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Induced transitions and energy of a damped oscillator

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Damped harmonic-oscillator pseudostationary-state wave functions are used to calculate transitions probabilities for a simple harmonic oscillator (SHO) subjected to damping from time $t=0$. The mean and variance of the energy operator are obtained in the state which was initially an SHO stationary state. This state is an instantaneous eigenstate of the energy at times when $\sin(\omega t)=0$, ω being the reduced frequency.

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Several authors have treated the damped harmonic oscillator (DHO) as a closed system by the Kanai-Caldirola Hamiltonian [1–5]. On the other hand, a large volume of literature has been devoted to the open-system, second-quantization approach, from which we select a few papers [6–11]. In this Brief Report the author would like to take the former method further by answering two important questions concerning the imposition of linear damping on a simple harmonic oscillator (SHO). The first question relates to transitions from one stationary state of the SHO to another owing to the perturbative effect of damping (however, perturbation theory will not be used). The other question concerns the possible result of an energy measurement while the SHO is still damped.

It is well known that a canonical treatment of the DHO, with damping constant γ, in either classical or quantum mechanics yields the (classical) energy, or the (quantum-mechanical) energy operator [5]

$$E(q,p,t)=e^{-2\gamma t}H_{\mathrm{KC}}(q,p,t)\ . \tag{1}$$

We shall find it convenient to use the Schrödinger representation with the dimensionless coordinate $x=(m_0\omega_0/\hbar)^{1/2}q$, and momentum, $p=-i\partial/\partial x$. Then

$$H_{\mathrm{KC}}(x,p,t)=\tfrac{1}{2}\hbar\omega_0\left[-e^{-2\gamma t}\left[\frac{\partial}{\partial x}\right]^2+e^{2\gamma t}x^2\right]\ , \tag{2}$$

where ω_0 is the SHO frequency. We shall need also the DHO frequency given by

$$\omega^2=\omega_0^2-\gamma^2\ . \tag{3}$$

From a practical point of view, a DHO has only a finite *effective* lifetime. Damping must be switched in at a certain time. Let us take this as $t=0$, and suppose that at times $t<0$ the system was an SHO in its nth eigenstate $\Phi_n^{\mathrm{SHO}}(x)e^{-i\omega_0 t}$. At times $t>0$ the system is a DHO with Hamiltonian H_{KC} as in Eq. (2), and is described by the Schrödinger wave equation

$$i\hbar\frac{\partial\psi}{\partial t}=H_{\mathrm{KC}}\psi\ . \tag{4}$$

With solutions corresponding to pseudostationary states [cf. Ref. [2], Eq. (3.5)]

$$\psi_m(x,t)=N_m(\omega/\omega_0)^{1/4}\exp[\tfrac{1}{2}\gamma t-\{(\omega+i\gamma)/(2\omega_0)\}e^{2\gamma t}x^2-i(m+(\tfrac{1}{2}))\omega t]\times\mathcal{H}_m(x(\omega/\omega_0)^{1/2}e^{\gamma t})\ , \tag{5a}$$

where ω is given by Eq. (3), and N_m is the normalization factor,

$$N_m=(\pi^{1/2}2^m m!)^{-1/2}\ , \tag{5b}$$

and $\mathcal{H}_m$ is the Hermite polynomial of degree m.

A particular solution of Eq. (4) describes the evolution of the state $\Phi_n^{\mathrm{SHO}}(x)$ (at time $t=0$) after damping is switched in. We denote this state by $\Phi_n(x,t)$. Thus

$$i\hbar\frac{\partial\Phi_n(x,t)}{\partial t}=H_{\mathrm{KC}}\Phi_n(x,t)\quad(t\geq 0)\ , \tag{6a}$$

$$\Phi_n(x,0)=\Phi_n^{\mathrm{SHO}}(x)\ . \tag{6b}$$

We note the correspondence

$$\Phi_m^{\mathrm{SHO}}(x)e^{-i(m+1/2)\omega_0 t}=\mathrm{Lim}_{\gamma\to 0}\psi_m(x,t)\ , \tag{7a}$$

but as $t\to 0$

$$\Phi_m^{\mathrm{SHO}}(x)\neq\psi_m(x,0)\ . \tag{7b}$$

The initial SHO eigenfunction may be expanded in terms of the complete set of orthonormal functions $\psi_m(x,0)$:

$$\Phi_n^{\mathrm{SHO}}(x)=\sum_{m=0}^{\infty}c_m^n\psi_m(x,0)\ . \tag{8a}$$

At a later time, since both $\Phi_n(x,t)$ and $\psi_m(x,t)$ satisfy Eq. (6a),

$$\Phi_n(x,t)=\sum_{m=0}^{\infty}c_m^n\psi_m(x,t)\ . \tag{8b}$$

The coefficients are given by

$$c_m^n=\int_{-\infty}^{\infty}\psi_m^*(x,0)\Phi_n^{\mathrm{SHO}}(x)dx\ . \tag{9}$$

Since the Hermite polynomials in Φ_n^{SHO} and ψ_m have different arguments the general evaluation of c_m^n is not easy. However, for specific n the evaluation is always possible. For simplicity we restrict ourselves to $n=0$ and write c_m^0 as c_m. If we define

$$I_m=\int_{-\infty}^{+\infty} e^{-au^2}\mathcal{H}_m(u)du \ , \quad a=(\omega+\omega_0-i\gamma)/\omega \ , \tag{10a}$$

and use the recurrence relations

$$\mathcal{H}'_m=2m\mathcal{H}_{m-1} \ , \quad 2u\mathcal{H}_m=\mathcal{H}_{m+1}+2m\mathcal{H}_{m-1} \ , \tag{10b}$$

then we can determine c_m from the relations

$$c_m=N_0N_m(\omega_0/\omega)^{1/4}I_m \ , \tag{11a}$$

$$I_{m+2}=(m+1)AI_m \ , \quad A=2i\gamma(\omega+i\gamma)/[\omega_0(\omega_0+\omega)] \ , \tag{11b}$$

$$I_0=(2\pi/a)^{1/2} \ , \quad I_1=0 \ . \tag{11c}$$

We can now answer our first question: if damping acts during the interval (0,t), what is the probability that a transition from the ground state to the nth exited state of the SHO can be observed after time t?

We find it convenient to use bra and ket notation. Then the answer to our question is

$$\text{Prob}(0\rightarrow n)=|\langle \Phi_0(x,t)|\Phi_n^{\text{SHO}}(x,t)\rangle|^2 = \left|\left\langle \sum_{m=0}^{\infty} c_m\psi_m(x,t) \right. \right. \times \left. \left| \Phi_n^{\text{SHO}}(x)e^{-i[n+(1/2)]\omega_0 t}\right\rangle\right|^2 . \tag{12}$$

Substituting $\Phi_n^{\text{SHO}}(x)=N_n\exp\{-(x^2/2)\}\mathcal{H}_n(x)$, we need to evaluate

$$J_m=\int_{-\infty}^{\infty} e^{-bu^2}\mathcal{H}_m(u)du \ , \quad b=(\omega+\omega_0e^{-2\gamma t}-i\gamma)/\omega \ . \tag{13}$$

The difference between a, in Eq. (10a), and b in Eq. (13) is due to the occurrence of $\psi_m(x,0)$ in Eq. (9) and $\psi_m(x,t)$ in Eq. (12). Corresponding to Eqs. (11c) and (11d) we have

$$J_{m+2}=(m+1)BJ_m \ ,$$
$$B=2[(\omega+i\gamma)^2-(\omega_0e^{-2\gamma t})^2]/\{\omega_0[(1+e^{-4\gamma t})\omega_0 +2\omega e^{-2\gamma t}]\} \ , \tag{14a}$$

$$J_0=(2\pi/b)^{1/2} \ , \quad J_1=0 \ . \tag{14b}$$

The probability of no transition is found to be

$$\text{Prob}(0\rightarrow 0)=N_0^4(\omega_0/\omega)e^{-\gamma t}|S|^2 \ , \tag{15a}$$

$$S=\sum_{m(\text{even})=0}^{\infty} N_m^2 e^{\pm im\omega t}I_m^*J_m \ , \tag{15b}$$

where the fact that only even values of m contribute follows from Eqs. (11) and (14). It may be shown that

$$S=2\pi^{1/2}(a^*b)^{-1/2}[1-(\tfrac{1}{4})e^{2i\omega t}A^*B]^{-1/2} \ . \tag{16}$$

Equations (15a) and (15b) then yield a lengthy result for Prob(0→0).

Times at which $\sin\omega t=0(e^{i\omega t}=1)$ are significant, because then, as we shall see, $\Phi_n(x,t)$ is an eigenfunction of the energy with eigenvalue given by Eq. (27) below. At these times we find

$$\text{Prob}(0\rightarrow 0)=C|[(1+e^{-4\gamma t})\omega_0 +2\omega e^{-2\gamma t}]/[(1+e^{-2\gamma t})^2 +i\gamma'(1-e^{-4\gamma t})]| \ , \tag{17a}$$

where

$$\gamma'=\gamma/(\omega+\omega_0) \ , \quad C=4\omega_0/(\omega^2|ab|) \ , \tag{17b}$$

and a and b are to be found in Eqs. (10a) and (13). It is readily verified that Prob(0→0)→1 as $\gamma\rightarrow 0$. We may subject our results to two further tests.

(a) $\gamma \ll \omega_0$ so that $\omega\approx\omega_0$. Let us wait for a long time until, with $\sin\omega t=0$, $e^{-\gamma t}=\varepsilon$ $(0<\varepsilon\ll 1)$. Then Eqs. (17a) and (17b) give

$$\text{Prob}(0\rightarrow 0)\approx 2\varepsilon \ . \tag{18a}$$

(b) Again $\gamma\ll\omega_0$, but we wait only until $\omega t=\pi$. Then $e^{-\gamma t}=1-\eta$ $(0<\eta\ll 1)$. In this case, working to second order in η, Eqs. (17a) and (17b) yield the very credible result

$$\text{Prob}(0\rightarrow 0)\approx 1-\tfrac{1}{2}\eta^2 \ . \tag{18b}$$

These together with further numerical tests convince us of the reliability of our calculation.

Turning our attention to Prob(0→1) we note that an extra factor of u occurs in the integral of Eq. (13). Let us call this integral J'_m; then using the second of Eqs. (10b) it follows that $J'_m=0$ for m even and the summation for S in Eq. (15b) vanishes. Thus

$$\text{Prob}(0\rightarrow 1)=0 \ .$$

Prob(0→2) can be calculated using $H_2(x)=4x^2-2$. The second relation of Eqs. (10b) is used twice to treat the $4x^2$ part. The result is too unwieldy to display.

We can continue this process substituting into Eq. (12) the appropriate Hermite polynomials. There is no difficulty in principle in calculating Prob($n\rightarrow n'$) for any n or n'. As stated in Ref. [5], and as we have seen for Prob(0→1), damping can induce transitions only between states of like parity. Unfortunately, the transition probabilities given in Ref. [5] are incorrect.

Digressing slightly before answering our second question, we calculate the mean and variance of the energy in the pseudostationary state $\psi_m(x,t)$. These follow easily from Eqs. (1), (2), and (5) and using

$$\sigma^2=\langle E^2\rangle-\langle E\rangle^2 \tag{19}$$

for the variance. We find for $m=0,1,2,\ldots,$

$$\langle \psi_m(x,t)|E(t)|\psi_m(x,t)\rangle=[m+(\tfrac{1}{2})](\hbar\omega_0^2/\omega)e^{-2\gamma t} \ , \tag{20a}$$

$$\sigma_m^2=\tfrac{1}{2}(m^2+m+1)(\hbar\gamma\omega_0/\omega)^2e^{-4\gamma t} \ , \tag{20b}$$

where $E(t)$ is given by Eqs. (1) and (2). Hence a pseudostationary state is never an eigenfunction of the energy (nor of the Hamiltonian). No special significance should be attached to Eqs. (20a) and (20b) as it is not feasible to prepare the system in the initial state $\psi_m(x,0)$. However, it is interesting to compare them with corresponding results for the evolved SHO state $\Phi_n(x,t)$. The Heisenberg form of $E(t)$ has been obtained in Eq. (22) of Ref. [5]:

$$E(t)=e^{-2\gamma t}\{[1+2(\gamma/\omega)^2\sin^2\omega t]E(0)-(\gamma/\omega)\sin^2\omega t L(0) +2\gamma(\omega_0/\omega)^2\sin^2\omega t S(0)\}\ , \tag{21}$$

where

$$\begin{aligned} E(0)&=p^2/(2m_0)+(\tfrac{1}{2})m_0\omega_0^2q^2\ , \\ L(0)&=p^2/2m_0-(\tfrac{1}{2})m_0\omega_0^2q^2\ , \\ S(0)&=(\tfrac{1}{2})(qp+pq)\ . \end{aligned} \tag{22}$$

Let us take the expectation value of $E(t)$ in the DHO state $\Phi_n(x,t)$. We need only expectation values in the initial SHO eigenstate $\Phi_n(x,0)$, i.e., we need

$$\begin{aligned} \langle n|E(0)|n\rangle&=\hbar\omega_0[n+(\tfrac{1}{2})]\ , \\ \langle n|L(0)|n\rangle&=-(\tfrac{1}{2})\hbar\omega_0\langle n|a^{+2}+a^2|n\rangle=0\ , \\ \langle n|S(0)|n\rangle&=(\tfrac{1}{2})i\hbar\langle n|a^{+2}-a^2|n\rangle=0\ . \end{aligned} \tag{23}$$

The result is

$$\langle\phi_n(x,t)|E(t)|\phi_n(x,t)\rangle =\hbar\omega_0[n+(\tfrac{1}{2})][1+2(\gamma/\omega)^2\sin^2\omega t]e^{-2\gamma t}\ . \tag{24}$$

The only nonvanishing mean values encountered when we square the right-hand side of Eq. (21) to form $\langle E^2\rangle$ are

$$\begin{aligned} \langle n|E^2(0)|n\rangle&=\hbar^2\omega_0^2[n+(\tfrac{1}{2})]^2\ , \\ \langle n|L^2(0)|n\rangle&(\tfrac{1}{2})\hbar^2\omega_0^2(n^2+n+1)\ , \\ \langle n|S^2(0)|n\rangle&=(\tfrac{1}{2})\hbar^2(n^2+n+1)\ . \end{aligned} \tag{25}$$

Then from Eq. (19) [cf. Eq. (20b)],

$$\sigma_n^2(t)=2(n^2+n+1)(\hbar\gamma\omega_0/\omega)^2\sin^2\omega t \times[1+(\gamma/\omega)^2\sin^2\omega t]e^{-4\gamma t}\ . \tag{26}$$

The uncertainty vanishes whenever $\sin\omega t=0$. Only at these times does it make sense to speak of energy eigenvalues (cf. Ref. [5]):

$$\varepsilon_n(t=r\pi/\omega)=[n+(\tfrac{1}{2})]\hbar\omega_0e^{-2\gamma t}\ ,\quad r=0,1,2,\ldots\ . \tag{27}$$

At intermediate times when $\sin^2\omega t=1$, we note that the difference

$$\langle E\{t=[r+(\tfrac{1}{2})]\pi/\omega\}\rangle-[n+(\tfrac{1}{2})]\hbar\omega_0e^{-2\gamma t} =2(\gamma/\omega)^2[n+(\tfrac{1}{2})]\hbar\omega_0e^{-2\gamma t}\ . \tag{28}$$

Equations (24) and (26) provide the answer to our second question. However, an instantaneous measurement of energy contradicts the time-energy uncertainty principle [12]. As damping increases, the period of oscillations lengthens and there would be time to observe the energy in the neighborhood of $\omega t=r\pi$ or of $\omega t=[r+(\tfrac{1}{2})]\pi$. To take an extreme case, suppose $\gamma=\omega_0(1-\varepsilon)$, $0<\varepsilon\ll1$. Then $\omega^2\approx2\varepsilon\omega_0^2$. For $t\approx r\pi/\omega$ we should find $E\approx[n+(\tfrac{1}{2})]\hbar\omega_0e^{-2\gamma t}$ with certainty, whereas for $t\approx(r+1)\pi/\omega$ we should expect to obtain $E=\varepsilon^{-1}[n+(\tfrac{1}{2})]\hbar\omega_0e^{-2\gamma t}$ with a large standard deviation of $\varepsilon^{-1}[(n^2+n+1)/2)]^{1/2}\hbar\omega_0e^{-2\gamma t}$.

Let us compare with the situation in classical mechanics. In Eq. (21) we need to write

$$\begin{aligned} E(0)&=(\tfrac{1}{2})m_0[\dot q^{\,2}(0)+\omega_0^2q^2(0)]\ , \\ L(0)&=(\tfrac{1}{2})m_0[\dot q^{\,2}(0)-\omega_0^2q^2(0)]\ , \\ S(0)&=m_0q(0)\dot q(0)\ . \end{aligned} \tag{29}$$

At times when $\sin\omega t=0$ or ±1,

$$E_{\rm cl}(t=r\pi/\omega)=E(0)e^{-2\gamma t}\ , \tag{30a}$$

$$E_{\rm cl}[t=(r+1)\pi/\omega]=[\{(\omega_0^2+\gamma^2)/\omega^2\}E(0) +\{2m_0\omega_0^2\gamma/\omega^2\}q(0)\dot q(0)]e^{-2\gamma t}\ . \tag{30b}$$

At the times of Eq. (30b),

$$E_{\rm cl}[t=(r+1)\pi/\omega]-E(0)e^{-2\gamma t} =\{2\gamma^2/\omega^2\}[E(0)+\{m_0\omega_0^2/\gamma\}q(0)\dot q(0)]e^{-2\gamma t}\ . \tag{31}$$

We know that the second term in the square bracket corresponds to $\{\omega_0^2/\gamma\}S(0)$ in quantum mechanics, with vanishing expectation value in a stationary state. Therefore we can drop this term in making a comparison with Eq. (28). The agreement is then exact, as required by Ehrenfest's theorem.

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