Nonideal lasers, nonclassical light, and deformed photon states

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We show that both super-Poissonian and sub-Poissonian photon statistics may be modeled by the use of the recently introduced M-type q-deformed coherent states, while P-type q-deformed coherent states exhibit nonclassical sub-Poissonian photon statistics. Applications to the characterization of the photon statistics of laser outputs reasonably close to threshold, single-atom resonance fluorescence, the micromaser field, and absorption by two-level atoms are considered.

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The states of an ideal laser are conventionally described by Glauber coherent states [1]. However, real lasers do not strictly adhere to this description; in particular, the photon number statistics of real lasers are not exactly Poissonian [2]. Furthermore, various nonlinear interactions give rise to well-defined deviations from the Poissonian distribution [3]. Recently, deformations of the commutation rules of boson operators have been considered both in the purely mathematical context of giving realizations of the so-called "quantum groups" (noncocommutative Hopf algebras) [4] but also as models for physical systems which deviate from the ideal cases [5]. We approach the problem of the "real" laser in this latter phenomenological spirit, and show that indeed a coherent state of the deformed boson (q-coherent state) provides a more accurate model of a nonideal laser, at least as far as the photon number statistics is concerned.

An ideal laser may be described as a normalized eigenstate of the photon annihilation operator a, where a and its Hermitian conjugate $a^\dagger$ (photon creation operator) satisfy

$$[a,a^\dagger]\equiv aa^\dagger - a^\dagger a = I. \tag{1}$$

The normalized eigenstate satisfying $a|\alpha\rangle=\alpha|\alpha\rangle$ is easily seen to be

$$|\alpha\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{2}$$

The number eigenstates are $|n\rangle$, and this coherent state gives rise to the Poisson distribution

$$P_n = |\langle n|\alpha\rangle|^2 = \exp(-|\alpha|^2)\frac{|\alpha|^{2n}}{n!}. \tag{3}$$

The factorial moments of this distribution are

$$\langle n\rangle = |\alpha|^2,$$
$$\langle n(n-1)\rangle = |\alpha|^4,$$
$$\langle n(n-1)(n-2)\rangle = |\alpha|^6,$$

etc., from which the variance is found to be

$$\sigma^2 = \langle n^2\rangle - \langle n\rangle^2 = |\alpha|^2.$$

A convenient measure of the deviation of a distribution from the Poisson distribution is the Mandel parameter

$$Q = \frac{\sigma^2}{\langle n\rangle} - 1 = \frac{\langle n(n-1)\rangle}{\langle n\rangle} - \langle n\rangle,$$

which vanishes for the Poisson distribution, is positive for a super-Poissonian distribution, and negative for a sub-Poissonian distribution.

The two main deformations of the canonical commutation relations, Eq. (1), which have been considered are

(a) "maths" boson,

$$aa^\dagger - qa^\dagger a = I. \tag{4}$$

This was introduced by Arik and Coon [6], who also described the corresponding q-coherent states. We refer to this deformed boson as a "maths" (or M) boson as the "basic" numbers [cf. Eq. (7)] and special functions, q functions, associated with this operator have been investigated in the mathematical literature for over 150 years; see, for example, Ref. [7].

(b) "physics" boson,

$$aa^\dagger - qa^\dagger a = q^{-N}. \tag{5}$$

The number operator N in Eq. (5) satisfies $[N,a]=-a$, just as for the usual (nondeformed) boson operators. This deformation was introduced [8,9] in order to provide a realization of the "quantum groups" [4] which arise naturally in the solution of certain lattice models [10]. The q-coherent states associated with these "physics" (or P) bosons have been investigated by several authors [9,11,12].

One may readily show that the normalized q-coherent state $|\alpha\rangle$ satisfying $a|\alpha\rangle=\alpha|\alpha\rangle$ is given by

$$|\alpha\rangle = \frac{1}{\sqrt{E_q(|\alpha|^2)}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{[n]!}}|n\rangle, \tag{6}$$

where $[n]$ (read "box n") is given by

$$[n] = \begin{cases} \dfrac{1-q^n}{1-q} & M \text{ case} \\[2ex] \dfrac{q^n-q^{-n}}{q-q^{-1}} & P \text{ case} \end{cases} \tag{7}$$

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and, in both cases,

$$E_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]!} \tag{8}$$

with

$$[n]! = [n][n-1]\cdots[1]. \tag{9}$$

As a model for a nonideal laser, this q-coherent state gives rise to the photon number distribution

$$P_n = \frac{1}{E_q(|\alpha|^2)} \frac{|\alpha|^{2n}}{[n]!}. \tag{10}$$

Note that the distribution in Eq. (10) depends on two parameters; $|\alpha|^2$ and the value of q (taken here to be real). We shall refer to Eq. (10) as the q-Poisson distribution.

One can easily check that the P-type q-Poisson distribution is sub-Poissonian ($Q \leq 0$) for all values of q, reducing to the conventional Poisson distribution for $q = 1$. On the other hand, the M-type q-Poisson distribution is super-Poissonian for $q < 1$ and sub-Poissonian for $q > 1$.

The q-Poissonian q-factorial moments are $\langle [n] \rangle = |\alpha|^2, \langle [n][n-1] \rangle = |\alpha|^4$, etc.

To evaluate the average number of photons and the Mandel parameter for the q-Poisson distribution we note that the corresponding factorial moments satisfy

$$\langle n \rangle = \frac{x}{E_q(x)} \frac{\partial E_q(x)}{\partial x} \bigg|_{x=|\alpha|^2},$$

$$\langle n(n-1) \rangle = \frac{x^2}{E_q(x)} \frac{\partial^2 E_q(x)}{\partial^2 x} \bigg|_{x=|\alpha|^2}.$$

These expressions were used to construct Fig. 1, which provides estimates of the q-Poissonian parameters q and $|\alpha|^2$ corresponding to a distribution which is specified in terms of given values of $\langle n \rangle$ and Q. The values of q corresponding to given pairs of values of $\langle n \rangle$ and Q are presented in Fig. 1(a), and the corresponding values of $|\alpha|^2$ are presented in Fig. 1(b).

For small deviations from a Poissonian distribution we define $q = e^{-s}$ and obtain in the M case

$$s = \frac{2Q}{\langle n \rangle},$$

which is positive (i.e., $q < 1$) for a super-Poissonian distribution and negative ($q > 1$) for a sub-Poissonian distribution. In the P case we obtain

$$s^2 = -\frac{3Q}{\langle n \rangle (\langle n \rangle + \frac{3}{2})}$$

so that only the sub-Poissonian distribution ($Q < 0$) corresponds to a real value of s (and q).

Another useful result is

$$\rho \equiv \lim_{\langle n \rangle \to 0} \frac{Q}{\langle n \rangle} = \begin{cases} \dfrac{1-q}{1+q} & M \text{ case} \\[2ex] \dfrac{2}{q+q^{-1}} - 1 & P \text{ case}. \end{cases} \tag{11}$$

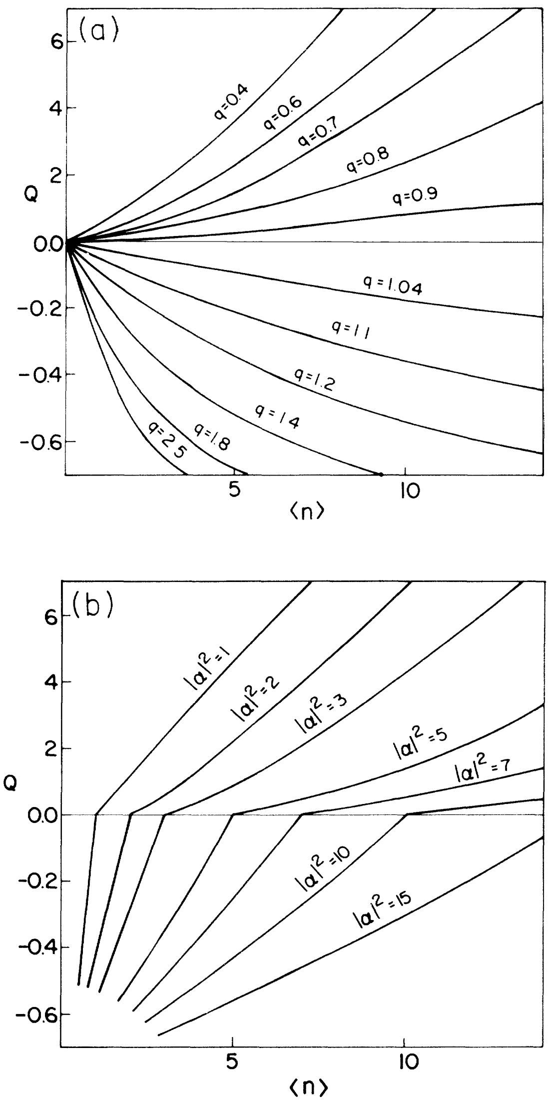

FIG. 1. Parameters of the q-Poissonian distribution corresponding to given values of the average photon number $\langle n \rangle$ and of the Mandel parameter Q. (a) The values of q. (b) The values of $|\alpha|^2$. Note the different scales for $Q > 0$ and $Q < 0$.

In the M case the range of ρ is $-1 < \rho < 1$, corresponding to a sub-Poissonian distribution for $\rho < 0$ and to a super-Poissonian distribution for $\rho > 0$. In the P case the range of ρ is $-1 < \rho \leq 0$, exhibiting only a sub-Poissonian distribution.

From Eq. (11) we obtain

$$q = \begin{cases} \dfrac{1-\rho}{1+\rho} & M \text{ case} \\[2ex] \dfrac{1}{1+\rho} \pm \sqrt{\dfrac{1}{(1+\rho)^2} - 1} & P \text{ case}. \end{cases} \tag{12}$$

Using the three highest peaks in the experimental data pertaining to the photon statistics of a He-Ne laser just above threshold [13] we obtain $\frac{P_2^2}{P_1 P_3} = \frac{[3]}{[2]} = 1.319$, which in the M case is a quadratic equation in q, yielding $q =$

0.747. Note that the corresponding equation for the P case can be shown to rule out the P boson as a model of this system since for all real and positive q the inequality $\frac{[3]}{[2]} \geq \frac{3}{2}$ holds.

In Fig. 2, we compare the best fit for the M-boson q-coherent state against the experimental data [13] and the ideal (Glauber) coherent state. The value of q corresponding to the best fit is 0.749, in very close agreement with the value estimated above using the highest three peaks. It is not surprising that a better fit is obtained with the q-coherent state, due to the extra parameter q. However, certain constraints are satisfied [for example, the convergence criterion for the M-type q-exponential function demands that $(1 - q)|\alpha|^2 \leq 1$ and is satisfied here] and, as we have already remarked, the P-boson model is ruled out.

Experimental studies of the photon statistics of a laser at different intensities above the threshold were reported in Refs. [15] and [16]. Since super-Poissonian statistics is exhibited, only M-type analysis is warranted. In both cases it is found that for counting times that are short relative to the intensity correlation time the distributions agree with q-Poissonian statistics, the value of q increasing from a value which could be close to zero at threshold to a value close to unity (Poissonian distribution) for intensities about an order of magnitude higher than the threshold intensity. At twice the threshold intensity values of q ranging between roughly 0.3 and 0.8 were obtained from the different sets of experimental data.

Another set of experimental data, exhibiting a sub-Poissonian distribution, involves the photons emitted by single-atom resonance fluorescence [14]. Using the data for P_0, P_1, P_2, we obtain $[2] = 3.44$, which corresponds to $q_M = 2.44$ or to $q_P = 3.12$ (or $\frac{1}{3.12} = 0.321$). This is in agreement with the estimate for q_M obtained using Eq. (12) and the data reported in Ref. [14], $\langle n \rangle = 6.23 \times 10^{-3}$ and $Q = -2.52 \times 10^{-3}$, from which $q_M = 2.36$.

Sub-Poissonian photon statistics was also established for the micromaser field [17]. For the set of data $\langle n \rangle = 13$ and $Q = -0.7$ we read off Fig. 1(a) the value $q = 1.3$. The corresponding value of $|\alpha|^2$ can similarly be read off Fig. 1(b). A sub-Poissonian distribution was also observed for photon absorption by two level atoms [18]. Using Fig. 1(a) for the values $\langle n \rangle = 2.69$ and $Q = -0.51$ we obtain $q = 1.8$.

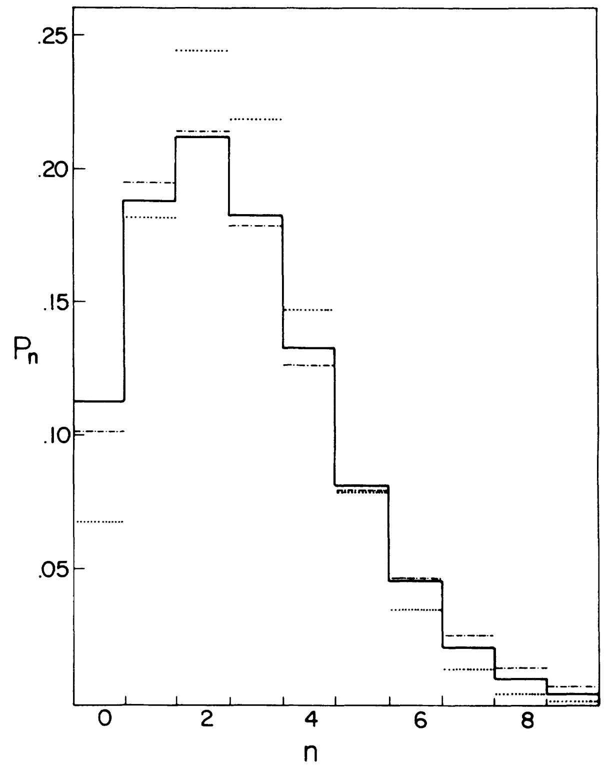

FIG. 2. Comparison of the Poissonian (dotted), q-Poissonian (dashed), and measured (Ref. [13]) photon number distribution for a laser just above threshold.

The q-Poisson distribution presently introduced provides yet another example, in this case from quantum optics, where a more accurate model of a physical system may be obtained by use of quantum group ideas.

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