Inertia as a zero-point-field Lorentz force

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(Received 8 February 1993)

Under the hypothesis that ordinary matter is ultimately made of subelementary constitutive primary charged entities or "partons" bound in the manner of traditional elementary Planck oscillators (a time-honored classical technique), it is shown that a heretofore uninvestigated Lorentz force (specifically, the magnetic component of the Lorentz force) arises in any accelerated reference frame from the interaction of the partons with the vacuum electromagnetic zero-point field (ZPF). Partons, though asymptotically free at the highest frequencies, are endowed with a sufficiently large "bare mass" to allow interactions with the ZPF at very high frequencies up to the Planck frequencies. This Lorentz force, though originating at the subelementary parton level, appears to produce an opposition to the acceleration of material objects at a macroscopic level having the correct characteristics to account for the property of inertia. We thus propose the interpretation that inertia is an electromagnetic resistance arising from the known spectral distortion of the ZPF in accelerated frames. The proposed concept also suggests a physically rigorous version of Mach's principle. Moreover, some preliminary independent corroboration is suggested for ideas proposed by Sakharov (Dokl. Akad. Nauk SSSR **177**, 70 (1968) [Sov. Phys. Dokl. **12**, 1040 (1968)]) and further explored by one of us [H. E. Puthoff, Phys. Rev. A **39**, 2333 (1989)] concerning a ZPF-based model of Newtonian gravity, and for the equivalence of inertial and gravitational mass as dictated by the principle of equivalence.

PACS number(s): 03.65.−w, 03.50.−z, 05.45.+b

I. INTRODUCTION

Inertia as formulated by Galileo (ca. 1638) was simply the property of a material object to either remain at rest or in uniform motion in the absence of external forces. In his first law of motion, Newton (ca. 1687) merely restated the Galilean proposition. However, in his second law, Newton expanded the concept of inertia into a fundamental quantitative property of matter. By proposing a relationship between external force acting upon an object and change in that object's velocity ($\mathbf{F}=m\mathbf{a}$), he defined and quantified the property of inertial mass. Since the time of Newton there has been only one noteworthy attempt to associate an underlying origin of inertia of an object with something external to that object: Mach's principle. Since motion would appear to be devoid of meaning in the absence of surrounding matter, it was argued by Mach (ca. 1883) that the local property of inertia must somehow asymptotically be a function of the cosmic distribution of all other matter. Mach's principle has remained, however, a philosophical statement rather than a testable scientific proposition. Thus apart from Mach's principle, the fact that matter has the property of inertia is a postulate of physics, and while special and general relativity both involve the inertial properties of matter, they provide no deeper insight into an origin of inertia than Newton's definition of inertia as a fundamental property of matter.

Recently one of us [2] analyzed a hypothesis of Sakharov [1,3] that Newtonian gravity could be interpreted as a van der Waals type of force induced by the electromagnetic fluctuations of the vacuum, the so-called zero-point fluctuations or zero-point field (ZPF). In that analysis ordinary neutral matter is treated as a collection of electromagnetically interacting polarizable particles made of charged point-mass subparticles (partons). This is a reasonable approach in ZPF analyses in which an ideal Planck oscillator serves as an analytical surrogate for more detailed representations of matter; or, more specifically, it is a simple model in which at ultrahigh (Planckian) energies matter appears as if formed of very small elementary constituents that respond like oscillators characterized by a radiation damping constant Γ and a characteristic frequency ω_0. The effect of the ZPF is to induce a *Zitterbewegung* motion in the parton in a manner entirely analogous to that of the bound oscillators used to represent the interaction of matter with electromagnetic radiation by Planck [4] and others. This has the consequence that the van der Waals force associated with the long-range radiation fields generated by the parton *Zitterbewegung* can be identified with the Newtonian gravitational field.

We have now found that the inertia of such a particle can also be calculated from the particle's interaction with the ZPF. For the idealized case we have analyzed, the $\mathbf{F}=m\mathbf{a}$ equation of motion appears to be related to the known distortion of the ZPF spectrum in an accelerated reference frame. This distortion of the ZPF spectrum due solely to acceleration gives rise to the well-known Davies-Unruh effect [5]. We show in this paper that there exists another effect, a heretofore unexplored electromagnetic Lorentz force (specifically the magnetic component of the Lorentz force) on an ideal charged particle, and since this ZPF force acts against the force giving rise to the acceleration and is proportional to the acceleration, it would appear to offer the interpretation of being the "cause" of the property of inertia. Stated another way, the resistance to acceleration which defines the inertia of matter appears to be an electromagnetic resistance (specifically Lorentz force) of the ZPF acting at the constituent particle (parton) level. This furthermore opens the possibility of specifying a causal basis and thus developing a scientific version of Mach's principle involving the universal ZPF, thereby offering deeper insight into what has been thought to be a fundamental, nonderivable property of matter, i.e., inertia.

The existence of an electromagnetic ZPF is a clear prediction of quantum theory resulting from quantization of the harmonically oscillating radiation modes in a *Hohlraum*. While quantum mechanics predicts a ZPF, there is, in fact, a minority view in modern physics that asserts that this situation might be turned around, and by assuming a ZPF *a priori* several quantum effects can be derived using classical formalism as a consequence of perturbation of elementary particles by such a random electromagnetic field. This approach, sometimes termed stochastic electrodynamics (SED), is a modern development of much earlier investigations by Planck [6], Nernst [7] and Einstein and Stern [8]. Considerable progress has been made in SED since the 1960's when this line of investigation was reopened by Marshall [9], Boyer [10], and others. A detailed account, with many references, of the development of this theory may be found in de la Peña [11] and in the brief update by Cole [12]. Given the relative ease and simplicity of the SED approach, and the fact that Milonni [13] has shown that for a broad class of problems (which includes the type of model being discussed here) quantum-mechanical and SED treatments are isomorphic, we shall use the SED approach here. In either case, quantum mechanics or SED, there appears a ubiquitous ZPF which can be regarded as a propagating electromagnetic field in free space with spectral energy density,

$$\rho(\omega)d\omega=\frac{\hbar\omega^3}{2\pi^2c^3}d\omega \ . \tag{1}$$

The issue of whether this field should be regarded as real or virtual has been an ongoing debate in quantum theory [14], whereas in SED the ZPF is by definition real [15]. Taking a pragmatic view, we use SED exclusively as a useful and convenient tool that is straightforward and intuitively clear, and which has been applied to the very real effects attributable to ZPF-matter interactions, such as the Casimir effect [16], the Lamb shift [17], the van der Waals forces [18], diamagnetism [19], spontaneous emission [20], and quantum noise [21]. That these effects are due to the ZPF is well known from QED and usually also from SED analyses such as those cited above; for discussion of many other specifically SED references concerning these and related effects, see de la Peña [11].

The ZPF spectrum of Eq. (1) is Lorentz invariant [22]. This has the consequence that motion through space at constant velocity does not, by virtue of a Doppler shift, change the ZPF spectral characteristics in any way so as to make the ZPF detectable. However, in an accelerated reference frame a manifestation of the ZPF does appear. It has been shown by Unruh, Davies, and others [23] using methods of quantum field theory, and then by Boyer [24] using SED formalism, that in a uniformly accelerated coordinate system with constant proper acceleration $\mathbf{a}$, a pseudo-Planckian spectrum will appear having a radiation temperature

$$T=\frac{\hbar a}{2\pi ck} \ . \tag{2}$$

In such a uniformly accelerated frame the spectral energy density takes the modified form

$$\rho(\omega)d\omega=\left[\frac{\omega^2}{\pi^2c^3}\right]\left[1+\left[\frac{a}{\omega c}\right]^2\right] \times\left[\frac{\hbar\omega}{2}+\frac{\hbar\omega}{\exp(2\pi c\omega/a)-1}\right]d\omega \ . \tag{3}$$

We have found that the associated modification of the ZPF as seen from an accelerated frame leads to a new result. Upon analyzing the force $\mathbf{F}$ that the ZPF exerts per constituent parton in an accelerated frame, it has been found that this force is directly proportional to and directed opposite to the acceleration vector $\mathbf{a}$. In other words, the acceleration process meets with a resistance from the ZPF which is a function of a radiation reaction damping constant Γ defining the interaction of the parton with the radiation field and of the acceleration $\mathbf{a}$. We interpret this as the inertia associated with the parton, i.e., the inertial mass of the particle (Planck's oscillator) containing the parton. This is equivalent to stating that Newton's law of motion, $\mathbf{F}=m\mathbf{a}$, may be formulated from the ordinary electrodynamics including the ZPF via the techniques of SED in the sense that the electrodynamic $\mathbf{F}(\mathbf{a})$ relationship predicts an inertial mass, per parton, of

$$m_i=\left[\frac{\hbar\omega_P}{c^2}\right](\Gamma\omega_P) \ , \tag{4}$$

where Γ is the Abraham-Lorentz damping constant of the underlying oscillating parton, and ω_P is the Planck frequency,

$$\omega_P=\left[\frac{c^5}{\hbar G}\right]^{1/2} . \tag{5}$$

We hasten to point out that although the Davies-Unruh effect and the inertia effect proposed herein are both due to the distortion of the ZPF as observed from an accelerated frame, the Davies-Unruh effect manifests itself primarily at low frequencies, as follows from the ω dependence of the second factor in the parentheses in Eq. (3), whereas on the contrary, as the derivation in Sec. II will show, the inertia effect here explored appears primarily because of the distortion of the ZPF vector components at very high frequencies. As follows from Eq. (3), the very-high-frequency distortions of the ZPF have a spectral energy density that, within the range of applicability of Eq. (3), grows linearly with ω. On the other hand, the purely thermal pseudo-Planckian part that constitutes the Davies-Unruh effect dies out exponentially at high frequencies.

In the next section we present a detailed mathematical derivation of Eq. (4). The last section presents some additional discussion.

II. NEWTON'S EQUATION OF MOTION, $\mathbf{F}=m\mathbf{a}$

The SED technique we use for calculating the effects of the electric and magnetic components of the ZPF on a parton is similar to the method introduced by Einstein and Hopf [25]. The particle model we use is that of Puthoff [2]. The particle acts as a harmonic oscillator with a characteristic frequency ω_0, free to vibrate in a plane perpendicular to the direction of acceleration. The relevant parameter for calculating the response of the oscillating particle to the driving force of the ZPF is the Abraham-Lorentz damping constant Γ of the parton (see discussion in Sec. III).

The ω_0 is a characteristic frequency in the manner of the Planck oscillators [4]. The aggregate of point charges in a finite object are not free, but rather bound to the whole. As our analysis will show, for the Planckian frequencies of interest ($\omega\approx\omega_P$), ω_0 will be negligible, i.e., partons are asymptotically free. In the case of the electron, for example, ω_0 would possibly be on the order of the Compton frequency, since this is roughly the frequency at which the center of charge oscillates in *Zitterbewegung* around the center of mass in conventional interpretations of QED [26]. The inclusion of ω_0 at this point affords physical clarity and will have mathematical advantages at the stage of locating and of separating poles in the process of contour integration. We do not need to specify any further constraints on ω_0 other than $\omega_0 \ll \omega_P$. Eventually ω_0 disappears from the calculations and the final result does not contain ω_0.

A. Acceleration relative to the ZPF

The formalism we start from corresponds to that of Boyer [5] for a small oscillator. Three coordinate systems are specified: I_*, I_τ, and S. We let the particle oscillator be subject to a force along the x axis of an inertial laboratory coordinate system, I_*, in such a way that the coordinate system of the particle, S, is accelerating with respect to this laboratory frame with a constant acceleration $\mathbf{a}$, as viewed from a moving but non-accelerating inertial coordinate system, I_τ, that coincides with S at proper time τ. We refer to time in I_* by t_*; τ refers to the particle proper time, i.e., its time in S. We consider the simple case of uniformly accelerated (in I_τ, the instantaneous frame of the particle) motion in the x direction, which yields so-called hyperbolic motion [27]. A lucid description of this situation and of the relations between quantities in the various frames is given by Boyer [5], whose notation we also follow.

We let $\beta_\tau = v_x(\tau)/c$ and $\gamma_\tau=(1-\beta_\tau^2)^{-1/2}$ and then use the Lorentz transformation to relate $\mathbf{E}_\tau^{\rm ZP}(0,\tau)$ in I_τ to the laboratory coordinates [28],

$$\mathbf{E}_\tau^{\rm ZP}(0,\tau) = \sum_{\lambda=1}^{2}\int d^3k\{\hat{\mathbf{i}}\epsilon_x+\hat{\mathbf{j}}\gamma_\tau[\epsilon_y-\beta_\tau(\hat{\mathbf{k}}\times\hat{\boldsymbol{\epsilon}})_z] + \hat{\mathbf{k}}\gamma_\tau[\epsilon_z+\beta_\tau(\hat{\mathbf{k}}\times\hat{\boldsymbol{\epsilon}})_y]\} \times H_{\rm ZP}(\omega)\cos[\mathbf{k}\cdot\mathbf{R}_*(\tau)-\omega t_*(\tau) - \theta(\mathbf{k},\lambda)] \,, \tag{6}$$

where $\mathbf{R}_*$ and t_* refer to the space and time coordinates of the central-force point of the oscillator in the laboratory frame I_* and $\theta(\mathbf{k},\lambda)$ is a family of random variables whose elements are mutually independent and where for each choice of $\mathbf{k}$ and λ there is a *different* random variable uniformly distributed between 0 and 2π, and

$$H_{\rm ZP}^2(\omega)=\frac{\hbar\omega}{2\pi^2} \,. \tag{7}$$

The expression for the field in Eq. (6) results after a Lorentz transformation from the random field $\mathbf{E}_*^{\rm ZP}(\mathbf{R}_*,t_*)$ at $\mathbf{R}_*(\tau),t_*(\tau)$, the equilibrium point of the oscillator in the laboratory inertial frame I_*. Since we will be interested in the Lorentz force we also require the Lorentz-transformed form of the magnetic field,

$$\mathbf{B}_\tau^{\rm ZP}(0,\tau)=\sum_{\lambda=1}^{2}\int d^3k\{\hat{\mathbf{i}}(\hat{\mathbf{k}}\times\hat{\boldsymbol{\epsilon}})_x+\hat{\mathbf{j}}\gamma_z[(\hat{\mathbf{k}}\times\hat{\boldsymbol{\epsilon}})_y+\beta_\tau\epsilon_z] + \hat{\mathbf{k}}\gamma_\tau[(\hat{\mathbf{k}}\times\hat{\boldsymbol{\epsilon}})_z-\beta_\tau\epsilon_y]\} \times H_{\rm ZP}(\omega)\cos[\mathbf{k}\cdot\mathbf{R}_*(\tau)-\omega t_*(\tau) - \theta(\mathbf{k},\lambda)] \,. \tag{8}$$

In Eqs. (6) and (8) we sum over the two possible polarizations $\lambda=1,2$ and integrate over the wave vector $\mathbf{k}$. The fact that $\boldsymbol{\epsilon}$ should read $\boldsymbol{\epsilon}_\lambda$ is understood and is omitted for simplicity of notation.

For constant acceleration as perceived by a particle we have the well-known case of hyperbolic motion in which the acceleration a enters as [29]

$$\beta_\tau=\tanh\left[\frac{a\tau}{c}\right] , \tag{9a}$$

$$\gamma_\tau=\cosh\left[\frac{a\tau}{c}\right] , \tag{9b}$$

and we can select space and time coordinates and orientation in I_* such that [5]

$$\mathbf{R}_*(\tau)\cdot\hat{\mathbf{i}}=\frac{c^2}{a}\cosh\left[\frac{a\tau}{c}\right], \tag{9c}$$

$$t_*=\frac{c}{a}\sinh\left[\frac{a\tau}{c}\right], \tag{9d}$$

and therefore

$$\mathbf{E}_\tau^{\mathrm{ZP}}(0,\tau)=\sum_{\lambda=1}^{2}\int d^3k\left\{\hat{\mathbf{i}}\epsilon_x+\hat{\mathbf{j}}\cosh\left[\frac{a\tau}{c}\right]\left[\epsilon_y-\tanh\left[\frac{a\tau}{c}\right](\hat{\mathbf{k}}\times\hat{\epsilon})_z\right]+\hat{\mathbf{k}}\cosh\left[\frac{a\tau}{c}\right]\left[\epsilon_z+\tanh\left[\frac{a\tau}{c}\right](\hat{\mathbf{k}}\times\hat{\epsilon})_y\right]\right\}H_{\mathrm{ZP}}(\omega)$$
$$\times\cos\left[k_x\frac{c^2}{a}\cosh\left[\frac{a\tau}{c}\right]-\left[\frac{\omega c}{a}\right]\sinh\left[\frac{a\tau}{c}\right]-\theta(\mathbf{k},\lambda)\right], \tag{10}$$

$$\mathbf{B}_\tau^{\mathrm{ZP}}(0,\tau)=\sum_{\lambda=1}^{2}\int d^3k\left\{\hat{\mathbf{i}}(\hat{\mathbf{k}}\times\hat{\epsilon})_x+\hat{\mathbf{j}}\cosh\left[\frac{a\tau}{c}\right]\left[(\hat{\mathbf{k}}\times\hat{\epsilon})_y+\tanh\left[\frac{a\tau}{c}\right]\epsilon_z\right]+\hat{\mathbf{k}}\cosh\left[\frac{a\tau}{c}\right]\left[(\hat{\mathbf{k}}\times\hat{\epsilon})_z-\tanh\left[\frac{a\tau}{c}\right]\epsilon_y\right]\right\}$$
$$\times H_{\mathrm{ZP}}(\omega)\cos\left[k_x\frac{c^2}{a}\cosh\left[\frac{a\tau}{c}\right]-\left[\frac{\omega c}{a}\right]\sinh\left[\frac{a\tau}{c}\right]-\theta(\mathbf{k},\lambda)\right]. \tag{11}$$

The ZPF is referenced with respect to the equilibrium point of the particle.

The equation of motion is a particular form of the Abraham-Lorentz-Dirac equation derived for a particle undergoing hyperbolic motion [30],

$$m_0\frac{d^2\mathbf{r}}{d\tau^2}=-m_0\omega_0^2\mathbf{r}+\frac{2}{3}\frac{e^2}{c^3}\left[\frac{d^3\mathbf{r}}{d\tau^3}-\frac{a^2}{c^2}\frac{d\mathbf{r}}{d\tau}\right]+e\mathbf{E}_\tau^{\mathrm{ZP}}(0,\tau), \tag{12}$$

where $\mathbf{r}$ is the vector displacement of the oscillating particle in the S frame. It is the additional term in a^2 that captures our attention, and as shown by Boyer [5] this term is a relativistic one. This is the reason that we develop relativistic expressions even though the particle velocity (from constant acceleration) may be extremely small. Inertia will be shown to be a relativistic effect, a situation not so surprising if inertia originates in *Zitterbewegung* and somewhat analogous to the ordinary electromagnetic Lorentz force being a relativistic phenomenon (resulting from the invariance of the equations of electrodynamics under Lorentz transformations). Due to the fact that the effect to be derived is mainly due to the very-high-frequency components of the ZPF, we do not need to include any coherence effects. Hence in Eq. (12) we may neglect the action of other particles. Our high-frequency analysis automatically excludes many-particle cooperative effects like those responsible for refractive behavior, i.e., high index of refraction at lower frequencies.

One can solve Eq. (12) using Fourier transforms. The assumed two dimensionality of the *Zitterbewegung* trajectories implies that the particle moves in a plane [2,26]. The instantaneous displacement of the parton in I_τ is taken to be in the yz plane, and so we write

$$\mathbf{r}(\tau)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}d\Omega\,\boldsymbol{\eta}_i(\Omega)\exp(-i\Omega\tau), \tag{13a}$$

$$\mathbf{E}_{\tau(yz)}^{\mathrm{ZP}}(0,\tau)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}d\Omega\,\boldsymbol{\Sigma}_{(yz)}(\Omega)\exp(-i\Omega\tau), \tag{13b}$$

from which we may also obtain $\mathbf{B}_{\tau(yz)}^{\mathrm{ZP}}(0,\tau)$.

The equation of motion (12), a particular version of the Abraham-Lorentz-Dirac equation, in the nonrelativistic case with constant acceleration has the form developed by Boyer [5] in his Eq. (14). The Fourier transform of Eq. (12) is

$$\left\{m_0(-i\Omega)^2-\frac{2e^2}{3c^3}\left[(-i\Omega)^3-(-i\Omega)\frac{a^2}{c^2}\right]+m_0\omega_0^2\right\}\times\boldsymbol{\eta}(\Omega)=e\boldsymbol{\Sigma}_{(yz)}(\Omega), \tag{14}$$

where m_0 is the bare mass of the parton associated with Γ, i.e., $m_0=2e^2/3\Gamma c^3$. Our Eq. (14) has the solution

$$\mathbf{r}(\tau)=(2\pi)^{-1/2}\times\int_{-\infty}^{\infty}d\Omega\frac{e}{m_0}\frac{\boldsymbol{\Sigma}_{(yz)}(\Omega)\exp(-i\Omega\tau)}{[\omega_0^2-\Omega^2-i\Gamma(\Omega^3+\Omega a^2/c^2)]}, \tag{15}$$

with

$$\mathbf{v}(\tau)=\frac{d\mathbf{r}}{d\tau}=\dot{\mathbf{r}}=(2\pi)^{-1/2}\times\int_{-\infty}^{\infty}d\Omega\frac{e}{m_0}\frac{(-i\Omega)\times\boldsymbol{\Sigma}_{(yz)}(\Omega)exp(-i\Omega\tau)}{[\omega_0^2-\Omega^2-i\Gamma(\Omega^3+\Omega a^2/c^2)]}, \tag{16}$$

$$\boldsymbol{\Sigma}_{(yz)}(\Omega)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}d\tau\,\mathbf{E}_{\tau(yz)}^{\mathrm{ZP}}(0,\tau)\exp(i\Omega\tau). \tag{17}$$

After laying out this formalism developed by Boyer [5], the next step is to calculate a specific kind of radiation pressure exerted by the ZPF in the accelerated frame S on the oscillating particle. We compute the Lorentz force on a parton oscillator and average over the random phases. The ZPF will exert a magnetic Lorentz force on the parton,

$$\mathbf{F}=e\frac{\mathbf{v}(\tau)}{c}\times\mathbf{B}_\tau^{\rm ZP}(0,\tau)\ , \tag{18}$$

which is the only one that will remain since the electric part of the Lorentz force will not contribute owing to the [soon to be performed, see Eqs. (30)–(32) below] averaging over the random phases,

$$\langle e^{i\theta(\mathbf{k},\lambda)}\rangle=0\ . \tag{19}$$

Because of symmetry we show below [Eq. (34)] that after averaging, the resulting averaged magnetic Lorentz force takes place only along the x axis; i.e., the direction of the acceleration since the average force vanishes along the other two directions. Three comments are then in order. (i) As discussed in Rindler [27], the resulting force is an ordinary three-force: It is the same in the I_* and I_τ systems because it is collinear with the relative velocity between the two systems. (ii) The technique of first calculating the velocity $\mathbf{v}(\tau)$ from the effect of the electric field and then proceeding to find the effect of the magnetic field, as in Eq. (18), constitutes the essence of the method of Einstein and Hopf [25,31]. (iii) When such techniques are applied to a genuinely *free* electromagnetically interacting particle, ZPF forces, in the absence of friction, yield an increase in the translational kinetic energy of random motion for the particle [32], whereas in the present case the particle oscillator is not free as it is constrained to undergo uniform acceleration by the applied external force.

B. Solution of the force equation

We need to consider only very short times τ, so that to first order in $(a\tau/c)$ outside the phases,

$$\beta_\tau\approx\left[\frac{a\tau}{c}\right]\ll 1\ , \tag{20}$$

$$\gamma_\tau\approx(1-\beta_\tau^2)^{-1/2}\approx 1\ , \tag{21}$$

and in the phases we go to second order in $(a\tau/c)$,

$$\cosh\left[\frac{a\tau}{c}\right]=1+\frac{1}{2!}\left[\frac{a\tau}{c}\right]^2+\cdots\approx 1+\frac{1}{2}\left[\frac{a\tau}{c}\right]^2\ , \tag{22}$$

$$\sinh\left[\frac{a\tau}{c}\right]=\left[\frac{a\tau}{c}\right]+\frac{1}{3!}\left[\frac{a\tau}{c}\right]^3+\cdots\approx\left[\frac{a\tau}{c}\right]\ . \tag{23}$$

Thus the expression for the phases reads

$$\begin{aligned}k_x\frac{c^2}{a}\cosh\left[\frac{a\tau}{c}\right]-\left[\frac{\omega c}{a}\right]\sinh\left[\frac{a\tau}{c}\right]-\theta(\mathbf{k},\lambda)&\approx k_x\frac{c^2}{a}+k_x\frac{c^2}{a}\frac{1}{2}\left[\frac{a\tau}{c}\right]^2-\left[\frac{\omega c}{a}\right]\left[\frac{a\tau}{c}\right]-\theta(\mathbf{k},\lambda)\\ &\approx k_x\frac{c^2}{a}+\frac{k_x}{2}a\tau^2-\omega\tau-\theta(\mathbf{k},\lambda)\ ,\end{aligned} \tag{24}$$

and therefore

$$\begin{aligned}\cos\left\{k_x\frac{c^2}{a}\cosh\left[\frac{a\tau}{c}\right]-\left[\frac{\omega c}{a}\right]\sinh\left[\frac{a\tau}{c}\right]-\theta(\mathbf{k},\lambda\right\}&\approx\cos\left\{k_x\frac{c^2}{a}+k_x\frac{a\tau^2}{2}-\omega\tau-\theta(\mathbf{k},\lambda)\right\}\\ &=\mathrm{Re}\left\{\exp\left[i\left[k_x\frac{c^2}{a}+k_x\frac{a\tau^2}{2}-\omega\tau-\theta(\mathbf{k},\lambda)\right]\right]\right\}\ .\end{aligned} \tag{25}$$

This explains why a relativistic formalism is relevant even for nonrelativistic displacement motions of the particle.

The velocity in the I_τ frame [Eq. (16)] can be written as

$$\begin{aligned}\mathbf{v}(\tau)&=(2\pi)^{-1/2}\int_{-\infty}^{\infty}d\Omega\frac{e}{m_0}\frac{(-i\Omega)e^{-i\Omega\tau}}{[\omega_0^2-\Omega^2-i\Gamma(\Omega^3+\Omega a^2/c^2)]}(2\pi)^{-1/2}\int_{-\infty}^{\infty}d\tau'\mathbf{E}_{\tau'(yz)}^{\rm ZP}(0,\tau')e^{i\Omega\tau'}\\ &=\frac{e}{2\pi m_0}\int_{-\infty}^{\infty}d\tau'\mathbf{E}_{\tau'(yz)}^{\rm ZP}(0,\tau')\int_{-\infty}^{\infty}d\Omega\frac{(-i\Omega)\exp[i\Omega(\tau'-\tau)]}{[\omega_0^2-\Omega^2-i\Gamma(\Omega^3+\Omega a^2/c^2)]}\ .\end{aligned} \tag{26}$$

In the yz plane of the oscillator motion the projection of the electric vector is [33]

$$\begin{aligned}\mathbf{E}_{\tau(yz)}^{\rm ZP}(0,\tau)&=\sum_{\lambda'=1}^{2}\int d^3k'(\hat{\mathbf{j}}\epsilon_y'+\hat{\mathbf{k}}\epsilon_z')H_{\rm ZP}(\omega')\exp\{i[\mathbf{k}'\cdot\mathbf{R}_*(\tau')-\omega' t_*(\tau')-\theta(\mathbf{k}',\lambda')]\}\\ &\approx\sum_{\lambda'=1}^{2}\int d^3k'(\hat{\mathbf{j}}\epsilon_y'+\hat{\mathbf{k}}\epsilon_z')H_{\rm ZP}(\omega')\exp\left[i\left[k_x'\frac{c^2}{a}-\theta(\mathbf{k}',\lambda')-\omega'\tau'+\frac{k_x' a\tau'^2}{2}\right]\right]\ ,\end{aligned} \tag{27}$$

where we have used the approximation $(a\tau/c)\approx 0$ outside the phases because $(a\tau/c)\ll 1$. Of paramount importance to this equation as well as to many of the following ones, see, e.g., Eq. (28) and (90) below, is the fact that only very high frequencies will be found to contribute to the inertia effect. We write the velocity as

$$\mathbf{v}(\tau)=\frac{e}{2\pi m_0}\int_{-\infty}^{\infty}d\tau'\sum_{\lambda'=1}^{2}\int d^3k'(\hat{\mathbf{j}}\epsilon'_y+\hat{\mathbf{k}}\epsilon'_z)H_{\rm ZP}(\omega')\exp\left[i\left[k'_x\frac{c^2}{a}-\theta(\mathbf{k}',\lambda')-\omega'\tau'+\frac{k'_x a\tau'^2}{2}\right]\right]$$
$$\times\int_{-\infty}^{\infty}d\Omega\frac{(-i\Omega)\exp[i\Omega(\tau'-\tau)]}{[\omega_0^2-\Omega^2-i\Gamma(\Omega^3+\Omega a^2/c^2)]}\ , \tag{28}$$

where we have been able to integrate over τ' from $-\infty$ to $+\infty$ because of the fast-wave approximation of physical optics. As the relevant contributions to the inertia effect come exclusively from very large ω', the large $|\tau'|$ part of the integral is irrelevant. Within the same order of approximation the magnetic field becomes

$$\mathbf{B}_\tau^{\rm ZP}(0,\tau'')=\sum_{\lambda''=1}^{2}\int d^3k''[\hat{\mathbf{i}}(\hat{\mathbf{k}}''\times\hat{\boldsymbol{\epsilon}}'')_x+\hat{\mathbf{j}}(\hat{\mathbf{k}}''\times\hat{\boldsymbol{\epsilon}}'')_y+\hat{\mathbf{k}}(\hat{\mathbf{k}}''\times\hat{\boldsymbol{\epsilon}}'')_z]H_{\rm ZP}(\omega'')$$
$$\times\exp\left[i\left[k''_x\frac{c^2}{a}-\theta(\mathbf{k}'',\lambda'')-\omega''\tau''+\frac{k''_x a\tau''^2}{2}\right]\right]\ . \tag{29}$$

Next we compute the (magnetic component of the) Lorentz force, where $\langle\ \rangle$ refers to the usual average over random phases. The proper times τ and τ'' must be the same in the force expression; however, we retain the formal distinction to allow us to more easily trace the origin of the various factors. Later we will set $\tau''=\tau$. The Lorentz force is

$$\mathbf{F}=e\left\langle\frac{\mathbf{v}(\tau)}{c}\times\mathbf{B}_\tau^{\rm ZP}(0,\tau'')\right\rangle$$
$$=\frac{1}{2}{\rm Re}\left[\frac{e}{c}\left\langle\frac{e}{2\pi m_0}\int_{-\infty}^{\infty}d\tau'\sum_{\lambda'=1}^{2}\int d^3k'(\hat{\mathbf{j}}\epsilon'_y+\hat{\mathbf{k}}\epsilon'_z)H_{\rm ZP}(\omega')\exp\left[i\left[k'_x\frac{c^2}{a}-\theta(\mathbf{k}',\lambda')-\omega'\tau'+\frac{k'_x a\tau'^2}{2}\right]\right]\right.\right.$$
$$\times\int_{-\infty}^{\infty}d\Omega\frac{(-i\Omega)\exp[i\Omega(\tau'-\tau)]}{\{\omega_0^2-\Omega^2-i\Gamma[\Omega^3+\Omega(a^2/c^2)]\}}$$
$$\otimes\sum_{\lambda''=1}^{2}\int d^3k''\{\hat{\mathbf{i}}(\hat{\mathbf{k}}''\times\boldsymbol{\epsilon}'')_x+\hat{\mathbf{j}}(\hat{\mathbf{k}}''\times\hat{\boldsymbol{\epsilon}}'')_y+\hat{\mathbf{k}}(\hat{\mathbf{k}}''\times\hat{\boldsymbol{\epsilon}}'')_z\}H_{\rm ZP}(\omega'')$$
$$\left.\left.\times\left\{\exp\left[i\left[k''_x\frac{c^2}{a}-\theta(\mathbf{k}'',\lambda'')-\omega''\tau''+\frac{k''_x a\tau''^2}{2}\right]\right]\right\}^*\right\rangle\right]\ , \tag{30}$$

where $\otimes$ denotes the vector cross product, and the asterisk the complex complementation. Noting that

$$\langle\exp i[\theta(\mathbf{k}',\lambda')+\theta(\mathbf{k}'',\lambda'')]\rangle\equiv 0\ , \tag{31}$$

$$\langle\exp i[\theta(\mathbf{k}',\lambda')-\theta(\mathbf{k}'',\lambda'')]\rangle=\delta_{\lambda\lambda}\delta(\mathbf{k}'-\mathbf{k}'')\ , \tag{32}$$

and now setting $\tau''=\tau$ we arrive at

$$\mathbf{F}=\frac{1}{2c}\frac{e^2}{2\pi m_0}{\rm Re}\left\{\int_{-\infty}^{\infty}d\tau'\sum_{\lambda'=1}^{2}\int d^3k'\int_{-\infty}^{\infty}d\Omega\frac{(-i\Omega)\exp[i\Omega(\tau'-\tau)]}{[\omega_0^2-\Omega^2-i\Gamma(\Omega^3+\Omega a^2/c^2)]}H_{\rm ZP}^2(\omega')\right.$$
$$\times\exp\left[-i\omega'(\tau'-\tau)+\frac{iak'_x}{2}(\tau'^2-\tau^2)\right]$$
$$\left.\times[-\hat{\mathbf{k}}\epsilon'_y(\hat{\mathbf{k}}'\times\hat{\boldsymbol{\epsilon}}')_x+\hat{\mathbf{i}}\epsilon'_y(\hat{\mathbf{k}}'\times\hat{\boldsymbol{\epsilon}}')_z+\hat{\mathbf{j}}\epsilon'_z(\hat{\mathbf{k}}'\times\hat{\boldsymbol{\epsilon}}')_x-\hat{\mathbf{i}}\epsilon'_x(\hat{\mathbf{k}}'\times\hat{\boldsymbol{\epsilon}}')_y]\right\}\ . \tag{33}$$

Now the F_y and F_z components of $\mathbf{F}$ vanish because of symmetry, i.e., the situation must be cylindrically symmetric around the x axis of acceleration, so we need only compute F_x, which is

$$F_x=\hat{\imath}\cdot\mathbf{F}=\frac{1}{2c}\frac{e^2}{2\pi m_0}{\rm Re}\left\{\int_{-\infty}^{\infty}d\tau'\sum_{\lambda'=1}^{2}\int d^3k'\int_{-\infty}^{\infty}d\Omega\frac{(-i\Omega)\exp[i\Omega(\tau'-\tau)]}{[\omega_0^2-\Omega^2-i\Gamma(\Omega^3+\Omega a^2/c^2)]}H_{\rm ZP}^2(\omega')\right.$$
$$\left.\times\exp\left[-i\omega'(\tau'-\tau)+\frac{iak'_x}{2}(\tau'^2-\tau^2)\right][\epsilon'_y(\hat{\mathbf{k}}'\times\hat{\boldsymbol{\epsilon}}')_z-\epsilon'_z(\hat{\mathbf{k}}'\times\hat{\boldsymbol{\epsilon}}')_y]\right\}$$
$$=\frac{1}{2c}\frac{e^2}{2\pi m_0}{\rm Re}\left\{\int_{-\infty}^{\infty}d\tau'\sum_{\lambda'=1}^{2}\int d^3k'\int_{-\infty}^{\infty}d\Omega\frac{(-i\Omega)\exp[i\Omega(\tau'-\tau)]}{[\omega_0^2-\Omega^2-i\Gamma(\Omega^3+\Omega a^2/c^2)]}H_{\rm ZP}^2(\omega')\right.$$
$$\left.\times\exp\left[-i\omega'(\tau'-\tau)+\frac{iak'_x}{2}(\tau'^2-\tau^2)\right]\{\hat{\mathbf{i}}\cdot[\hat{\boldsymbol{\epsilon}}'\times(\hat{\mathbf{k}}'\times\hat{\boldsymbol{\epsilon}}')]\}\right\}\ . \tag{34}$$

Observe that

$$\hat{\epsilon}'\times(\hat{\mathbf{k}}'\times\hat{\epsilon}')=\hat{\mathbf{k}}'(\hat{\epsilon}'\cdot\hat{\epsilon}')-\hat{\epsilon}'(\hat{\mathbf{k}}'\cdot\hat{\epsilon}')=\hat{\mathbf{k}}' , \tag{35}$$

hence taking the x axis as the azimuth axis we have

$$F_x=\frac{1}{2c}\frac{e^2}{2\pi m_0}\mathrm{Re}\left\{\int_{-\infty}^{\infty}d\tau'\sum_{\lambda'=1}^{2}\int d^3k'\cos\theta'\int_{-\infty}^{\infty}d\Omega\frac{(-i\Omega)\exp[i\Omega(\tau'-\tau)]}{[\omega_0^2-\Omega^2-i\Gamma(\Omega^3+\Omega a^2/c^2)]}H_{\mathrm{ZP}}^2(\omega')\right.$$
$$\left.\times\exp\left[-i\omega'(\tau'-\tau)+\frac{iak_x'}{2}(\tau'^2-\tau^2)\right]\right\} . \tag{36}$$

To compute the angular part we do the following:

$$I\equiv\int d^3k'\cos\theta'\exp\left[\frac{iak'}{2}\cos(\theta')(\tau'^2-\tau^2)\right]$$
$$=\int_0^{\infty}k'^2dk'\int_0^{2\pi}d\phi'\int_0^{\pi}d\theta'\sin\theta'\cos\theta'\exp\left[\frac{iak'}{2}(\tau'^2-\tau^2)\cos\theta'\right]$$
$$=2\pi\int_0^{\infty}k'^2dk'\int_{-1}^{1}d\mu\,\mu\exp(Q\mu) , \tag{37}$$

where

$$Q\equiv\frac{iak'}{2}(\tau'^2-\tau^2) . \tag{38}$$

Since [34]

$$\int\mu e^{Q\mu}d\mu=e^{Q\mu}\left[\frac{\mu}{Q}-\frac{1}{Q^2}\right] \tag{39}$$

we find

$$I=2\pi\int_0^{\infty}k'^2dk'\left.\right|_{-1}^{1}e^{Q\mu}\left[\frac{\mu}{Q}-\frac{1}{Q^2}\right]$$
$$=2\pi\int_0^{\infty}k'^2dk'(2i)\left\{\frac{\sin[(a/2)k'(\tau'^2-\tau^2)]}{[(a/2)k'(\tau'^2-\tau^2)]^2}-\frac{\cos[(a/2)k'(\tau'^2-\tau^2)]}{[(a/2)k'(\tau'^2-\tau^2)]}\right\} \tag{40}$$

and thus

$$F_x=\frac{e^2}{2cm_0}\mathrm{Re}\left\{\int_{-\infty}^{\infty}d\tau'(4i)\int_0^{\infty}k'^2dk'H_{\mathrm{ZP}}^2(\omega')\left[\frac{\sin[(a/2)k'(\tau'^2-\tau^2)]}{[(a/2)k'(\tau'^2-\tau^2)]^2}-\frac{\cos[(a/2)k'(\tau'^2-\tau^2)]}{[(a/2)k'(\tau'^2-\tau^2)]}\right]\right.$$
$$\left.\times\int_{-\infty}^{\infty}d\Omega\frac{(-i\Omega)\exp[i\Omega(\tau'-\tau)]}{[\omega_0^2-\Omega^2-i\Gamma(\Omega^3+\Omega a^2/c^2)]}e^{-i\omega'(\tau'-\tau)}\right\} . \tag{41}$$

Converting from wave vector k' to angular frequency ω' and using

$$H_{\mathrm{ZP}}^2(\omega')=\frac{\hbar\omega'}{2\pi^2} \tag{42}$$

we find

$$F_x=\frac{\hbar e^2}{\pi^2cm_0}\mathrm{Re}\left\{\int_{-\infty}^{\infty}d\tau'\int_0^{\infty}\frac{\omega'^3d\omega'}{c^3}\left[\frac{\sin\left[\frac{a}{2}\frac{\omega'}{c}(\tau'^2-\tau^2)\right]}{\left[\frac{a}{2}\frac{\omega'}{c}(\tau'^2-\tau^2)\right]^2}-\frac{\cos\left[\frac{a}{2}\frac{\omega'}{c}(\tau'^2-\tau^2)\right]}{\left[\frac{a}{2}\frac{\omega'}{c}(\tau'^2-\tau^2)\right]}\right]\right.$$
$$\left.\times e^{-i\omega'(\tau'-\tau)}\int_{-\infty}^{\infty}d\Omega\frac{\Omega\exp[i\Omega(\tau'-\tau)]}{[\omega_0^2-\Omega^2-i\Gamma(\Omega^3+\Omega a^2/c^2)]}\right\} . \tag{43}$$

The last integral,

$$J=\int_{-\infty}^{\infty} d\Omega \frac{\Omega \exp[i\Omega(\tau'-\tau)]}{[\omega_0^2-\Omega^2-i\Gamma(\Omega^3+\Omega a^2/c^2)]} , \quad (44)$$

can be computed by contour integration on the complex plane using residue theory. Letting $\Omega \equiv x$, $\tau'-\tau \equiv \alpha$, and $a^2/c^2 \equiv \phi^2$, we may write

$$J=\int_{-\infty}^{\infty} \frac{xe^{i\alpha x}dx}{[-i\Gamma x^3-x^2-i\Gamma\phi^2 x+\omega_0^2]} . \quad (45)$$

We know that

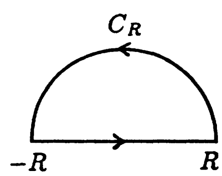
$$\mathcal{J}_c \equiv \oint \frac{ze^{i\alpha z}dz}{[-i\Gamma z^3-z^2-i\Gamma\phi^2 z+\omega_0^2]} = 2\pi i \sum_{\{j\}} \text{Res}\left[\frac{ze^{i\alpha z}}{[-i\Gamma z^3-z^2-i\Gamma\phi^2 z+\omega_0^2]} ;z_j\right] , \quad (46)$$

where $\{j\}$ represents the collection of poles that are located inside the area enclosed by the path. Since z, $e^{i\alpha z}$, and $[-i\Gamma z^3-z^2-i\Gamma\phi^2 z+\omega_0^2]$ are analytic everywhere, the only poles occur in the zeros of the denominator. We thus find the roots of the polynomial,

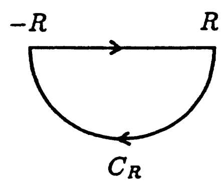
$$-i\Gamma z^3-z^2-i\Gamma\phi^2 z+\omega_0^2=0 , \quad (47)$$

where $\Gamma,\phi,\omega_0 \in R$, and $z \in C$.

Since the coefficients are in general complex, from the d'Alembert-Gauss fundamental theorem of algebra we know there are in general three possibly different complex roots, z_1, z_2, and z_3. Given the positions of the poles to be found below, there are reasonable options for the closed contours of integration,



$$\oint_C = \lim_{R\to\infty}\left[\int_{-R}^{R}+\int_{C_R;\theta=0}^{\theta=+\pi}\right] , \quad \alpha>0 , \quad (48)$$



$$\oint_C = \lim_{R\to\infty}\left[\int_{-R}^{R}+\int_{C_R;\theta=0}^{\theta=-\pi}\right] , \quad \alpha<0 , \quad (49)$$

$$\mathcal{J}_C=J_R+J_{CR} . \quad (50)$$

We know that the integral we need is

$$\lim_{R\to\infty} J_R \to J \quad (51)$$

and thus we want to have that

$$\lim_{R\to\infty} J_{CR}=0 . \quad (52)$$

This may be ascertained by means of Jordan's lemma. Therefore if $\alpha=\tau'-\tau<0$ we select the lower complex plane path (taking care of the minus sign because of change in chirality of the integral), and if $\alpha=\tau'-\tau>0$ we select a path in the upper plane.

We make use of the physical fact that the bare mass m_0 is very large and thus $\Gamma=2e^2/3m_0c^3$ is very small. In the limit $\Gamma\to 0$ the equation becomes

$$-z^2+\omega_0^2=0 , \quad \Gamma\to 0 , \quad (53)$$

with solutions $z=\pm\omega_0$. Hence for small Γ we expect that the approximate forms of the two roots would be

$$z_1 \approx \omega_0 - i\delta , \quad (54)$$

$$z_2 \approx -\omega_0 - i\epsilon , \quad (55)$$

where $\delta,\epsilon \in R$. We may check the roots z_1 and z_2 and derive some internal consistency conditions that δ and ϵ must satisfy,

$$-i\Gamma z_1^3-z_1^2-i\Gamma\phi^2 z_1+\omega_0^2=0 , \quad (56)$$

and from Eq. (54),

$$-i\Gamma[\omega_0^3-3\omega_0 i\delta+O(\delta^2)]-[\omega_0^2-2\omega_0 i\delta+O(\delta^2)] -i\Gamma\phi^2(\omega_0-i\delta)+\omega_0^2=0 , \quad (57)$$

where we are neglecting second-order terms in δ and thus

$$-i\Gamma\omega_0^3+2\omega_0 i\delta-3\Gamma\omega_0^2\delta-i\Gamma\phi^2\omega_0-\Gamma\phi^2\delta=0 \quad (58)$$

and so, to first order in δ, from the real and imaginary parts we get

$$i[-\Gamma\omega_0^3+2\omega_0\delta-\Gamma\phi^2\omega_0]=0\ , \quad (59)$$

$$-\Gamma\delta[3\omega_0^2+\phi^2]=0\ . \quad (60)$$

This implies that the level of approximation is exact to the order of neglecting

$$\delta\Gamma=\Gamma\delta\approx 0\ . \quad (61)$$

We are thus left with

$$\delta=\frac{\Gamma}{2}(\omega_0^2+\phi^2)>0\ . \quad (62)$$

We repeat this for $z_2=-\omega_0-i\epsilon$. Exactly the same approach yields that to the level of approximation in which we can neglect $\Gamma\epsilon$, i.e.,

$$\Gamma\epsilon\approx 0\ , \quad (63)$$

analogously we obtain

$$\epsilon=\frac{\Gamma}{2}[\omega_0^2+\phi^2]=\delta>0\ . \quad (64)$$

We thus have to $O(\delta^2)=O(\epsilon^2)$,

$$z_{1,2}=\pm\omega_0-i\delta=\pm\omega_0-i\frac{\Gamma}{2}[\omega_0^2+\phi^2]\ . \quad (65)$$

Next we multiply as follows:

$$\begin{aligned}(z-z_1)(z-z_2)&=z^2-(z_1+z_2)z+z_1z_2\\&=z^2+2i\delta z-(\omega_0^2+\delta^2)\\&\approx z^2+2i\delta z-\omega_0^2\ \ [\text{to } O(\delta^2)]\ . \end{aligned} \quad (66)$$

From the fundamental theorem of algebra we know that the original cubic equation (47) may be factored as

$$\begin{aligned}-i\Gamma z^3-z^2-i\Gamma\phi^2 z+\omega_0^2&=-i\Gamma(z-z_1)(z-z_2)(z-z_3)\\&\approx -i\Gamma(z-z_3)(z^2+2i\delta z-\omega_0^2)\\&=-i\Gamma z^3+i\Gamma z_3 z^2+2\Gamma\delta z^2-2\Gamma\delta z z_3+i\Gamma\omega_0^2 z-i\Gamma\omega_0^2 z_3\end{aligned} \quad (67)$$

and from (61) then

$$z^2(1+i\Gamma z_3)+i\Gamma(\omega_0^2+\phi^2)z-\omega_0^2(1+i\Gamma z_3)=0 \quad (68)$$

or, because of (62)

$$z^2(1+i\Gamma z_3)+2i\delta z-\omega_0^2(1+i\Gamma z_3)=0\ . \quad (69)$$

For this last identity to hold we need to have δ very small and

$$1+i\Gamma z_3\approx 0\ . \quad (70)$$

Therefore we have for the three poles

$$z_3\approx\frac{i}{\Gamma}\ , \quad (71)$$

$$z_2\approx-\omega_0-i\delta\ , \quad (72)$$

$$z_1\approx\omega_0-i\delta\ . \quad (73)$$

We now replace the z_3 solution of (71) in Eq. (47) in order to determine the error and we obtain that

$$\phi^2+\omega_0^2\approx 0\ . \quad (74)$$

This is the error in the fit. It is a very small number in comparison to Γ^{-2}.

The integrand of $\bar{J}_c$ is of the form

$$\frac{F}{G}=\frac{ze^{i\alpha z}}{[-i\Gamma z^3-z^2-i\Gamma\phi^2 z+\omega_0^2]} \quad (75)$$

so that

$$G'=\frac{dG}{dz}=-3i\Gamma z^2-2z-i\Gamma\phi^2\ . \quad (76)$$

Thus we evaluate the residues around each pole by means of the well-known theorem that for integrals of the form $F(z)/G(z)$, where G has only simple zeros, the residue at z_j is $F(z_j)/G'(z_j)$, where the z_j must be simple zeros of G.

$$\begin{aligned}\text{Res}\left\{\frac{F(z)}{G(z)}\ ;z=z_1=\omega_0-i\delta\right\}&=\frac{F(z_1)}{G'(z_1)}=\frac{F(\omega_0-i\delta)}{G'(\omega_0-i\delta)}\\&=\frac{(\omega_0-i\delta)\exp[i\alpha(\omega_0-i\delta)]}{-3i\Gamma(\omega_0-i\delta)^2-2(\omega_0-i\delta)-i\Gamma\phi^2}\ ,\end{aligned} \quad (77)$$

$$\text{Res}\left\{\frac{F(z)}{G(z)};z=z_2=-\omega_0-i\delta\right\}=\frac{F(z_2)}{G'(z_2)}=\frac{F(-\omega_0-i\delta)}{G'(-\omega_0-i\delta)}$$
$$=\frac{(-\omega_0-i\delta)\exp[i\alpha(-\omega_0-i\delta)]}{-3i\Gamma(-\omega_0-i\delta)^2-2(-\omega_0-i\delta)-i\Gamma\phi^2}, \quad (78)$$

$$\text{Res}\left\{\frac{F(z)}{G(z)};z=z_3=\frac{i}{\Gamma}\right\}=\frac{F(z_3)}{G'(z_3)}=\frac{\exp(-\alpha/\Gamma)}{1-(\phi\Gamma)^2}\approx 0 \quad \text{for } \alpha>0 . \quad (79)$$

Next we examine the evaluation of $\bar{J}_c$ as stipulated by Jordan's lemma. As $|z|\to\infty$ we have

$$\frac{z}{-i\Gamma z^3-z^2-i\Gamma\phi z+\omega_0^2}\to 0 , \quad |z|\to\infty . \quad (80)$$

Hence if $\alpha>0$, we integrate over the upper semicircular contour [Eq. (48)] and Jordan's lemma guarantees that

$$J=(2\pi i)\text{Res}\left\{\frac{F(z)}{G(z)};z=z_3=\frac{i}{\Gamma}\right\} . \quad (81)$$

So for $\alpha>0$, because of (79)

$$J=\int_{-\infty}^{\infty}\frac{xe^{i\alpha x}dx}{-i\Gamma x^3-x^2-i\Gamma\phi x+\omega_0^2}\approx 0 \quad (82)$$

or more precisely,

$$J=(2\pi i)\frac{\exp(-\alpha/\Gamma)}{1-(\phi\Gamma)^2}\to 0 \quad (83)$$

for τ' larger than τ by several Γ. Therefore we must have $\tau'\le\tau$ for the force F_x not to vanish; or more properly, we should integrate over τ' the contributions from $-\infty$ to τ since the part from τ to ∞ essentially vanishes.

We then look at the case $\alpha<0$. Jordan's lemma guarantees convergence in the lower complex plane. However, the integral performed over the contour of Eq. (49) has opposite chirality and the residue theorem reads

$$\bar{J}_c=-(2\pi i)\sum_{\{j\}}\text{Res}\{z_j\} , \quad (84)$$

where j refers to the poles of the integrand inside C. We then have

$$\bar{J}_c=\int_{-R}^{R}+\int_{C_R;\theta=0,|z|=R}^{-\pi}$$
$$=-(2\pi i)(\text{Res}\{z_1\}+\text{Res}\{z_2\}) \quad (85)$$

in the limit when $R\to\infty$, and the integral C_R goes to zero because of Jordan's lemma. We obtain

$$J=-2\pi i\left\{\frac{(\omega_0-i\delta)\exp[i\alpha(\omega_0-i\delta)]}{-3i\Gamma(\omega_0-i\delta)^2-2(\omega_0-i\delta)-i\Gamma\phi^2}+\frac{(-\omega_0-i\delta)\exp[i\alpha(-\omega_0-i\delta)]}{-3i\Gamma(-\omega_0-i\delta)^2-2(-\omega_0-i\delta)-i\Gamma\phi^2}\right\} . \quad (86)$$

Let $J\equiv\mathcal{J}_{\text{num}}/\mathcal{J}_{\text{den}}$, where $\mathcal{J}_{\text{num}}$ and $\mathcal{J}_{\text{den}}$ denote, respectively,

$$\mathcal{J}_{\text{num}}=\frac{2\pi}{3\Gamma}e^{\alpha\delta}\left\{\left[(\omega_0^2+\delta^2)(\omega_0+i\delta)+\frac{2i}{3\Gamma}(\omega_0^2+\delta^2)+\frac{\phi^2}{3}(\omega_0-i\delta)\right]e^{i\alpha\omega_0}\right.$$
$$\left.-\left[(\omega_0^2+\delta^2)(\omega_0-i\delta)-\frac{2i}{3\Gamma}(\omega_0^2+\delta^2)+\frac{\phi^2}{3}(\omega_0+i\delta)\right]e^{-i\alpha\omega_0}\right\}$$
$$=\frac{2\pi}{3\Gamma}(\omega_0^2+\delta^2)e^{\alpha\delta}\left\{\left[(\omega_0+i\delta)+\frac{2i}{3\Gamma}+\frac{\phi^2}{3}\frac{(\omega_0-i\delta)}{(\omega_0^2+\delta^2)}\right]e^{i\alpha\omega_0},\right.$$
$$\left.-\left[(\omega_0-i\delta)-\frac{2i}{3\Gamma}+\frac{\phi^2}{3}\frac{(\omega_0+i\delta)}{(\omega_0^2+\delta^2)}\right]e^{-i\alpha\omega_0}\right\}, \quad (87)$$

$$\mathcal{J}_{\text{den}}=(\omega_0^2+\delta^2)^2+\left[\frac{2}{3\Gamma}\right]^2(\omega_0^2+\delta^2)+\frac{\phi^4}{9}+\frac{2i}{3\Gamma}(\omega_0^2+\delta^2)(\omega_0-i\delta)-\frac{2i}{3\Gamma}(\omega_0^2+\delta^2)(\omega_0+i\delta)$$
$$+\frac{\phi^2}{3}[(\omega_0-i\delta)^2+(\omega_0+i\delta)^2]-\frac{\phi^2}{3}\frac{2i}{3\Gamma}(\omega_0-i\delta)+\frac{\phi^2}{3}\frac{2i}{3\Gamma}(\omega_0+i\delta)$$
$$\approx\left[\frac{2\omega_0}{3\Gamma}\right]^2 . \quad (88)$$

The denominator simplifies considerably via the dominance of certain large terms. Recall that the δ^2 terms could be neglected $[\delta=(\Gamma/2)(\omega_0^2+\delta^2)]$. Moreover, terms with Γ^{-1} are much larger than terms with ω_0 or ω_0^2. All in all, the Γ^{-2} term winds up determining the denominator, leading to the simplification. Thus for $\alpha<0$ the expression becomes

$$J \approx \frac{2\pi}{3\Gamma}\omega_0^2 e^{\alpha\delta}\frac{\left[\frac{2i}{3\Gamma}e^{i\alpha\omega_0}+\frac{2i}{3\Gamma}e^{-i\alpha\omega_0}\right]}{\left[\frac{2\omega_0}{3\Gamma}\right]^2} = \pi i\left[e^{i(\omega_0-i\delta)(\tau'-\tau)}+e^{-i(\omega_0+i\delta)(\tau'-\tau)}\right], \tag{89}$$

whereas for $\alpha>0, J\approx 0$, from (82).

Thus the expression for the force becomes

$$F_x=\frac{\hbar e^2}{\pi^2 c m_0}\mathrm{Re}\left\{\int_{-\infty}^{\tau} d\tau'\int_0^\infty \frac{\omega'^3 d\omega'}{c^3}\left[\frac{\sin\left[\frac{a}{2}\frac{\omega'}{c}(\tau'^2-\tau^2)\right]}{\left[\frac{a}{2}\frac{\omega'}{c}(\tau'^2-\tau^2)\right]^2}-\frac{\cos\left[\frac{a}{2}\frac{\omega'}{c}(\tau'^2-\tau^2)\right]}{\left[\frac{a}{2}\frac{\omega'}{c}(\tau'^2-\tau^2)\right]}\right]\right.$$

$$\left.\times(\pi i)\left[e^{-i[(\omega'-\omega_0)+i\delta](\tau'-\tau)}+e^{-i[(\omega'+\omega_0)+i\delta](\tau'-\tau)}\right]\right\} \tag{90}$$

since only the case $\tau'-\tau<0$ contributes. At high frequencies—which are the only ones that substantially contribute to the final result because the frequency integration over ω' peaks near a frequency ω_c to be introduced in (108) below—the exponentials in the $\omega'\pm\omega_0$ introduce rapid oscillations in frequency for $|\tau'-\tau|$ sufficiently large. This is a sufficient reason to justify the claim that the only case producing a nonvanishing result is $\tau'\approx\tau$. However, there are additional reasons. The exponent $\exp[-i(i\delta)(\tau'-\tau)]=\exp[\delta(\tau'-\tau)]$ strongly damps the expressions for $(\tau'-\tau)\delta<0$ when $|\tau'-\tau|\gg\delta^{-1}$. Moreover, the sine and cosine expressions oscillate strongly with frequency in the frequency integration when τ' differs considerably from τ. As a consequence, $\tau'\approx\tau$ is the only case in which a contribution may be expected.

Let

$$\mu\equiv\left[\frac{a}{2}\frac{\omega'}{c}(\tau'^2-\tau^2)\right]. \tag{91}$$

Clearly for τ' in the immediate neighborhood of τ, μ is very small and then

$$\left[\frac{\sin\left[\frac{a}{2}\frac{\omega'}{c}(\tau'^2-\tau^2)\right]}{\left[\frac{a}{2}\frac{\omega'}{c}(\tau'^2-\tau^2)\right]^2}-\frac{\cos\left[\frac{a}{2}\frac{\omega'}{c}(\tau'^2-\tau^2)\right]}{\left[\frac{a}{2}\frac{\omega'}{c}(\tau'^2-\tau^2)\right]}\right]=\frac{\sin\mu}{\mu^2}-\frac{\cos\mu}{\mu}=\frac{1}{\mu}\left[\frac{\sin\mu}{\mu}-\cos\mu\right]\approx\frac{\mu}{3} \tag{92}$$

for small μ, after neglecting terms of $O(\mu^3)$ and higher. Thus

$$\left[\frac{\sin\mu}{\mu^2}-\frac{\cos\mu}{\mu}\right]\approx\frac{\mu}{3}=\frac{1}{3}\left[\frac{a\omega'}{2c}\right](\tau'^2-\tau^2)=\frac{1}{3}\left[\frac{a\omega'}{2c}\right](\tau'-\tau)(\tau'+\tau)=\frac{1}{3}\left[\frac{a\omega'}{2c}\right][\alpha^2+2\tau\alpha], \tag{93}$$

where α was defined just before Eq. (45) above.

We thus find for F_x

$$F_x=\frac{1}{\pi}\frac{2e^2}{3m_0c^3}\frac{\hbar}{c^2}\frac{a}{2}\frac{1}{2}\mathrm{Re}\left\{i\int_{-\infty}^0 d\alpha\int_0^\infty \gamma(\omega')\omega'^4 d\omega'[\alpha^2+2\tau\alpha][e^{-i[(\omega'-\omega_0)+i\delta]\alpha}+e^{-i[(\omega'+\omega_0)+i\delta]\alpha}]\right\} \tag{94}$$

where $\gamma(\omega')$ is a form factor obtained by Rueda [35] to represent the fact that components of the ZPF electromagnetic radiation whose wavelengths are smaller than the size of the electromagnetically interacting particle (which in the present case is the parton) are ineffective in producing any translational motion of the interacting particle (parton) as a whole. Wavelengths shorter than the interacting particle diameter can only yield internal deformations and vibrations of such a particle. We recall furthermore that partons cannot be smaller than a fundamental minimal length, the Planck length λ_P [36], and for concreteness we assume (within a parameter of order one that we omit for simplicity) that this is actually the size of these fundamental entities. Given this limit on the parton size, it is not necessary to assume a cutoff of the ZPF itself which would destroy the Lorentz invariance of the spectrum. The cutoff originates in the size of the parton, in that ZPF frequencies with wavelengths less than the parton size would cease to result in effective translational interactions.

Letting $v \equiv -\alpha$, $dv = -d\alpha$,

$$F_x = \frac{1}{2\pi}\Gamma\frac{\hbar}{c^2}\frac{a}{2}\mathrm{Re}\left\{ i\int_0^\infty dv \int_0^\infty \gamma(\omega')\omega'^4 d\omega'[v^2-2\tau v][e^{i[(\omega'-\omega_0)+i\delta]v}+e^{i[(\omega'+\omega_0)+i\delta]v}]\right\} . \tag{95}$$

We define

$$\nu \equiv \omega' v , \tag{96}$$

$$T \equiv \frac{\tau}{\omega'} , \tag{97}$$

$$p \equiv \frac{\omega_0}{\omega'} , \tag{98}$$

$$q \equiv \frac{\delta}{\omega'} \tag{99}$$

to serve the purpose of normalizing the expressions. Since ω' is positive, the limits of integration in ν are the same as for v, namely, $-\infty$ and $+\infty$. Thus

$$F_x = \frac{\Gamma}{4\pi}\frac{\hbar}{c^2} a \,\mathrm{Re}\left\{ i\int_0^\infty \gamma(\omega')\omega' d\omega' \int_0^\infty [\nu^2-2T\nu]d\nu[e^{i[(1-p)+iq]\nu}+e^{i[(1+p)+iq]\nu}]\right\} . \tag{100}$$

We may proceed to integrate over ν.

We evaluate $\mathrm{Re}\{i\int_0^\infty \gamma(\omega')\omega' d\omega' K\}$ where

$$\begin{aligned} K &\equiv \int_0^\infty d\nu(\nu^2-2T\nu)[e^{i[(1-p)+iq]\nu}+e^{i[(1+p)+iq]\nu}] \\ &= \int_0^\infty d\nu(\nu^2-2T\nu)e^{-q\nu}\{\cos[(1-p)\nu]+\cos[(1+p)\nu]+i\sin[(1-p)\nu]+i\sin[(1+p)\nu]\} , \end{aligned} \tag{101}$$

but

$$\begin{aligned} \mathrm{Re}\left\{ i\int_0^\infty \gamma(\omega')\omega' d\omega' K\right\} &= \int_0^\infty \gamma(\omega')\omega' \mathrm{Re}\{iK\} d\omega' \\ &= -\int_0^\infty \gamma(\omega')\omega' d\omega' \int_0^\infty d\nu(\nu^2-2T\nu)e^{-q\nu}\{\sin[(1-p)\nu]+\sin[(1+p)\nu]\} . \end{aligned} \tag{102}$$

We use Gradshteyn and Ryzhik [37] to obtain

$$\int_0^\infty x^{\mu-1}e^{-\beta x}\sin(\delta x)\,dx = \frac{\Gamma(\mu)}{(\beta^2+\delta^2)^{\mu/2}}\sin\left[\mu \arctan\frac{\delta}{\beta}\right] \quad \text{for } \mu,\beta,\delta \in R,\ \mu > -1 ,\ \beta > 0 , \tag{103}$$

hence

$$\int_0^\infty \nu^2 \sin[(1\pm p)\nu]e^{-q\nu}d\nu = \frac{2}{[q^2+(1\pm p)^2]^{3/2}}\sin\left[3\arctan\left[\frac{1\pm p}{q}\right]\right] \tag{104}$$

and

$$\int_0^\infty -2T\nu\sin[(1\pm p)\nu]e^{-q\nu}d\nu = \frac{-2T}{[q^2+(1\pm p)^2]}\sin\left[2\arctan\left[\frac{1\pm p}{q}\right]\right] . \tag{105}$$

Observe that as mainly $\omega' \gg \omega_0$, expressions like $p = \omega_0/\omega'$ and $q = \delta/\omega' = (\Gamma/2\omega')(\omega_0^2 + a^2/c^2)$ are extremely small (negligible by many orders of magnitude) for all regions of ω' of any relevance in the ω' integration. The parameter a/c cannot be much larger than ω_0 and most likely is much smaller even for collisions in particle physics experiments where a takes high (negative) values. So $(\Gamma\omega_0)\omega_0 \ll \omega_c$ and $(\Gamma a/c)a/c \ll \omega_c$, where ω_c is a frequency bound introduced below. Hence we can neglect p and q in comparison to unity at this stage, and thus are left with

$$\mathrm{Re}\left\{ i\int_0^\infty \gamma(\omega')\omega' d\omega' K\right\} = -\int_0^\infty \gamma(\omega')\omega' d\omega' \left[-4\sin\left[3\arctan\frac{1}{q}\right]+4T\sin\left[2\arctan\frac{1}{q}\right]\right] . \tag{106}$$

However,

$$\lim_{q\to 0^+} \arctan\frac{1}{q} = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots \tag{107}$$

and thus for all practical purposes it can be shown that the form factor $\gamma(\omega')$ effectively produces a cutoff at a frequency ω_c [35], so that we write

$$\mathrm{Re}\left\{ i\int_0^\infty \gamma(\omega')\omega' d\omega' K \right\} \approx -\int_0^{\omega_c} \omega' d\omega' 4\sin\left[\pm\frac{3\pi}{2}\right]$$
$$= \pm 4\int_0^{\omega_c} \omega' d\omega' = \pm 2\omega_c^2 \;. \tag{108}$$

Therefore finally we arrive at

$$F_x = -\frac{1}{2\pi}\Gamma\omega_c\frac{\hbar\omega_c}{c^2}a \;, \tag{109}$$

where the negative sign is selected because the action of the vacuum on the accelerated body is an opposing action, in other words, the inertia. The ZPF-determined inertial mass associated with the parton oscillator is

$$m_i = \frac{\Gamma\hbar\omega_c^2}{2\pi c^2} \;. \tag{110}$$

This corresponds to the rest mass associated with the subelementary particle that is the parton or Planck oscillator. The cutoff ω_c comes from the ineffectiveness of the ZPF in producing any translational motion of the parton at wavelengths smaller than the parton size, and *not* from the introduction of any *ad hoc* cutoff in the ZPF spectrum. It is important to note that this leaves the Lorentz invariance of the theory intact.

A simple estimate, using this value of m_i, as done by Puthoff [2] gives $\omega_c = (2\pi)^{1/2}\omega_P$, where ω_P is the Planck frequency, $\omega_P = (c^5/\hbar G)^{1/2}$ and G is the Newtonian gravitational constant. Hence

$$m_i = (\Gamma\omega_P)\frac{\hbar\omega_P}{c^2} = \frac{2}{3}\alpha\frac{m_P^2}{m_0} \;, \tag{111}$$

where α is the fine structure constant, $\alpha = e^2/\hbar c$, and $m_P = \hbar\omega_P/c^2$ is the so-called Planck mass. This holds, provided the parton has a charge e equal in magnitude to the electronic charge. Relation (111) corresponds to that in Ref. [24] of Puthoff [2]. However, if we consider both the parton's charge e and its bare mass m_0, as free parameters (a more general viewpoint given the fractional charges of quarks and the possible existence of further structure at the very high energies under discussion) then there is a relationship between the particle's inertial mass m_i and the parton's charge e and bare mass m_0 predicted by our model, $m_i \propto e^2/m_0$.

III. DISCUSSION AND CONCLUSIONS

It is useful at this point to give an overview of our arguments and assumptions. Because of the $1/r$ singularity of the Coulomb potential, any charged particle suffers an infinite self-energy problem arising from its own electrostatic field. For this and other reasons (e.g., the elementary fact that a neutron may decay into a proton and an electron) models have been developed which assume that "large-scale" particles, whether charged or neutral—such as protons and neutrons—actually consist of smaller, more elementary charged particles, such as quarks. The situation for the electron is still less clear [38]. This leads to the distinction between the external "dressed mass" of a particle and the aggregate internal "bare mass" of the constituent elementary particles.

Our model does not address differences between the various types of fundamental particles. We simply assume that material objects at the most fundamental level are made of positively and negatively charged entities, referred to as partons, capable of interacting effectively with the ZPF at all frequencies up to those corresponding to the size of the parton, assumed to be the Planck length λ_P. These partons are simply the oscillators developed by Planck [4] and extended to the smallest possible size scale. In the SED analysis an equation of motion is set up for the response of the parton to the driving forces of the ZPF electromagnetic waves. The unknown free parameter is, of course, the parton mass m_0, or, entirely equivalently, the Abraham-Lorentz damping constant, $\Gamma = 2e^2/3m_0c^2$.

Puthoff [2] found that the mass equivalent of the *Zitterbewegung* motion of his parton resulted in a mass m_g which could be interpreted as the gravitational mass of the associated particle, i.e., the gravitational "dressed mass," because that mass, m_g, was the one involved in the gravitational interaction. This mass appears to be twice as large as the inertial mass, m_i, derived herein in Eq. (110). However, in Appendix A we discuss a possible resolution of this factor of 2 discrepancy, and argue that in fact $m_i = m_g$. The significance of our result is that by tracing a totally different effect of ZPF-matter interaction—the Lorentz force that appears simply as a result of coordinate transformation to an accelerated system—we derive an identical mass as in the gravity case for a particle consisting of an oscillating parton. This mass, m_i, appears to be an inertial "dressed mass" because it is precisely a resistance to acceleration resulting from ZPF forces.

The linkage of inertia with the ZPF sheds light on Mach's principle. In its simplest terms Mach's principle states that the inertial mass of a body cannot in principle have an operational meaning in the absence of the rest of the matter in the universe. This is due to the fact that the acceleration of a body, wherein inertial mass comes into play, implies acceleration relative to some frame with respect to which the acceleration can be measured. In standard parlance, this concept is operationalized in terms of acceleration relative to "the fixed stars." Embedded in this measurement-of-acceleration concept is the deeper implication that not only is the existence of the inertial mass of a body in question dependent on the "fixed stars," but its magnitude must somehow be dependent on the aggregate mass of those stars, since asymptotic elimination of those stars would of necessity result in asymptotic diminution to zero of any meaningful mass concept.

In this article we have demonstrated how the ZPF may be shown to give rise to the inertial mass of a particle.

The ZPF could thus serve as the Machian cosmic reference frame. This may in turn be related to the cosmic distribution of matter in the context of the model of dynamically balanced absorption and reemission of ZPF radiation by mass distributed over cosmological space recently proposed [39]. We propose that this quantitative ZPF-based formulation of Mach's principle answers objections such as those posed by Jennison and Drinkwater [40] to a nonlocal origin of inertia.

An interesting point is that the bulk of the contribution to the effect, in this case the inertial mass, comes from the very-high-frequency components of the ZPF, a fact that explains why it has taken so long to recognize a relationship between inertia and electromagnetism. It is frequencies not far below the Planck frequency that are relevant in the integrations leading to Eqs. (109) and (110) [41].

An additional comment is that, as pointed out in the derivation, the inertial effect is indeed a relativistic one although it is used here only to obtain the nonrelativistic limit of the equation of motion, as can be seen from the initial steps of Sec. II and from the analysis and discussion in Boyer [5]. This is perhaps not so surprising once one realizes that the inertial effect comes from the Lorentz force, Eq. (18), and that the ordinary Lorentz force is relativistic, i.e., can be cast in covariant form [42].

In conclusion (i) it appears that a magnetic component of the Lorentz force arises in ZPF-matter interactions in accelerating reference frames such that the property of resisting acceleration which defines inertia could be attributed to this interaction. (ii) Newton's equation of motion $\mathbf{F}=m\mathbf{a}$ thus appears to be made explainable directly by ZPF electrodynamics. (iii) The equivalence of the ZPF inertial mass derived here and the ZPF gravitational mass in the Sakharov-Puthoff model of Newtonian gravity would appear to provide some corroboration to this aspect of the principle of equivalence. (iv) Alternatively, if the principle of equivalence is taken as given, our argument for inertia expounded here seems to provide some independent collateral support for the concept of ZPF-based Newtonian gravity developed in the Sakharov-Puthoff model. (v) Finally, a causal and quantifiable basis for Mach's principle is suggested.

ACKNOWLEDGMENTS

Useful comments were received from the following researchers: Giancarlo Cavalleri, Ana Maria Cetto, Daniel Cole, Joseph Eberly, Kirk McDonald, Adrian Melissinos, Peter Milonni, Franco Persico, Howard R. Reiss, and Paul Wesson. One of us (A.R.) received partial support from the Offices of Academic Affairs and of University Research at California State University, Long Beach. This work was also partially supported by the Lockheed Independent Research Program. B.H. wishes to thank Professor J. Trümper and the Max-Planck-Institut für Extraterrestrische Physik for gracious hospitality during several stays.

APPENDIX A: EQUALITY OF m_i AND m_g

We address the apparent factor of 2 discrepancy between Puthoff's [2] gravitational mass, $m_g=\Gamma\hbar\omega_c^2/\pi c^2$, and our inertial mass, $m_i=\Gamma\hbar\omega_c^2/2\pi c^2$.

Puthoff's nonrelativistic calculation of the *Zitterbewegung* used a technique of Rueda [43] for calculating the $\langle \dot{x}^2\rangle$ in his Eq. (12)–(22). Starting from $\ddot{x}$, he integrated over time, introducing the form $\dot{x}=\int_0^T \ddot{x}\,dt$. This assumes implicitly that $\dot{x}(0)=0$ and preselects a frame of reference. Our approach is different. We calculated $\dot{x}(t)$ via Fourier expansion without requiring $\dot{x}(t)$ to vanish at $t=0$. Let us call this Fourier method "method I" and the one used in Rueda [43] and Puthoff [2] "method II." Then starting from the simple expression

$$\ddot{x}=\frac{e}{m_0}E_x^{\rm ZP}, \tag{A1}$$

where for simplicity of notation we have used $E_x^{\rm ZP}$ to refer to the usual

$$E_x^{\rm ZP}=\mathrm{Re}\left\{\sum_{\lambda=1}^{2}\int d^3k\,\hat{\epsilon}H_{\rm ZP}(\omega)\times\exp[i\hat{\mathbf{k}}\cdot\mathbf{r}-i\omega t+i\theta(\mathbf{k},\lambda]\right\}, \tag{A2}$$

we may compare the results for $\langle \dot{x}\rangle$ in the two methods.

Method I. Taking

$$\dot{x}=\frac{e}{m_0}\mathrm{Re}\left\{\sum_{\lambda=1}^{2}\int d^3k(\hat{\epsilon}\cdot\hat{\mathbf{x}})H_{\rm ZP}(\omega)\frac{i}{\omega}\times\exp[i\mathbf{k}\cdot\mathbf{r}-i\omega t+i\theta(\mathbf{k},\lambda)]\right\}, \tag{A3}$$

straightforward calculation with averaging over the phases as in Eq. (32) yields

$$\langle \dot{x}^2\rangle=\frac{1}{2}\frac{e^2}{m^2}\frac{8\pi}{3}\frac{\hbar\omega_c^2}{4\pi c^3}. \tag{A4}$$

Method II. In order to calculate $\dot{x}$ from $\ddot{x}$ we integrate from 0 to t:

$$\begin{aligned}\dot{x}(t)&=\frac{e}{m_0}\mathrm{Re}\left\{\int_0^t dt'\sum_{\lambda=1}^{2}\int d^3k(\hat{\epsilon}\cdot\hat{\mathbf{x}})H_{\rm ZP}(\omega)\exp[i\mathbf{k}\cdot\mathbf{r}-i\omega t'+i\theta(\mathbf{k},\lambda)]\right\}\\&=\frac{e}{m_0}\mathrm{Re}\left\{\sum_{\lambda=1}^{2}\int d^3k(\hat{\epsilon}\cdot\hat{\mathbf{x}})H_{\rm ZP}(\omega)\left[\frac{1-e^{-i\omega t}}{i\omega}\right]\exp[i\mathbf{k}\cdot\mathbf{r}+i\theta(\mathbf{k},\lambda)]\right\},\end{aligned} \tag{A5}$$

and again a straightforward calculation yields

$$\langle \dot{x}^2 \rangle = \frac{e^2}{m^2} \frac{8\pi}{3} \frac{\hbar\omega_c^2}{4\pi c^3} , \tag{A6}$$

which is twice as large as (A4).

Both methods I and II can be considered correct given the reasonable order of approximation in the gravity analysis, but there is a subtle difference between the meaning of $\dot{x}$. In method I, only the purely fluctuating part of the Fourier expression is captured. In method II, on the other hand, one begins with an *a priori* assumption that $\dot{x}(0)=0$ which automatically introduces, as in Eq. (A5), a sort of phasing of each Fourier component individually by requiring each one to adjust the velocity to a zero value at $t=0$.

Restating this, in method I we average the Fourier components around their means. Say we take for simplicity $\cos\omega t = \mathrm{Re}\, e^{i\omega t}$, then

$$\langle \cos\omega t \rangle = 0 , \tag{A7a}$$

$$\langle \cos^2\omega t \rangle = \tfrac{1}{2} , \tag{A7b}$$

or equivalently,

$$\langle e^{i\omega t} \rangle = 0 , \tag{A8a}$$

$$\langle (e^{i\omega t})^2 \rangle = \tfrac{1}{2}\mathrm{Re} \left[\frac{1}{2\pi} \int_0^{\omega t = 2\pi} e^{i\omega t} e^{i\omega t *} d(\omega t) \right] = \tfrac{1}{2} , \tag{A8b}$$

where the first term on the left-hand side should be taken in the symbolic sense. In method II we are rephasing the Fourier components to adjust to the zero velocities value at $t=0$, so that we take, say, $\cos\omega t - 1$ instead of $\cos\omega t$, and then

$$\langle e^{i\omega t} - 1 \rangle = -1 , \tag{A9a}$$

and symbolically again,

$$\langle (e^{i\omega t} - 1)^2 \rangle = \tfrac{1}{2}\mathrm{Re} \left[\frac{1}{2\pi} \int_0^{\omega t = 2\pi} (e^{i\omega t} - 1)(e^{i\omega t} - 1)^* \times d(\omega t) \right] = 1 , \tag{A9b}$$

hence the factor of 2.

Physically, method I is appropriate when the purely fluctuating part of an effect is thought to be the relevant one, as should be the case for *Zitterbewegung*. Method II is the appropriate one to use for analyzing a systematic increase, as, for example, when calculating the growth of a velocity after a reference time $t=0$ for a particle subject to secular ZPF acceleration as in Rueda [43]. For a fluctuating effect resulting in zero average velocity, method I would appear to be the more appropriate, and, in retrospect, this would be the method of choice for the gravity analysis of Puthoff [2]. We therefore propose that

$$m_g = m_i = \frac{\Gamma\hbar\omega_c^2}{2\pi c^2} . \tag{A10}$$

APPENDIX B: BARE MASS m_0

In [2] the mass m_i is interpreted as the energy associated with *Zitterbewegung*. In Eq. (11) we found that m_i depends on the damping constant Γ or entirely equivalently, on the "mass" m_0. This last is a free parameter introduced in the dynamic Abraham-Lorentz-type equation (12). Its interpretation here is that of an "internal mass" for the description of the internal dynamics in response to the ZPF. If the inertia effect is exclusively due to the ZPF then m_0 per se does not contribute to the inertia. We call m_0, above, a "bare mass" but since it also involves the electromagnetic or Coulomb component, such a name may not be entirely appropriate. The mass m_0 is really the superposition of the "bare" plus the "electric" mass components, and it may be positive or negative. We assumed it to be very large and positive, however, its "bare mass" designation may suggest that it is very large in absolute value but negative. This last case for the m_0 parameter does not affect our derivation. Recall that in Eq. (109) we selected the negative sign. However, if m_0 and hence Γ happened to be negative we would select the positive sign in (109) so that the inertia effect consistently opposes the acceleration. Furthermore, if we inspect Eq. (12) within the electromagnetic viewpoint of this article and assume that Hooke's recovery force is of electric origin and depends linearly on the electrical charge e, then changing the sign of m_0 is entirely equivalent to not changing the sign of m_0 but instead switching the sign of the charge from $+e$ to $-e$. A final comment in this regard is that the dynamical meaning of the "internal mass" m_0 is somewhat similar to the quark masses of some quark particle models whose quarks have masses much larger than the mass of the corresponding particle.

APPENDIX C: RADIATIVE MASS SHIFT

An interesting point with respect to our inertial rest mass m_i of Eq. (110) is its remarkable similarity to the so-called radiative mass shift. This similarity appears to be more than just a coincidence. Under an intense electromagnetic field (e.g., laser or microwave maser irradiation) it has been theoretically predicted that the electron experiences an increase in its rest mass of the form [44]

$$\Delta m = \frac{e^2 \langle \mathbf{A}^2 \rangle}{2m_0} , \tag{C1}$$

where m_0 represents now the ordinary rest mass of the particle and $\mathbf{A}$ is the vector potential. This analogy was discovered by Dr. Peter Milonni and he did the following short derivation [45]. Let H_A be the part of the particle's electrodynamic Hamiltonian that after expansion of the $(\mathbf{p} - e\,\mathbf{A})^2$ factor has the form

$$H_A = \frac{e^2}{2m_0 c^2} \langle \mathbf{A}^2 \rangle . \tag{C2}$$

Then after expanding $\mathbf{A}$ in terms of its creation and annihilation operations and averaging in the standard fashion we get

$$H_A = \frac{e^2}{2m_0c^2}\sum_{k,\lambda}\left[\frac{2\pi\hbar c^2}{\omega_k V}\right]\langle a_{k\lambda}a^\dagger_{k\lambda}\rangle$$
$$=2\left[\frac{e^2}{2m_0c^2}\right]\left[\frac{2\pi\hbar c^2}{V}\right]\left[\frac{V}{(2\pi)^3}\right]\int\frac{d^3k}{\omega_k}$$
$$=\frac{e^2\hbar}{2\pi m_0c^3}\omega_c^2 , \qquad \text{(C3)}$$

which, once the standard form for $\Gamma=2e^2/3m_0c^3$ is replaced, is seen to coincide (modulo a factor of order unity) with our Eq. (110). The cavity normalization volume is denoted by V and the cutoff frequency is ω_c. This analogy, we believe, is not a coincidence. It reinforces the view made evident by our classical approach that inertia is generated by the vacuum that opposes the externally imposed acceleration and tends to throw back the system into an inertial frame. It is tempting to explore modifications of our model in a manner that would yield m_i as a radiative mass shift. However, in order to make the two approaches coincide fully we would need to leave the exclusively electromagnetic viewpoint of SED and assume other interactions whose corresponding zero-point fields generate the inertia of the nonradiative mass m_0. This is beyond the scope of the present paper.

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[15] Among objections to the reality of the ZPF, the two major ones appear to be our apparent inability to perceive such a seemingly intense (hence one would think obvious) radiation field and the "cosmological-constant problem" [Λ problem, see P. S. Wesson, Astrophys. J. **378**, 466 (1991); or Space Sci. Rev. **59**, 365 (1992) for the most recent discussion]. With regard to the first, perception, and indeed any type of measurement, is a matter of response to differences. While it might be argued that the eye, or an instrument, is perfectly capable of registering the presence of a featureless surrounding, uniformly intense surface brightness, such a perception would depend on there being a net flux into the detector, i.e., a difference between the outside and the inside of a detector, the inside of the detector being shielded from the outside, hence dark and absorbing with respect to the external radiation field. A radiation field that is uniform everywhere in space, including inside the would-be detector, would appear to be undetectable. Only if matter could be arranged so as to somehow create nonuniformities in this radiation field would there be a mechanism for detection. In fact, conducting plates can be used to alter the ZPF, and this does indeed make the ZPF detectable, in the form of the well-known Casimir force [see C. L. Sukenkik *et al.*, Phys. Rev. Lett. **70**, 560 (1993)]. The Sakharov-Puthoff model of gravity may answer the Λ problem, which goes as follows. The total, frequency-integrating energy density of the ZPF is, from Eq. (1), simply,

$$\rho_E=(\hbar/8\pi^2c^3)\omega_c^4 ,$$

where ω_c is the cutoff frequency for ZPF-parton interactions. (As discussed elsewhere in this paper, this need not be an actual cutoff in the spectrum, simply an interaction cutoff, which would therefore not destroy the Lorentz invariance of the spectrum.) In accordance with conventional physics, the mass equivalent of ρ_E is assumed to be a gravitational source and hence to contribute to the curvature of the universe. Now the average density of visible matter in the universe is approximately one-tenth the "critical density" required to just contain the expansion of the universe. Taking as a (clearly high) upper limit a mass equivalent for the ZPF equal to the critical density $\rho_c \approx 10^{-29}$ g cm^{-3}, would limit the cutoff frequency to $\omega_c < 7\times10^7$ s^{-1}. Anything greater than that would result in a huge effective Λ. Since it is assumed that ω_c must actually be on the order of the Planck frequency, $\omega_P=[c^5/\hbar G]^{1/2}\approx 3\times10^{43}$ s^{-1}, this results in the Λ problem, i.e., that the radius of curvature of the universe would be expected to be orders of magnitude smaller than an atomic nucleus—which it clearly is not—if the ZPF were real. However, in the Sakharov-Puthoff model the overall ZPF does not itself gravitate, since it is the perturbation of the ZPF in the presence of matter that results in the gravitational force. In this model, a uniform ZPF, whether real or virtual, is not a gravitational source, hence not a contributor to Λ.

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