Eigenvectors of two particles' relative position and total momentum

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We give the explicit form of the common eigenvectors of the relative position Q_1-Q_2 and the total momentum P_1+P_2, of two particles which were considered by Einstein, Podolsky, and Rosen [Phys. Rev. **47**, 777 (1935)] in their argument that the quantum-mechanical state vector is not complete. Orthonormality and completeness of such eigenvectors, as well as their use in constructing the unitary operator for simultaneously squeezing Q_1-Q_2 and P_1+P_2, are derived by using the technique of integration within an ordered product of operators.

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I. INTRODUCTION

Traditionally, when studying quantum-mechanical models, one diagonalizes a complete commuting set of coordinate operators and works with the associated δ-function normalized eigenvectors. On rare occasions, one may wish to diagonalize some coordinates and some other (commuting) momenta variables and work with the relevant δ-function normalized eigenstates. Such states may appear, for example, in Maslov's semiclassical quantization scheme [1], on in an even more famous example, in the scheme proposed in 1935 by Einstein, Podolsky, and Rosen (EPR) in their study of correlated systems and the significance on the outcome of a second, noncausally connected measurement to the results of a first measurement [2]. While this experiment has undergone refinement in detail, as well as experimental conformation fully in accord with quantum mechanics [3], these facts do not diminish interest in properties of the original set of operators and their eigenvectors as considered by EPR. Specifically, let Q_j, P_j, $j=1,2$, be a standard pair of Heisenberg variables for which $[Q_j,P_k]=i\delta_{jk}$. Then the new variables of interest are a relative coordinate Q_1-Q_2 and a total momentum P_1+P_2, which evidently commute and thus can be simultaneously diagonalized.

Our goal in this paper is to study constructing such simultaneous eigenstates in terms of conventional creation and annihilation operators, as well as to reexpress several operators that employ these states in Dirac-like representations involving integrations over the eigenvalues with and without dilations. The construction of the operators of interest is greatly facilitated by the use of the integration within an ordered product (IWOP) technique that has also been previously used in other problems [4]. The present example illustrates once again the power and utility of the IWOP technique.

II. THE COMMON EIGENSTATE OF Q_1-Q_2 AND P_1+P_2 IN A TWO-MODE FOCK SPACE

In the two-mode fock space spanned by

$$|nm\rangle=\frac{a^{\dagger n}b^{\dagger m}}{\sqrt{n!m!}}|00\rangle\ , \tag{1}$$

where $[a,a^\dagger]=[b,b^\dagger]=1$, $|00\rangle$ is the ground state, we shall prove that the common eigenstates of Q_1-Q_2 and P_1+P_2 are all given by

$$|\eta\rangle=\exp\left[-\frac{|\eta|^2}{2}+\eta a^\dagger-\eta^* b^\dagger+a^\dagger b^\dagger\right]|00\rangle \tag{2}$$

in which $\eta=\eta_1+i\eta_2$ is an arbitrary complex number. In fact, acting with a and b on $|\eta\rangle$, respectively, gives us

$$a|\eta\rangle=(\eta+b^\dagger)|\eta\rangle\ ,\quad b|\eta\rangle=(-\eta^*+a^\dagger)|\eta\rangle\ . \tag{3}$$

It then follows that

$$(a-b^\dagger)|\eta\rangle=\eta|\eta\rangle\ ,\quad (b-a^\dagger)|\eta\rangle=-\eta^*|\eta\rangle \tag{4}$$

The sum and difference of these two equations lead to

$$\frac{1}{\sqrt{2}}[(a+a^\dagger)-(b+b^\dagger)]|\eta\rangle=\sqrt{2}\eta_1|\eta\rangle=(Q_1-Q_2)|\eta\rangle\ , \tag{5}$$

$$\frac{1}{\sqrt{2}i}[(a-a^\dagger)+(b-b^\dagger)]|\eta\rangle=\sqrt{2}\eta_2|\eta\rangle=(P_1+P_2)|\eta\rangle\ , \tag{6}$$

where

$$Q_1=\frac{a+a^\dagger}{\sqrt{2}}\ ,\quad Q_2=\frac{b+b^\dagger}{\sqrt{2}}\ ;\qquad P_1=\frac{a-a^\dagger}{\sqrt{2}i}\ ,\quad P_2=\frac{b-b^\dagger}{\sqrt{2}i} \tag{7}$$

are two-mode coordinate and momentum operators.

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III. SOME PROPERTIES OF $|\eta\rangle$

We now examine whether or not $|\eta\rangle$ satisfies a completeness relation. This can be very easily studied with the IWOP technique [4]. Using the normal ordering form of the two-mode vacuum projector

$$|00\rangle\langle 00| = :e^{-a^\dagger a - b^\dagger b}: \tag{8}$$

we have

$$\int \frac{d^2\eta}{\pi}|\eta\rangle\langle\eta| = \int \frac{d^2\eta}{\pi} :e^{-|\eta|^2+\eta a^\dagger-\eta^* b^\dagger + a^\dagger b^\dagger - a^\dagger a - b^\dagger b + \eta^* a - \eta b + ab}:$$
$$= :\exp\{(a^\dagger - b)(a-b^\dagger) + a^\dagger b^\dagger + ab - a^\dagger a - b^\dagger b\}: = 1 \ . \tag{9}$$

Next we calculate the overlap $\langle\eta'|\eta\rangle$. With the aid of (4) we see that

$$\langle\eta|(a^\dagger - b) = \eta^*\langle\eta| \ , \quad \langle\eta|(b^\dagger - a) = -\eta\langle\eta| \ . \tag{10}$$

Therefore, from (10) and (4) we have

$$\begin{aligned}\langle\eta'|(a-b^\dagger)|\eta\rangle &= \eta\langle\eta'|\eta\rangle = \eta'\langle\eta'|\eta\rangle \ , \\ \langle\eta'|(b-a^\dagger)|\eta\rangle &= -\eta'^*\langle\eta'|\eta\rangle = -\eta^*\langle\eta'|\eta\rangle \ ,\end{aligned} \tag{11}$$

which together with (9) implies that

$$\langle\eta'|\eta\rangle = \pi\delta^{(2)}(\eta'-\eta) \ . \tag{12}$$

Hence, $|\eta\rangle$ is an orthonormal eigenstate of $(Q_1 - Q_2)$ and (P_1+P_2), which as given by (2) has apparently not been considered in the literature before.

IV. THE EIGENSTATE OF Q_1+Q_2 AND P_1-P_2

On the other hand, we can also derive the common eigenstates of Q_1+Q_2 and P_1-P_2. They are all given by

$$(Q_1+Q_2)|\xi\rangle = \xi_1|\xi\rangle \ , \quad (P_1-P_2)|\xi\rangle = \xi_2|\xi\rangle \tag{13}$$

$$\begin{aligned}|\xi\rangle &= \exp\left[-\frac{|\xi|^2}{2} + \xi a^\dagger + \xi^* b^\dagger - a^\dagger b^\dagger\right]|00\rangle \ , \\ \xi &= \xi_1 + i\xi_2\end{aligned} \tag{14}$$

where ξ is also an arbitrary complex number. Similarly, using the IWOP we can prove that

$$\int \frac{d^2\xi}{\pi}|\xi\rangle\langle\xi| = \int \frac{d^2\xi}{\pi} :e^{-|\xi|^2+\xi(a^\dagger+b)+\xi^*(b^\dagger+a)-(a^\dagger+b)(b^\dagger+a)}: = 1 \ , \tag{15}$$

$$\langle\xi'|\xi\rangle = \pi\delta^{(2)}(\xi'-\xi) \ . \tag{16}$$

Let us recall the definition of two-index Hermite polynominals [5]

$$H_{m,n}(\xi,\xi^*) = \sum_{l=0}^{\min(m,n)} \frac{m!n!}{l!(m-l)!(n-l)!} \times(-1)^l \xi^{m-l}\xi^{*n-l} \tag{17}$$

or

$$H_{m,n}(\xi,\xi^*) = \left.\frac{\partial^{n+m}}{\partial t^m \partial t'^n} e^{-tt'+t\xi+t'\xi^*}\right|_{t=t'=0} \tag{18}$$

whose generating function is

$$\sum_{m,n=0}^{\infty} \frac{t^m t'^n}{m!n!} H_{m,n}(\xi,\xi^*) = e^{-tt'+t\xi+t'\xi^*} \ . \tag{19}$$

We can expand the exponential, which is within the symbol : :, as

$$e^{-|\xi|^2} :e^{-a^\dagger b^\dagger + \xi a^\dagger + \xi^* b^\dagger} e^{-ab+\xi b+\xi^* a} e^{-a^\dagger a - b^\dagger b}:$$
$$= e^{-|\xi|^2} \sum_{m,n=0}^{\infty} \sum_{m',n'=0}^{\infty} \frac{a^{\dagger m} b^{\dagger n}}{m!n!} H_{m,n}(\xi,\xi^*) :e^{-a^\dagger a - b^\dagger b}: \frac{a^{m'} b^{n'}}{m'!n'!} H^*_{m',n'}(\xi,\xi^*) \ . \tag{20}$$

Thus, the completeness relation can be rewritten as

$$\int \frac{d^2\xi}{\pi}|\xi\rangle\langle\xi| = \int \frac{d^2\xi}{\pi} e^{-|\xi|^2} \sum_{m,n,m',n'=0}^{\infty} |mn\rangle\langle m'n'| H_{m,n}(\xi,\xi^*) H^*_{m',n'}(\xi,\xi^*) = 1 \tag{21}$$

which tells us the important property that $H_{m,n}(\xi,\xi^*)$ possesses, namely,

$$\int \frac{d^2\xi}{\pi} e^{-|\xi|^2} H_{m,n}(\xi,\xi^*) H^*_{m'n'}(\xi,\xi^*) = \sqrt{m!n!m'!n'!}\,\delta_{m,m'}\delta_{n,n'} \ . \tag{22}$$

This expression is the generalization of the simple integral

$$\int \frac{d^2\xi}{\pi} e^{-|\xi|^2} \xi^{*m}\xi^n = \delta_{m,n}\sqrt{n!m!} \tag{23}$$

used in the Bargmann representation construction. This relation holds because

$$H_{m,0}(\xi,\xi^*)=\xi^m\ ,\quad H_{0,n}(\xi,\xi^*)=\xi^{*n}\ ,\quad H_{m,n}(\xi,\xi^*)=H^*_{n,m}(\xi,\xi^*)\ . \tag{24}$$

V. SIMULTANEOUS SQUEEZING UNITARY TRANSFORMATION FOR Q_1-Q_2 AND P_1+P_2

In this section we derive the squeezing unitary operator for both Q_1-Q_2 and P_1+P_2, since they commute each other. Note that $\eta=\eta_1+i\eta_2$, so we may reexpress $|\eta\rangle$ as

$$|\eta\rangle=\exp\{-\tfrac{1}{2}(\eta_1^2+\eta_2^2)+\eta_1(a^\dagger-b^\dagger)+i\eta_2(a^\dagger+b^\dagger)+a^\dagger b^\dagger\}|00\rangle\equiv|\eta_1,\eta_2\rangle\ . \tag{25}$$

Let us introduce the following integral-form unitary operator,

$$U\equiv\sqrt{\mu\nu}\int\frac{d^2\eta}{\pi}|\mu\eta_1,\nu\eta_2\rangle\langle\eta_1,\eta_2|\ , \tag{26}$$

where μ,ν are two independent positive numbers. Using (25) and (8) and the IWOP technique, we can perform the integration in (26) to get

$$\begin{aligned}U&=\sqrt{\mu\nu}\int\frac{d^2\eta}{\pi}{:}\exp\left\{-\frac{\eta_1^2}{2}(1+\mu^2)-\frac{\eta_2^2}{2}(1+\nu^2)+\eta_1[\mu(a^\dagger-b^\dagger)+(a-b)]\right.\\&\qquad\left.+i\eta_2[\nu(a^\dagger+b^\dagger)-(a+b)]+a^\dagger b^\dagger+ab-a^\dagger a-b^\dagger b\right\}{:}\\&=\frac{2\sqrt{\mu\nu}}{\sqrt{(1+\mu^2)(1+\nu^2)}}\exp\left\{\frac{(\mu^2-\nu^2)(a^{\dagger 2}+b^{\dagger 2})+2(1-\mu^2\nu^2)a^\dagger b^\dagger}{2L}\right\}{:}\exp\left\{(a^\dagger b^\dagger)\left[\frac{g}{L}-1\right]\begin{bmatrix}a\\b\end{bmatrix}\right\}{:}\\&\quad\times\exp\left\{\frac{(\nu^2-\mu^2)(a^2+b^2)+2(\mu^2\nu^2-1)ab}{2L}\right\}\end{aligned} \tag{27}$$

where 1 is a 2×2 unit matrix and

$$L=(1+\mu^2)(1+\nu^2)\ ,\quad g=\begin{bmatrix}(\mu+\nu)(1+\mu\nu) & (\mu-\nu)(\mu\nu-1)\\(\mu-\nu)(\mu\nu-1) & (\mu+\nu)(1+\mu\nu)\end{bmatrix}\ . \tag{28}$$

Note that

$$\det\left[\frac{g}{L}\right]=\frac{4\mu\nu}{(1+\mu^2)(1+\nu^2)}\ ,\quad\left[\frac{g}{L}\right]^{-1}=\begin{bmatrix}(\mu+\nu)(\mu\nu+1) & (\mu-\nu)(1-\mu\nu)\\(\mu-\nu)(1-\mu\nu) & (\mu+\nu)(\mu\nu+1)\end{bmatrix}\frac{1}{4\mu\nu}\ . \tag{29}$$

With the use of (29), we know how a and b change under the U transformation, i.e.,

$$UaU^{-1}=\frac{1}{4\mu\nu}\{(\mu\nu+1)[(\mu+\nu)a+(\nu-\mu)a^\dagger]+(1-\mu\nu)[(\mu-\nu)b-(\mu+\nu)b^\dagger]\}\ , \tag{30}$$

$$UbU^{-1}=\frac{1}{4\mu\nu}\{(\mu\nu+1)[(\mu+\nu)b+(\nu-\mu)b^\dagger]+(1-\mu\nu)[(\mu-\nu)a-(\mu+\nu)a^\dagger]\}\ . \tag{31}$$

It then follows that

$$U(Q_1-Q_2)U^{-1}=\frac{1}{\mu}(Q_1-Q_2)\ , \tag{32}$$

$$U(P_1+P_2)U^{-1}=\frac{1}{\nu}(P_1+P_2)\ . \tag{33}$$

Note that μ and ν are independent, as $[Q_1-Q_2, P_1+P_2]=0$. From Eqs. (30) and (31) we can also derive

$$U(P_1-P_2)U^{-1}=\mu(P_1-P_2)\ , \tag{34}$$

$$U(Q_1+Q_2)U^{-1}=\nu(Q_1+Q_2)\ . \tag{35}$$

So, U is indeed an operator which simultaneously squeezes Q_1-Q_2 and P_1+P_2.

In summary, we have found the Fock representation of the common eigenstate of Q_1-Q_2 and P_1+P_2, the importance of which was first considered by EPR in 1935 [2]. The orthonormal and completeness relations of the eigenstates are derived, and the squeezing operator for both Q_1-Q_2 and P_1+P_2 is also obtained, by virtue of the IWOP technique.

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