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Exact solution of the two-dimensional Dirac oscillator

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In the present article we have found the complete energy spectrum and the corresponding eigenfunctions of the Dirac oscillator in two spatial dimensions. We show that the energy spectrum depends on the spin of the Dirac particle.

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Recently, Moshinsky and Szczepaniak [1] proposed a new type of interaction in the Dirac equation which, besides the momentum, is also linear in the coordinates. They called the resulting Dirac equation the Dirac oscillator because in the nonrelativistic limit it becomes a harmonic oscillator with a very strong spin-orbit coupling term. Namely, the correction to the free Dirac equation

$$i\frac{\partial\Psi_c}{\partial t} = (\beta\gamma\mathbf{p} + \beta m)\Psi_c \tag{1}$$

reads

$$\mathbf{p} \to \mathbf{p} - im\omega\beta\mathbf{r}. \tag{2}$$

After substituting (2) into (1) we get a Hermitian operator such as linear in both $\mathbf{p}$ and $\mathbf{r}$. Recently, the Dirac oscillator has been studied in spherical coordinates and its energy spectrum and the corresponding eigenfunctions have been obtained [2]. A generalization of the one-dimensional version of the Dirac oscillator has been proposed by Domínguez-Adame [3]. In this case, the modification of the free Dirac equation, written in Cartesian coordinates, is made by means of the substitution $m \to m - i\gamma^0\gamma^1 V(x_1)$. Obviously, for $V(x_1) = m\omega x_1$ we have the standard Dirac oscillator. Here, as well as for the three-dimensional Dirac oscillator [2], bound states are present.

An interesting framework for discussing the Dirac oscillator is a 2+1 space-time. The absence of a third spatial coordinate permits a series of interesting physical and mathematical phenomena such as fractional statistics [4] and Chern-Simmons gauge fields among others. Since we are interested in studying the Dirac oscillator in a two-dimensional space, a suitable system of coordinates for writing the harmonic interaction is the polar ρ and ϑ coordinates. In this case the radial component of the modified linear momentum takes the form $p_\rho - im\omega\beta\rho$. It is the purpose of the present paper to analyze the solutions and the energy spectrum of the 2+1 Dirac oscillator expressed in polar coordinates.

One begins by writing the Dirac equation (1) in a given representation of the γ matrices. Since we are dealing with two component spinors it is convenient to introduce the following representation in terms of the Pauli matrices [5]:

$$\beta\gamma_1 = \sigma_1,\ \beta\gamma_2 = s\sigma_2,\ \beta = \sigma_3, \tag{3}$$

where the parameter s takes the values ± 1 ($+1$ for spin up and -1 for spin down). Then, the Dirac equation (9) written in polar coordinates reads

$$iE\Psi = \left[\sigma^1\partial_\rho + \sigma^2\left(\frac{ik_\vartheta s}{\rho} - m\omega\rho\right) + i\sigma^3 m\right]\Psi \tag{4}$$

with

$$\Psi = \Psi_0(\rho)e^{i(k_\vartheta\vartheta - Et)},$$

where the spinor Ψ is expressed in the (rotating) diagonal gauge, related to the Cartesian (fixed) gauge by means of the transformation $S(\rho,\vartheta)$ [6]

$$\Psi_c = S(\rho,\vartheta)^{-1}\Psi, \tag{5}$$

where the matrix transformation $S(\rho,\vartheta)$ can be written as

$$S(\rho,\vartheta) = \frac{1}{\sqrt{\rho}}\exp\left(-i\frac{\vartheta}{2}\sigma^3\right). \tag{6}$$

Noticing that $S(\rho,\vartheta)$ satisfies the relation

$$S(\rho,\vartheta + 2\pi) = -S(\rho,\vartheta) \tag{7}$$

we obtain

$$\Psi(\vartheta + 2\pi) = -\Psi(\vartheta), \tag{8}$$

so we have $k_\vartheta = N + 1/2$, where N is an integer number.

Using the representation (3), the spinor equation (4) can be written as system of two first-order coupled differential equations

$$i(E - m)\Psi_1 = \left(\frac{d}{d\rho} + \frac{k_\vartheta s}{\rho} - im\omega\rho\right)\Psi_2, \tag{9}$$

$$i(E+m)\Psi_2 = \left(\frac{d}{d\rho} - \frac{k_\vartheta s}{\rho} + im\omega\rho\right)\Psi_1, \quad (10)$$

where

$$\Psi_0 = \begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}. \quad (11)$$

Substituting (10) into (9) and vice versa we arrive at

$$\left[\frac{d^2}{d\rho^2} - \frac{(k_\vartheta s)(k_\vartheta s \mp 1)}{\rho^2} + m\omega(2k_\vartheta s \pm 1) - m^2\omega^2\rho^2 + (E^2 - m^2)\right]\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = 0. \quad (12)$$

It is not difficult to see that the solution of the second-order equation (12) for Ψ_1 can be expressed in terms of associated Laguerre polynomials $L_k^s(x)$ [7,8] as follows:

$$\Psi_1 = c_1 \exp(-x/2) x^{\frac{1}{2}(1/2+\mu)} L_n^\mu(x), \quad (13)$$

where we have made the change of variables

$$x = m\omega\rho^2, \quad (14)$$

where μ satisfies the relation

$$\mu = \pm(k_\vartheta s - 1/2) \quad (15)$$

and the natural number n satisfies the relation

$$\frac{E^2 - m^2}{m\omega} + (1 \mp 1)(2k_\vartheta s - 1) = 4n. \quad (16)$$

Since the function Ψ_1 must be regular at the origin, we obtain that the sign of μ in (13) is determined by the sign of s. In fact, for $k_\vartheta s > 0$ we have that Ψ_1 reads

$$\Psi_1 = c_1 \exp(-x/2) x^{k_\vartheta s/2} L_n^{k_\vartheta s - 1/2}(x). \quad (17)$$

Substituting (17) into (10) we arrive at

$$\Psi_2 = 2ic_1 \frac{(m\omega)^{1/2}}{E+m} \exp(-x/2) x^{(k_\vartheta s+1)/2} L_{n-1}^{k_\vartheta s+1/2}(x), \quad (18)$$

where c_1 is an arbitrary constant.

Analogously, we obtain that the regular solutions for, $k_\vartheta s < 0$ are

$$\Psi_2 = c_2 \exp(-x/2) x^{-k_\vartheta s/2} L_n^{-k_\vartheta s - 1/2}(x), \quad (19)$$

$$\Psi_1 = 2ic_2 \frac{(m\omega)^{1/2}}{E+m} \exp(-x/2) x^{(1-k_\vartheta s)/2} L_n^{1/2-k_\vartheta s}(x), \quad (20)$$

where c_1 is a normalization constant. The expression (16) can be rewritten as follows:

$$E^2 - m^2 = 4\left[n - \Theta(-k_\vartheta s)(k_\vartheta s - 1/2)\right] m\omega, \quad (21)$$

where $\Theta(x)$ is the Heaviside step function. Then from the relation (21) it is clear that the energy spectrum of the 2+1 Dirac oscillator depends on the value of s. Notice that for positive values of $k_\vartheta s$ there is no degeneration of the energy spectrum. For $k_\vartheta s < 0$ we observe that all the states with $(n \pm l,\ k_\vartheta s - 1/2 \pm l)$, where l is an integer, have the same energy. In this direction there are some differences with the spherical Dirac oscillator [2]. Despite the fact that in both cases bound states are obtained, for the 2+1 Dirac oscillator the energy spectrum is degenerate only for negative values of $k_\vartheta s$. In order to get a deeper understanding of the dependence of the energy spectrum on the spin we can take the nonrelativistic limit of the Dirac equation (4). In order to do that, it is advisable to work with Eq. (12). The Galilean limit is obtained by setting $E = m + \varepsilon$ and considering $\varepsilon \ll m$. Taking into account that the first two terms in Eq. (12) are associated with the operator P^2, we obtain in the nonrelativistic limit

$$\frac{P^2}{2m} - \omega\left(k_\vartheta s \pm \frac{1}{2}\right) + \frac{m^2\omega^2\rho^2}{2} = \varepsilon. \quad (22)$$

Notice that Eq. (22) corresponds to the Schrödinger Hamiltonian of a harmonic oscillator with an additional spin dependent term given by $-\omega(k_\vartheta s \pm \frac{1}{2})$. This contribution is proportional to the frequency of the oscillator.

It would be interesting to analyze the Dirac oscillator in more complex configurations where electromagnetic and gravitational interactions are present. Regretfully in this direction the possibilities of finding exactly solvable examples are limited to those where the Dirac equation with an anomalous moment is soluble [9]. A detailed discussion of this problem will be the objective of a forthcoming publication.

[1] M. Moshinsky and A. Szczepaniak, J. Phys. A **22**, L817 (1989).
[2] J. Benítez, R. P. Martínez y Romero, H. N. Núñez-Yépez, and A. L. Salas-Brito, Phys. Rev. Lett. **64**, 1643 (1990).
[3] F. Domínguez-Adame, Phys. Lett. A **162**, 18 (1992).
[4] F. Wilczek, Phys. Rev. Lett. **48**, 1144 (1982).
[5] C. R. Hagen, Phys. Rev. Lett. **64**, 503 (1990) .
[6] G. V. Shishkin and V. M. Villalba, J. Math. Phys. **30**, 2373 (1989).
[7] E. Kamke, *Differentialgleichungen Lösungsmethoden und Lösungen* (Verlag, Leipzig, 1959).
[8] *Handbook of Mathematical Functions,* Natl. Bur. Stand. Appl. Math. Ser. No. 55, edited by M. Abramowitz and I. Stegun (U.S. GPO, Washington, DC, 1965).
[9] V. G Bagrov and D. Gitman, *Exact Solutions of Relativistic Wave Equations* (Kluwer, Dordrecht, 1990).