase delay time works for classical scalar waves as well. This is consistent with heuristic derivations of Friedel's identity [15] (page 233) [27] (page 343), which involve a counting of the number of nodes of the wave function.

Finally, we want to comment on the difference between the spectral distributions $S(E,p)$ and $S^*(E,p)$. We have shown that the latter describes the true distribution of levels (E,p) in the random medium. The difference between both is particularly large away from the energy shell $E(p)$, and thus signifies contributions near or even inside the scatterers. This makes it plausible that $S(E,p)$ can be regarded as a distribution of traveling waves, whereas $S^*(E,p)$ represents the one for both traveling and standing waves. As a result $S(E,p)$ is nevertheless a relevant distribution in transport theory. In Sec. IV we argue that the use of $S^*(E,p)$ is sometimes essential, for instance in the Einstein relation for the conductivity.

IV. THE EINSTEIN RELATION AND THE THOULESS CRITERION

The electric (longitudinal) conductivity is an extremely important quantity in condensed matter theory, since it is accessible by experiment, and is extremely sensitive for details in electron transport. The dc longitudinal conductivity describes a stationary electron current in the presence of a homogeneous static electric field.

One important property of this dc conductivity is the absence of energy derivatives [15,28]. The Z-renormalization factor,

$$Z(E) = \left(1 - \frac{\partial \operatorname{Re} \Sigma(E,p)}{\partial E} \right)_{E=p^2} , \tag{21}$$

often called *the* mass-enhancement factor, is one example. This Z factor is actually the first term in Eq. (7) generalized for higher densities, and cancels against the other two mass-enhancement factors in this equation. Consequently, no such factors appear in the diffusion constant for electron-impurity scattering [22]. From Eq. (4) it is evident that such factors do not show up in the DOS either [recall that $J(k)$ is not considered to be a mass-enhancement]. By the Einstein relation for the electric conductivity,

$$\sigma(E) = \rho(E) \, D(E) , \tag{22}$$

these factors must therefore indeed be absent in the conductivity. In the present section we make it plausible that mass-enhancement factors also cancel in the conductivity for scalar waves, despite the fact that the cancellation theorem, as expressed by Eq. (7), no longer applies for classical scalar waves.

The last fact causes mass-enhancement factors to show up in the classical-wave diffusion constant. In terms of $J^*(E)$ defined in Eq. (10), the formula for the classical-wave diffusion constant reads [13,14]

$$D^*(E) = \frac{D_0(E)}{1 + 2\pi^2 n\, J^*(E)/E^2}\,. \tag{23}$$

Here $D_0 \sim \frac{1}{3}\, c_0\, \ell$. For classical waves no linear response theory can be applied to define the (Kubo) longitudinal conductivity. We can therefore define the classical-wave conductivity for a slab using a Landauer-type formula [29],

$$\sigma^* \equiv T \frac{L}{A}, \tag{24}$$

where T is the stationary diffusive "all-channel-in all-channel-out" transmission coefficient for slab with length L and front surface A, and is known to be proportional to $k^2 A\ell/L$ [30]. The Friedel identity (12) for classical waves shows that the DOS is renormalized as $\rho^*(E) = \rho_0(E)\left[1 + 2\pi^2 n\, J^*(E)/E^2\right]$. Filling in $\rho_0 \sim k^2/c_0$ and $D_0 \sim c_0\, \ell$ we find that

$$\sigma^*(E) \sim \rho^*(E)\, D^*(E) \tag{25}$$

apart from a numerical factor of order unity. The Einstein relation thus holds true, at least in the low-density regime under consideration. Its validity is more surprising for classical waves since it relates two quantities subject to mass-enhancement to one that is not affected by these corrections. For simplicity, we have assumed that $J(E)$ is negligible compared to $J^*(E)$, which is typically true near resonances. The exact microscopic outcome shows that c_0 in Eq. (23) must be replaced by c_0^2/v_p (v_p being the phase velocity) and J^* by $J^* - J$. In the context of other work we note that the cancellation of the transport velocity in the conductivity is not unrelated to the cancellation of the *group* velocity in one-dimensional multimode waveguides [9].

We discuss one important case where the conductivity and the Einstein relation play an important role: the Thouless criterion for strong localization [31,32]. When applied to classical waves this criterion asserts that states start to become localized whenever the dimensionless conductance $g^*(E) \sim T$ becomes equal to some universal critical value g_c of order unity. Since T is a time-integrated transmission and thus dc, we learn from this criterion that dc transmission experiments in principle provide sufficient and conclusive evidence for the onset of strong localization of classical waves. Applying the Einstein relation (25) gives

$$g^*(E) \sim N^*(E) \frac{D^*(E)}{L^2}\,. \tag{26}$$

Here $N^*(E) = \rho^*(E) \cdot AL$ is the total number of states at frequency E. Equation (26) is the well-known Thouless ratio of level spacing $1/N^*(E)$ and energy uncertainty D^*/L^2 from the diffusive transport. Localization thus implies that the level spacing exceeds the diffusive uncertainty. Since we have made it plausible that the same mass-enhancement factors enter in numerator and denominator of Eq. (26) these factors are not expected to modify the criterion for localization. This suggests that the correct classical-wave spectral distribution (17) is crucial in the experimental application of Eq. (26). If we would have used the spectral function in Eq. (14) the dimensionless conductivity would have suffered from mass-enhancement factors. In Ref. [13] this was a reason to reformulate Thouless's criterion in terms of uncertainties in wave numbers, rather than energies. If we incorporate the states inside the dielectric scatterers, such a procedure becomes redundant.

The criterion (26) is often used to discuss the role of both the diffusion constant and the DOS separately. As we will point out this can be misleading. Contrary to what is sometimes believed, the existence of a small diffusion coefficient may not be sufficient to decide for the onset of localization since it suffers from both (weak) localization corrections and mass-enhancement factors. The latter can be quite large ($n\, J^* \approx$5–10 in some experiments [12,33,34]).

On the other hand, DOS arguments for the onset of localization should also be carefully applied. It does not seem to be generally true that spectral regions with low DOS are the best suited to find localization of classical waves. Since the DOS suffers from mass-enhancement factors, it is large near scattering resonances. Nevertheless, it is often stated that these spectral regions are likely to be subject to localization [35,36] since the mean free path is small and the Ioffe-Regel criterion $k\ell \approx 1$ may be satisfied. We think that DOS arguments for localization should only concern the DOS for the traveling waves (thus ρ and not ρ^*) or even the one for the parent system (the one without disorder and thus ρ_0). This last statement is put forward by John [37].

V. CONCLUSIONS

In this paper we have discussed some fundamental differences between scalar classical waves and Schrödinger waves related to energy-dependent potentials. We have shown that, in spite of these differences and by explicitly making use of them, some very fundamental statements remain analogous, such as Friedel's argument to find the change in density of states from the phase shifts, the cancellation of mass-enhancement factors in the conductivity, and the Einstein relation. We have shown that resonances in classical-wave scattering result in small diffusion constants and large density of states. The quantity unambiguously sensible for localization is the dc conductivity or, for classical waves, the time-integrated transmission coefficient, where all these microscopic delay effects cancel.

Some other results in condensed matter theory do not seem to have an analogy in optics. These are often related to the absence of "photon-photon" interactions at the level of classical wave physics. For instance, the Friedel sum rule [27],

$$\int_0^{k_F} dk\, J(k) = \frac{1}{\pi} \sum_{\ell=0}^{\infty} (2\ell + 1)\, \phi_\ell(k_F) = Z\,, \tag{27}$$

Z being the atomic number of the impurity and k_F the Fermi wave number, tells that (spin 1/2) electrons scatter from a self-consistently screened (spherical) potential that decays faster than $1/r$ by charge neutrality. It does not seem to have a classical-wave counterpart.

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[1] P.W. Anderson, Philos. Mag. B **52**, 505 (1985).
[2] M. Kaveh, Philos. Mag. B **56**, 693 (1987).
[3] M.P. van Albada and A. Lagendijk, Phys. Rev. Lett. **55**, 2692 (1985); E. Akkermans, P.E. Wolf, and R. Maynard, *ibid.* **56**, 1471 (1986); M. Kaveh, M. Rosenbluh, I. Edrei, and I. Freund, *ibid.* **57**, 2049 (1986).
[4] E. Kogan, R. Baumgartner, R. Berkovits, and M. Kaveh, Physica B (to be published).
[5] E. Kogan, R. Baumgartner, R. Berkovits, and M. Kaveh, Phys. Rev. B **48**, 9404 (1993).
[6] E. Yablonovitch, in *Analogies in Optics and Micro Electronics*, edited by W. van Haeringen and D. Lenstra (Kluwer, Dordrecht, 1990); for a recent review, see E. Yablonovitch, J. Phys. C **5**, 2443 (1993).
[7] D. Lenstra, in *Huygens' Principle 1969-1990, Theory and Applications*, edited by H. Blok, H.A. Ferweda, and H.K. Kuiken (Elsevier, New York, 1992).
[8] A. Ranfagnie, D. Mugnai, P. Fabeni, G.P. Pazzi, G. Naletto, and C. Sozzi, Physica B **175**, 283 (1991).
[9] E.A. Montie, E.C. Cosman, G.W. 't Hooft, M.B. van der Mark, and C.W.J. Beenakker, Nature (London) **350**, 594 (1991).
[10] R.L. Weaver, Phys. Rev. B **47**, 1077 (1993).
[11] M. Kaveh and E. Kogan, in *Photonic Band Gaps and Localization*, edited by C. M. Soukoulis (Plenum, New York, 1993), p. 187.
[12] M.P. van Albada, B.A. van Tiggelen, A. Lagendijk, and A. Tip, Phys. Rev. Lett. **66**, 3132 (1991).
[13] B.A. van Tiggelen, A. Lagendijk, M.P. van Albada, and A. Tip, Phys. Rev. B **45**, 12 233 (1992).
[14] E. Kogan and M. Kaveh, Phys. Rev. B **46**, 10 636 (1992).
[15] G.D. Mahan, *Many-Particle Physics* (Plenum, New York, 1981).
[16] F. Friedel, F. Gautier, A.A. Gomèz, and P. Lenglart, in *Quantum Theory of Atoms, Molecules, and the Solid State*, edited by P.O. Löwdin (Academic, New York, 1966).
[17] J.M. Jauch, K.B. Sinha, and B.N. Misra, Helv. Phys. Acta **45**, 398 (1972).
[18] E.P. Wigner, Phys. Rev. **70**, 606 (1946).
[19] For a recent application of Friedel's identity, see E. Akkermans, Europhys. Lett. **15**, 709 (1991).
[20] E.N. Economou, *Green's Functions in Quantum Physics* (Springer-Verlag, Berlin, 1979).
[21] U. Frisch, *Probabilistic Methods in Applied Mathematics, Vol. I*, edited by A.T. Bharucha-Reid (Academic, New York, 1968).
[22] D. Vollhardt and P. Wölfle, Phys. Rev. B **22**, 4666 (1980).
[23] In the second mass-enhancement factor of Eq. (7) we can replace the partial energy derivative by a total derivative. The argument follows from time-reversal symmetry but is not given here. A related proof is given by B.J. Baumgartl, Z. Phys. **198**, 148 (1967), Eqs. (57)–(59).
[24] G. Cwilich and Y. Fu, Phys. Rev. B **46**, 12 015 (1992). In this reference mass-enhancement corrections to diffusion constant are obtained from momentum derivatives, and not from energy (frequency) derivatives as in this paper.
[25] B.A. van Tiggelen, A. Tip, and A. Lagendijk, J. Phys. A **26**, 1731 (1993).
[26] B.A. van Tiggelen, A. Lagendijk, and A. Tip, Phys. Rev. Lett. **71**, 1284 (1993).
[27] C. Kittel, *Quantum Theory of Solids* (Wiley, New York, 1963).
[28] G. Grimvall, Phys. Scr. **14**, 63 (1976).
[29] D.S. Fisher and P.A. Lee, Phys. Rev. B **23**, 6851 (1981).
[30] Th.M. Nieuwenhuizen and J.M. Luck, Phys. Rev. E **48**, 569 (1993).
[31] J.T. Edwards and D.J. Thouless, J. Phys. C **5**, 807 (1972).
[32] E. Abrahams, P.W. Anderson, D.C. Licciardello, and T.V. Ramakrishnan, Phys. Rev. Lett. **42**, 673 (1979).
[33] J.M. Drake and A.Z. Genack, Phys. Rev. Lett. **63**, 259 (1989).
[34] J.F. de Boer, M.P. van Albada, and A. Lagendijk, Phys. Rev. B **45**, 658 (1992).
[35] D. Sornette and B. Souillard, Europhys. Lett. **7**, 269 (1988).
[36] P. Sheng and Z.Q. Zhang, Phys. Rev. Lett. **57**, 1879 (1986).
[37] S. John, Comments Condens. Matter Phys. **4**, 193 (1988).