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Quasicontinuous measurements of photon number

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A quasicontinuous measurement incorporating lossless beam splitters and photodetectors is proposed. Here the measurement by photodetectors is assumed to obey the projection postulate. The photon statistics in the output state of the signal mode of a beam splitter is investigated. It is shown that under certain conditions the measurement yields results equivalent to those obtained by the continuous measurement of photon number described by the quantum Markov process. Furthermore, measurements using a nondegenerate parametric amplifier and four-wave mixer, instead of beam splitter, are considered. The relations to the continuous measurement of photon number with a quantum counter and the continuous quantum nondemolition measurement of photon number are discussed.

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I. INTRODUCTION

The beam splitter is a key component of optical measurements, mathematically equivalent to the Mach-Zehnder interferometer and linear directional coupler. These devices are characterized by SU(2) symmetry in the absence of dissipation. The phase sensitivity in the output state of the Mach-Zehnder interferometer has been investigated for bosonic [1] and fermionic [2,3] systems. The photon statistics in the output states of lossless beam splitters [4,5] and linear directional couplers [6,7] have been considered for various input states. In these studies [4–7], the quantum effects such as sub-Poissonian statistics and the squeezing of fluctuations in the output state were investigated.

In this paper, we propose a quasicontinuous measurement in terms of lossless beam splitters and photodetectors used to measure the photon number from one of the two output ports of the beam splitter. Here the measurement by the photodetector is assumed to be subject to the von Neumann projection postulate and thus to be the first kind of measurement [8,9]. We investigate the photon statistics in the output state under the condition that the photodetector exhibits the m-photon state. Then we show that under certain conditions our measurement yields results equivalent to those obtained by the continuous measurement of photon number [10–13] described by the quantum Markov process [14]. Furthermore, we consider measurements with a nondegenerate parametric amplifier and a four-wave mixer, instead of beam splitter. It is shown that a measurement by means of parametric amplifier and photodetector subject to the projection postulate can yield the same results as those obtained in the continuous measurement of photon number with quantum counter [15,16]. It is also found that a measurement in terms of a four-wave mixer and a photodetector can give results equivalent to those obtained in the continuous quantum nondemolition measurement of photon number [17].

This paper is organized as follows. In Sec. II we explain a measurement by means of beam splitter and photodetector, and we calculate an output state of signal mode. In Sec. III we investigate the photon statistics in the output state of the beam splitter for several input states, under the condition that the photodetector exhibits an m-photon state. We find an oscillatory behavior of photon statistics for even and odd coherent state inputs. In Sec. IV we discuss the relation between the measurement we proposed and the continuous measurement of photon number described by the quantum Markov process. In Sec. V we consider a measurement using nondegenerate parametric amplifier, instead of beam splitter, and photodetector, and we discuss the relation to the continuous measurement of photon number with quantum counter. We also consider a measurement with a four-wave mixer and a photodetector, and we show the equivalence to the continuous quantum nondemolition measurement of photon number. A summary is given in Sec. VI.

II. MEASUREMENT WITH BEAM SPLITTERS

Let us consider the measurement setup shown in Fig. 1. It is assumed here that the beam splitter is lossless and that the measurement performed by the photodetector obeys the projection postulate [8,9]. In the Heisenberg picture, the beam splitter is characterized by the relation

$$\begin{pmatrix} a_{\rm out} \\ b_{\rm out} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a_{\rm in} \\ b_{\rm in} \end{pmatrix}, \tag{2.1}$$

where $(a_{\rm in}, a_{\rm in}^\dagger)$ and $(b_{\rm in}, b_{\rm in}^\dagger)$ are the annihilation and creation operators of the input signal and reference modes, and $(a_{\rm out}, a_{\rm out}^\dagger)$ and $(b_{\rm out}, b_{\rm out}^\dagger)$ are those of the output modes. Here we have ignored the phase shift in each mode due to the beam splitter since such phase shifts are not important for our purpose. The transmittance τ and reflectance r of the beam splitter are given by

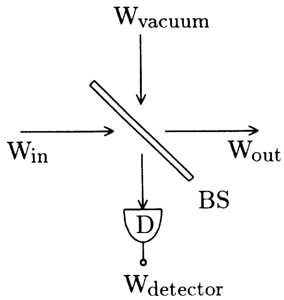

FIG. 1. A schematic representation of the measurement setup where BS is a lossless beam splitter and D is a photodetector with which the measurement obeys the projection postulate. $W_{\rm in}$ is the density matrix of the signal mode input and $W_{\rm vacuum}$ is the density matrix of the reference mode input assumed to be in a vacuum state. $W_{\rm detector}$ represents the output state of the reference mode exhibited by the photodetector. $W_{\rm out}$ indicates the output state of the signal mode under the condition that the photodetector exhibits the state $W_{\rm detector}$.

$$\tau = \cos^2\theta, \quad r = \sin^2\theta. \tag{2.2}$$

In the following, we assume that the input state of the reference mode is the vacuum state. In the Schrödinger picture, the output state $\hat{W}_{\rm out}$ of the beam splitter is given by

$$\hat{W}_{\rm out} = V(W_{\rm in} \otimes W_{\rm vacuum})V^\dagger, \tag{2.3a}$$
$$V = \exp[-\theta(J_+ - J_-)], \tag{2.3b}$$

where $W_{\rm in}$ is the density matrix of the signal mode input and $W_{\rm vacuum}$ is the density matrix of the reference mode input that satisfies $bW_{\rm vacuum} = W_{\rm vacuum}b^\dagger = 0$. In (2.3b), $J_\pm$ and J_0 are the generators of su(2) Lie algebra given by

$$J_+ = a^\dagger b, \tag{2.4a}$$
$$J_- = ab^\dagger, \tag{2.4b}$$
$$J_0 = \tfrac{1}{2}(a^\dagger a - b^\dagger b), \tag{2.4c}$$

which satisfy the commutation relations, $[J_+, J_-] = 2J_0$ and $[J_0, J_\pm] = \pm J_\pm$. Here we set $(a, a^\dagger) = (a_{\rm in}, a^\dagger_{\rm in})$ and $(b, b^\dagger) = (b_{\rm in}, b^\dagger_{\rm in})$. The relation (2.3) shows the well known fact that a lossless beam splitter is characterized by su(2) Lie algebra [1–7].

Suppose that the photodetector for the output of the reference mode exhibits an m-photon state such that $W_{\rm detector} = |m\rangle\langle m|$, where $b^\dagger b|m\rangle = m|m\rangle$. According to the projection postulate, the non-normalized output state of the signal mode is given by

$$\begin{aligned} W_{\rm out}(m) &= {\rm Tr}_b[\hat{W}_{\rm out}W_{\rm detector}] \\ &= \frac{1}{m!}\left(\frac{r}{\tau}\right)^m a^m \tau^{a^\dagger a/2} W_{\rm in} \tau^{a^\dagger a/2}(a^\dagger)^m, \end{aligned} \tag{2.5}$$

where ${\rm Tr}_b$ is the trace operation over the Hilbert space of the reference mode. Now we introduce the generators $K_\pm$ and K_0 of su(1,1) Lie algebra as

$$K_+X = a^\dagger X a, \tag{2.6a}$$
$$K_-X = aXa^\dagger, \tag{2.6b}$$
$$K_0X = \tfrac{1}{2}(a^\dagger aX + Xa^\dagger a + X) \tag{2.6c}$$

for an arbitrary operator X [18,19], where $[K_-, K_+] = 2K_0$ and $[K_0, K_\pm] = \pm K_\pm$ are satisfied. Then $W_{\rm out}(m)$ is expressed as follows:

$$W_{\rm out}(m) = U(\mu, m)W_{\rm in}, \tag{2.7a}$$
$$U(\mu, m) = \frac{(e^\mu - 1)^m}{m!}K_-^m \exp[-\mu(K_0 - \tfrac{1}{2})], \tag{2.7b}$$

where μ is defined by $\tau = e^{-\mu}$. Note that μ is positive since $1 > \tau > 0$. The normalized output state of the signal mode is given by $\tilde{W}_{\rm out}(m) = U(\mu, m)W_{\rm in}/{\rm Tr}_a[U(\mu, m)W_{\rm in}]$, where ${\rm Tr}_a$ is the trace operation over the Hilbert space of the signal mode. The relations (2.7) show that the output state of the signal mode of the beam splitter is described by su(1,1) Lie algebra under the condition that the output state of the reference mode is an m-photon state. It is interesting to note that the relationship between the two input modes and the two output modes is unitary and characterized by SU(2) symmetry while the reduced description of the signal input-output relation is nonunitary and characterized by SU(1,1) symmetry. The quantity $P_\mu(m) = {\rm Tr}_a[U(\mu, m)W_{\rm in}]$ represents the probability distribution of the photodetector registering m photons in the output state of the reference mode. This probability distribution is calculated as

$$P_\mu(m) = \frac{(e^\mu - 1)^m}{m!}\sum_{n=m}^{\infty}\frac{n!}{(n-m)!}e^{-\mu n}\langle n|W_{\rm in}|n\rangle, \tag{2.8}$$

which satisfies $\sum_{m=0}^{\infty} P_\mu(m) = {\rm Tr}W_{\rm in} = 1$.

It is interesting to remark that the output state (2.5) [or (2.7)] is equal to the state of photon system, after a photon counter registered m photons, in the continuous measurement of photon number described by the quantum Markov process [10–13]. The probability (2.8) is equivalent to the photon counting probability of the counter registering m photons. It should be noted here that the measurement by the photodetector in Fig. 1 obeys the projection postulate. However, the continuous measurement by the photon counter does not obey it. Later we will discuss further the similarity between these measurements.

When we do not refer to the result exhibited by the photodetector for the output of the reference mode, the output state of the signal mode becomes

$$\begin{aligned} \tilde{W}_{\rm out} &= \sum_{m=0}^{\infty} W_{\rm out}(m) \\ &= \exp[\mu(K_- - K_0 + \tfrac{1}{2})]W_{\rm in}, \end{aligned} \tag{2.9}$$

where ${\rm Tr}_a\tilde{W}_{\rm out} = {\rm Tr}_a W_{\rm in} = 1$ is satisfied. It is easily seen that this state is the same as that of the output from the thermal reservoir at a temperature of $T = 0$ [20]. Here the relations $\mu = 2\kappa t$ and $\tau = e^{-2\kappa t}$ are established, where κ is the damping constant of photons due to the reservoir and t is the transit time of the photon passing through the reservoir. Indeed, we have $\langle a^\dagger a\rangle_{\rm out} = e^{-\mu}\langle a^\dagger a\rangle_{\rm in}$. Thus, if we do not refer to the output state of the reference mode, the beam splitter with the vacuum input of the reference mode is equivalent to the thermal reservoir at $T = 0$.

III. PHOTON STATISTICS IN THE OUTPUT STATE

In this section, we consider the photon statistics in the output state of the signal mode for several input states, under the condition that the photodetector for the reference mode output shows an m-photon state. Using the density matrix $\tilde{W}_{\rm out}(m) = U(\mu, m)W_{\rm in}/{\rm Tr}_a[U(\mu, m)W_{\rm in}]$ of the signal mode output, we can calculate the average and second-order moment of the photon number and the Mandel Q factor [21],

$$\langle a^\dagger a\rangle_m = \frac{f_{m+1}}{f_m}, \tag{3.1a}$$

$$\langle (a^\dagger a)^2\rangle_m = \frac{f_{m+2}}{f_m} + \frac{f_{m+1}}{f_m}, \tag{3.1b}$$

$$Q_m = \frac{f_{m+2}}{f_{m+1}} - \frac{f_{m+1}}{f_m}, \tag{3.1c}$$

where we set $\langle\,\rangle_m = {\rm Tr}_a[\cdots \tilde{W}_{\rm out}(m)]$ and f_m is given by

$$f_m = \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} e^{-\mu n}\langle n|W_{\rm in}|n\rangle. \tag{3.2}$$

Thus, making use of (3.1) and (3.2), we obtain the following results.

(*a*) *The number eigenstate.* When the input state of the signal mode is the eigenstate of photon number, such that $W_{\rm in} = |n_0\rangle\langle n_0|$ with $n_0 > m$, we obtain

$$\langle a^\dagger a\rangle_m = n_0 - m, \tag{3.3a}$$
$$\langle (a^\dagger a)^2\rangle_m = (n_0 - m)^2, \tag{3.3b}$$
$$Q_m = -1, \tag{3.3c}$$
$$\langle a^\dagger a\rangle_m = \langle a^\dagger a\rangle_{m=0} - m. \tag{3.3d}$$

These are trivial results indicating that n_0 input photons are divided into m reference output photons and $n_0 - m$ signal output photons by the beam splitter. In particular, (3.3c) and (3.3d) show the sub-Poissonian photon statistics and the antibunching correlation of the photon numbers in the number eigenstate.

(*b*) *The binomial state.* The binomial state [22] is given by

$$|N; b\rangle = \sum_{n=0}^{N} P(N, n)^{1/2}|n\rangle, \tag{3.4a}$$

$$P(N, m) = \frac{N!}{n!(N-n)!}p^n(1-p)^{N-n}. \tag{3.4b}$$

Thus, for $W_{\rm in} = |N; b\rangle\langle b; N|$ $(N > m)$, we obtain the results

$$\langle a^\dagger a\rangle_m = (N-m)\frac{\tau p}{1-rp}, \tag{3.5a}$$

$$\langle (a^\dagger a)^2\rangle_m = (N-m)\frac{\tau p}{1-rp}\left[(N-m-1)\frac{\tau p}{1-rp} + 1\right], \tag{3.5b}$$

$$Q_m = -\frac{\tau p}{1-rp}, \tag{3.5c}$$

$$\langle a^\dagger a\rangle_m = \left(1 - \frac{m}{N}\right)\langle a^\dagger a\rangle_{m=0}. \tag{3.5d}$$

We find the antibunching correlation of photon numbers in the binomial state since the average photon number in the output state of the signal mode decreases if the photodetector for the output state of the reference mode registers the photons, $\langle a^\dagger a\rangle_m \le \langle a^\dagger a\rangle_{m=0}$. We also see the sub-Poissonian photon statistics from $Q_m < 0$.

(*c*) *The coherent state.* When the input state is the coherent state, such that $W_{\rm in} = |\alpha\rangle\langle\alpha|$ with $a|\alpha\rangle = \alpha|\alpha\rangle$ and $\bar{n}_0 = |\alpha|^2$, we obtain

$$\langle a^\dagger a\rangle_m = \tau\bar{n}_0, \tag{3.6a}$$
$$\langle (a^\dagger a)^2\rangle_m = (\tau\bar{n}_0)(\tau\bar{n}_0 + 1), \tag{3.6b}$$
$$Q_m = 0, \tag{3.6c}$$
$$\langle a^\dagger a\rangle_m = \langle a^\dagger a\rangle_{m=0}. \tag{3.6d}$$

Thus the property of the output state is independent of that how many photons the photodetector of the reference output registers. This is characteristic of the coherent state.

(*d*) *The thermal state.* When the input signal mode is in the thermal state, $W_{\rm in} = \sum_{n=0}^{\infty} p_n|n\rangle\langle n|$ with $p_n = \bar{n}_0^n/(1+\bar{n}_0)^{n+1}$, we get the results

$$\langle a^\dagger a\rangle_m = \frac{(m+1)\tau\bar{n}_0}{1+r\bar{n}_0}, \tag{3.7a}$$

$$\langle (a^\dagger a)^2\rangle_m = \frac{(m+1)\tau\bar{n}_0}{1+r\bar{n}_0}\left[\frac{(m+2)\tau\bar{n}_0}{1+r\bar{n}_0} + 1\right], \tag{3.7b}$$

$$Q_m = \frac{\tau\bar{n}_0}{1+r\bar{n}_0}, \tag{3.7c}$$

$$\langle a^\dagger a\rangle_m = (m+1)\langle a^\dagger a\rangle_{m=0}. \tag{3.7d}$$

Thus we find the bunching correlation of photon numbers in the thermal state since the average photon number in the output state increases if the photodetector registers the photons, $\langle a^\dagger a\rangle_m \ge \langle a^\dagger a\rangle_{m=0}$. We also have the super-Poissonian photon statistics ($Q_m > 0$).

(*e*) *The even and odd coherent states.* Finally, we consider even and odd coherent states $|\alpha_+\rangle$ and $|\alpha_-\rangle$ as input states of the signal mode,

$$|\alpha_\pm\rangle = \frac{1}{\sqrt{\mathcal{N}_\pm}}\left(|\alpha\rangle \pm |-\alpha\rangle\right), \tag{3.8a}$$

$$\mathcal{N}_\pm = 2\left(1 \pm e^{-2\bar{n}_0}\right), \tag{3.8b}$$

where $|\alpha\rangle$ is the usual coherent state and $\bar{n}_0 = |\alpha|^2$. These states are eigenstates of a^2, $a^2|\alpha_\pm\rangle = \alpha^2|\alpha_\pm\rangle$. The even and odd coherent states shows the sub-Poissonian photon statistics and quadrature squeezing under certain conditions [23]. Using (3.1), we obtain

$$\langle a^\dagger a\rangle_m = (\tau\bar{n}_0)\left[\tanh(\tau\bar{n}_0)\right]^{\sigma_m}, \tag{3.9a}$$

$$\langle (a^\dagger a)^2\rangle_m = (\tau\bar{n}_0)^2 + (\tau\bar{n}_0)\,[\tanh(\tau\bar{n}_0)]^{\sigma_m}\,, \tag{3.9b}$$

$$Q_m = \sigma_m \frac{2\tau\bar{n}_0}{\sinh(2\tau\bar{n}_0)} \tag{3.9c}$$

for even coherent state $W_{\rm in} = |\alpha_+\rangle\langle\alpha_+|$ and

$$\langle a^\dagger a\rangle_m = (\tau\bar{n}_0)\,[\tanh(\tau\bar{n}_0)]^{\sigma_{m+1}}\,, \tag{3.10a}$$

$$\langle (a^\dagger a)^2\rangle_m = (\tau\bar{n}_0)^2 + (\tau\bar{n}_0)\,[\tanh(\tau\bar{n}_0)]^{\sigma_{m+1}}\,, \tag{3.10b}$$

$$Q_m = \sigma_{m+1} \frac{2\tau\bar{n}_0}{\sinh(2\tau\bar{n}_0)} \tag{3.10c}$$

for odd coherent state $W_{\rm in} = |\alpha_-\rangle\langle\alpha_-|$. Here we set $\sigma_m = (-1)^m$. We also find the following relations:

$$\langle a^\dagger a\rangle^{\rm even}_{2m} = \langle a^\dagger a\rangle^{\rm odd}_{2n+1}, \tag{3.11a}$$

$$\langle (a^\dagger a)^2\rangle^{\rm even}_{2m} = \langle (a^\dagger a)^2\rangle^{\rm odd}_{2n+1}, \tag{3.11b}$$

$$Q^{\rm even}_{2m} = Q^{\rm odd}_{2n+1}, \tag{3.11c}$$

and

$$\langle a^\dagger a\rangle^{\rm even}_{2m+1} = \langle a^\dagger a\rangle^{\rm odd}_{2n}, \tag{3.12a}$$

$$\langle (a^\dagger a)^2\rangle^{\rm even}_{2m+1} = \langle (a^\dagger a)^2\rangle^{\rm odd}_{2n}, \tag{3.12b}$$

$$Q^{\rm even}_{2m+1} = Q^{\rm odd}_{2n}, \tag{3.12c}$$

where m and n are non-negative integers, and $\langle\,\rangle^{\rm even}_m$ and $\langle\,\rangle^{\rm odd}_m$ stand for average values calculated for the even and odd coherent state inputs. From (3.9) and (3.10), we find the following. For the even coherent state input, the photon statistics in the output state of the signal mode becomes super-Poissonian or sub-Poissonian, if the photodetector for the output of the reference mode exhibits a $(2m)$-photon state or a $(2m+1)$-photon state. On the other hand, for the odd coherent state input, a $(2m)$-photon state or a $(2m+1)$-photon state of the reference mode output leads the sub-Poissonian or super-Poissonian photon statistics in the output state of the signal mode.

IV. RELATION TO THE CONTINUOUS MEASUREMENT

Let us now discuss the similarity between the measurement shown in Fig. 1 and the continuous measurement of photon number described by the quantum Markov process [10–13]. We assume that the reflectance of the beam splitter is sufficiently low ($r \ll 1$) so that the probability of the photodetector for the output of the reference mode registering more than one photon is negligible. Under this assumption, we can approximate $U(\mu,m)$ as

$$U(\mu,0) = S(\delta t), \tag{4.1a}$$

$$U(\mu,1) = \mathcal{J}S(\delta t)\delta t, \tag{4.1b}$$

$$U(\mu,m>1) = 0, \tag{4.1c}$$

where we set $\mu = \lambda\delta t$ and use $\mu \approx r$, and the operators $S(t)$ and $\mathcal{J}$ are defined by

$$S(t) = \exp[-\lambda t(K_0 - \tfrac{1}{2})], \tag{4.2a}$$

$$\mathcal{J} = \lambda K_-. \tag{4.2b}$$

From the results given in Refs. [10–13], we find the following. The nonunitary operator $S(t)$ is equal to the time evolution operator in the continuous measurement of photon number described by the quantum Markov process [14]. This operator describes the time evolution of the photon system under the condition that no photon is registered by the counter during time t. The operator $\mathcal{J}$ describes the change in the state of the photon system when the counter registers one photon and $\mathrm{Tr}_a[\mathcal{J}W_{\rm in}]\delta t$ represents the probability of one photon being registered in $[0,\delta t)$. The parameter λ corresponds to the coupling between the counter and photons and λ^{-1} is a measure of the average time that elapses before the counter registers the presence of the photons. Here it is important to note that the continuous measurement of photon counting does not obey the projection postulate. However, the photodetector used in the measurement in Fig. 1 does obey it. In the continuous measurement of photon number, $P_0(t) = \mathrm{Tr}_a[S(t)W_{\rm in}]$ is the probability that the counter does not register photons in $[0,t)$ and $P_1(t) = \mathrm{Tr}_a[\mathcal{J}S(t)W_{\rm in}]\delta t$ is the probability that the counter registers the first photon in $[t, t+\delta t)$.

We further consider such a similarity. Suppose a series of n lossless beam splitters such that the output signal of the kth beam splitter is used as the input signal of the $(k+1)$th beam splitter. Here it should be noted that the result exhibited by the kth photodetector is taken into account in the input state of the $(k+1)$th beam splitter. The setup is shown in Fig. 2. Then we assume that the input state of the reference mode of each beam splitter is vacuum and that the output of the reference mode is observed by the photodetector subject to the projection postulate. If the kth photodetector exhibits the m_k-photon state, the non-normalized output state of the signal mode of the last (nth) beam splitter is given by

$$W_{\rm out}(m_n, m_{n-1}, \ldots, m_1) = U(\mu_n, m_n)U(\mu_{n-1}, m_{n-1})\cdots U(\mu_1, m_1)W_{\rm in}, \tag{4.3}$$

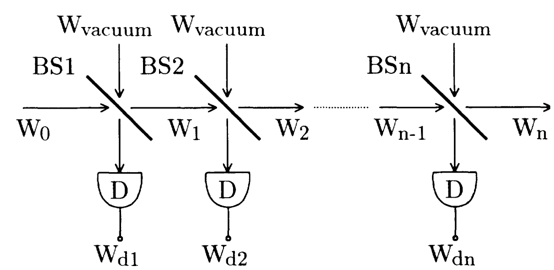

FIG. 2. A schematic representation of the measurement setup in terms of n lossless beam splitters and n photodetectors. BS_k is the kth beam splitter whose transmittance and reflectance are τ_k and r_k, and the D's are photodetectors with which the measurement is subject to the projection postulate. W_0 is the density matrix of the signal mode input, and the states of all reference inputs are assumed to be vacuum states, $W_{\rm vacuum} = |0\rangle\langle 0|$. W_k is the output state of the signal mode from the kth beam splitter under the condition that the kth photodetector exhibits the state W_{d_k} and W_k becomes the input state of the $(k+1)$th beam splitter.

where $U(\mu, m)$ is defined by (2.7b), $\tau_k = e^{-\mu_k}$ is the transmittance of the kth beam splitter, and $W_{\rm in}$ is the input state of the signal mode of the first beam splitter. The probability that each photodetector exhibits the m_k-photon state ($k = 1, \ldots, n$) is given by

$$P(m_n, m_{n-1}, \ldots, m_1) = {\rm Tr}_a W_{\rm out}(m_n, m_{n-1}, \ldots, m_1). \tag{4.4}$$

If none of the results exhibited by the n photodetectors are referred, the output state of the signal mode becomes

$$\tilde{W}_{\rm out} = \exp[\bar{\mu}(K_- - K_0 + \tfrac{1}{2})]W_{\rm in}, \tag{4.5}$$

where we have defined $\bar{\mu} = \sum_{k=1}^{n} \mu_k$. This is the same as the output state of the thermal reservoir with $T = 0$.

Now we assume that each of the n beam splitters has a sufficiently low reflectance ($r_k \approx \mu_k \ll 1$) and that the probability of the photodetector for the output state of the reference mode registering more than one photon is negligible. Then the probability that the given m photon detectors, for example, l_1th, l_2th, $\ldots$, l_mth detectors, exhibit one-photon state and the other $n - m$ photodetectors do not register a photon is obtained from $P(m_n, m_{n-1}, \ldots, m_1)$,

$$P(\overbrace{\underbrace{0,1,1,\ldots,0,1,0}_{(n-m)\text{ zeros}}}^{m \text{ ones}}) = P_e(t; t_{l_m}, t_{l_{m-1}}, \ldots, t_{l_1})\delta t_{l_m}\delta t_{l_{m-1}} \cdots \delta t_{l_1}, \tag{4.6a}$$

$$\begin{aligned} P_e(t; t_{l_m}, t_{l_{m-1}}, \ldots, t_{l_1}) &= {\rm Tr}_a[S(t - t_{l_m})\mathcal{J}S(t_{l_m} - t_{l_{m-1}})\mathcal{J} \cdots \mathcal{J}S(t_{l_1} - t_0)W_{\rm in}] \\ &= \lambda^m \exp[-\lambda \textstyle\sum_{j=1}^{m} t_{l_j}]{\rm Tr}_a \left\{\exp[-\lambda t a^\dagger a]a^m W_{\rm in}(a^\dagger)^m\right\}, \end{aligned} \tag{4.6b}$$

where we set $\mu_k = \lambda \delta t_k$ and $t_{l_k} = \sum_{j=1}^{l_k} \delta t_j$ with $t_n = t$ and $t_0 = 0$, and $S(t)$ and $\mathcal{J}$ are given by (4.2). It is seen that the quantity $P_e(t; t_m, t_{m-1}, \ldots, t_1)$ is equal to the elementary probability distribution in the continuous measurement of photon number [10,11], which is the probability that one photon is registered by the photon counter at each of the times $t_{l_m} > t_{l_{m-1}} > \cdots > t_{l_1}$ and no photon is registered in the rest of the measurement interval. The normalized output state of the signal mode is given by

$$\begin{aligned} \hat{W}_{\rm out}(t; t_{l_m}, t_{l_{m-1}}, \ldots, t_{l_1}) &= \frac{S(t - t_{l_m})\mathcal{J}S(t_{l_m} - t_{l_{m-1}})\mathcal{J} \cdots \mathcal{J}S(t_{l_1} - t_0)W_{\rm in}}{{\rm Tr}_a[S(t - t_{l_m})\mathcal{J}S(t_{l_m} - t_{l_{m-1}})\mathcal{J} \cdots \mathcal{J}S(t_{l_1} - t_0)W_{\rm in}]} \\ &= \frac{\exp[-\frac{1}{2}\lambda t a^\dagger a]a^m W_{\rm in}(a^\dagger)^m \exp[-\frac{1}{2}\lambda t a^\dagger a]}{{\rm Tr}_a\{\exp[-\lambda t a^\dagger a]a^m W_{\rm in}(a^\dagger)^m\}}, \end{aligned} \tag{4.7}$$

which is equal to that obtained in the continuous measurement of photon number. Therefore, it is found that the measurement in terms of a series of lossless beam splitters with very low reflectance and photodetectors subject to the projection postulate can lead to results equivalent to those obtained by the continuous measurement of photon number described by the quantum Markov process.

V. MEASUREMENTS WITH PARAMETRIC AMPLIFIERS AND FOUR-WAVE MIXERS

So far we have considered the quasicontinuous measurement of photon number with beam splitters. However, the discussion of the measurement with beam splitters can be applied to measurements by means of other optical devices. In this section, we consider quasicontinuous measurements with nondegenerate parametric amplifier and four-wave mixer. We first consider the measurement using the nondegenerate parametric amplifier and the photodetector shown in Fig. 3. The Hamiltonian of the nondegenerate parametric amplifier with the classical pump field is given by

$$H = \omega_A a^\dagger a + \omega_B b^\dagger a + i\mu(a^\dagger b^\dagger e^{-i\omega} - ab e^{i\omega t}), \tag{5.1}$$

where $(a, a^\dagger)$ and $(b, b^\dagger)$ are the signal and reference (idler) modes and $\mu e^{-i\omega t}$ stands for the classical pump field [24]. If we assume $\omega = \omega_A + \omega_B$, we obtain the time-translation generator of states in the interaction representation

$$H_{\rm int} = i\mu(a^\dagger b^\dagger - ab). \tag{5.2}$$

When it is assumed that the measurement performed by the photodetector is subject to the projection postulate, the output state $\hat{W}_{\rm out}$ of the nondegenerate parametric amplifier is given by

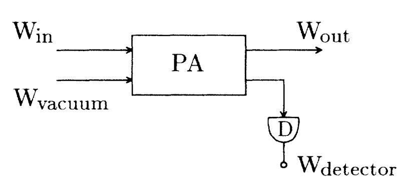

FIG. 3. A schematic representation of the measurement setup where PA is a nondegenerate parametric amplifier and D is a photodetector with which the measurement is subject to the projection postulate. $W_{\rm in}$ is the density matrix of the signal mode input and $W_{\rm vacuum}$ is the density matrix of the reference mode input assumed to be in a vacuum state. $W_{\rm detector}$ represents the output state of the reference mode exhibited by the photodetector. $W_{\rm out}$ indicates the output state of the signal mode under the condition that the photodetector exhibits the state $W_{\rm detector}$.

$$\hat{W}_{\rm out} = V(W_{\rm in} \otimes W_{\rm vacuum})V^{\dagger}, \tag{5.3a}$$
$$V = \exp[\theta(L_{+} - L_{-})], \tag{5.3b}$$

where $L_{\pm}$ and L_0 are the generators of su(1,1) Lie algebra,

$$L_{+} = a^{\dagger}b^{\dagger}, \tag{5.4a}$$
$$L_{-} = ab, \tag{5.4b}$$
$$L_0 = \tfrac{1}{2}(a^{\dagger}a + b^{\dagger}b + 1), \tag{5.4c}$$

and we set $\theta = \mu\tau$, where τ is the transit time of the photon passing through the parametric amplifier. Thus, if the photodetector for the output of the reference mode exhibits an m-photon state given by $W_{\rm detector} = |m\rangle\langle m|$, according to the projection postulate, we obtain the nonnormalized output state of the signal mode,

$$W_{\rm out}(m) = \hat{U}(s,m)W_{\rm in}, \tag{5.5a}$$
$$\hat{U}(s,m) = \frac{(1-e^{-s})^m}{m!}K_{+}^{m}\exp[-s(K_0 + \tfrac{1}{2})], \tag{5.5b}$$

where we set $e^{-s} = 1 - \tanh^2\theta$ $(s > 0)$, and $K_{\pm}$ and K_0 are given by (2.6). It should be noted that $W_{\rm out}(m)$ is equivalent to the state considered by Agarwal and Tara to investigate nonclassical character without squeezing or sub-Poissonian statistics [25]. When we do not refer to the state of the photodetector, the output state of the signal mode becomes

$$\tilde{W}_{\rm out} = \exp[s(K_{+} - K_0 - \tfrac{1}{2})]W_{\rm in}, \tag{5.6}$$

which is equivalent to the output state of linear amplifier. Indeed, we obtain the relation $\langle aa^{\dagger}\rangle_{\rm out} = G\langle aa^{\dagger}\rangle_{\rm in}$, where the gain constant is given by $G = e^{s}$.

Suppose the weak parametric coupling ($\theta \ll 1$) so that we can approximate $\hat{U}(s,m)$ as

$$\hat{U}(s,0) = \hat{S}(\delta t), \tag{5.7a}$$
$$\hat{U}(s,1) = \hat{\mathcal{J}}\hat{S}(\delta t)\delta t, \tag{5.7b}$$
$$\hat{U}(s,m>1) = 0, \tag{5.7c}$$

where $\hat{S}(t)$ and $\hat{\mathcal{J}}$ are defined by

$$\hat{S}(t) = \exp[-\lambda t(K_0 + \tfrac{1}{2})], \tag{5.8a}$$
$$\hat{\mathcal{J}} = \lambda K_{+}, \tag{5.8b}$$

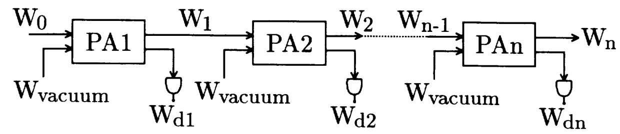

FIG. 4. A schematic representation of the measurement setup using n parametric amplifiers and n photodetectors. W_0 is the density matrix of the signal mode input, and the states of all reference inputs are assumed to be vacuum states, $W_{\rm vacuum} = |0\rangle\langle 0|$. W_k is the output state of the signal mode from the kth parametric amplifier under the condition that the kth photodetector exhibits the state W_{d_k} and W_k becomes the input state of the $(k+1)$th parametric amplifier.

and we set $s = \lambda\delta t$ ($\ll 1$). It is found from these expressions that the operator $\hat{\mathcal{J}}$ describes the change in the state of photons when a photon is registered by quantum counter [15,16]. The nonunitary operator $\hat{S}(t)$ describes the time evolution of the system without photons being registered by the quantum counter. The quantum counter characterized by $\hat{S}(t)$ and $\hat{\mathcal{J}}$ measures the photon number by making use of stimulated emission, while the usual photon counter characterized by $S(t)$ and $\mathcal{J}$ measures it by photon absorption. The quantum counter was proposed originally by Mandel to measure antinormally ordered correlation functions. Normally ordered correlation functions are measured by the usual photon counter.

When we consider a measurement in terms of a series of nondegenerate parametric amplifiers with weak parametric coupling constants and photodetectors shown in Fig. 4, using the same discussion as that derived (4.6) and (4.7), we can obtain the probability distribution $\hat{P}_e(t; t_{k_m}, t_{k_{m-1}}, \ldots, t_{k_1})$ that the k_1th, k_2th,…,k_mth photodetectors exhibit a one-photon state (the other $n-m$ photodetectors do not register a photon) and the output state $\hat{W}_{\rm out}(t; t_m, t_{m-1}, \ldots, t_1)$ of photons. Here we set $s_k = \theta_k^2 = \lambda\delta t_k$ and $t_{k_j} = \sum_{i=1}^{k_j}\delta t_i$ with $t_n = t$, where θ_k is the coupling constant of the kth parametric amplifier. Since $\hat{P}_e(t; t_{k_m}, t_{k_{m-1}}, \ldots, t_{k_1})$ and $\hat{W}_{\rm out}(t; t_{k_m}, t_{k_{m-1}}, \ldots, t_{k_1})$ are obtained by replacing $S(t)$ and $\mathcal{J}$ in (4.6) and (4.7) with $\hat{S}(t)$ and $\hat{\mathcal{J}}$, we have

$$\begin{aligned}\hat{P}_e(t; t_{k_m}, t_{k_{m-1}}, \ldots, t_{k_1}) &= {\rm Tr}_a[\hat{S}(t - t_{k_m})\hat{\mathcal{J}}\hat{S}(t_{k_m} - t_{k_{m-1}})\hat{\mathcal{J}}\cdots\hat{\mathcal{J}}S(t_{k_1})W_{\rm in}] \\ &= \lambda^m \exp\left[\lambda\sum_{j=1}^{m} t_{k_j}\right] {\rm Tr}_a\left\{\exp[-\lambda t a a^{\dagger}](a^{\dagger})^m W_{\rm in} a^m\right\}\end{aligned} \tag{5.9}$$

and

$$\begin{aligned}\hat{W}_{\rm out}(t; t_{k_m}, t_{k_{m-1}}, \ldots, t_{k_1}) &= \frac{\hat{S}(t - t_{k_m})\hat{\mathcal{J}}\hat{S}(t_{k_m} - t_{k_{m-1}})\hat{\mathcal{J}}\cdots\hat{\mathcal{J}}\hat{S}(t_{k_1})W_{\rm in}}{{\rm Tr}_a[\hat{S}(t - t_{k_m})\hat{\mathcal{J}}\hat{S}(t_{k_m} - t_{k_{m-1}})\hat{\mathcal{J}}\cdots\hat{\mathcal{J}}\hat{S}(t_{k_1})W_{\rm in}]} \\ &= \frac{\exp[-\frac{1}{2}\lambda t a a^{\dagger}](a^{\dagger})^m W_{\rm in} a^m \exp[-\frac{1}{2}\lambda t a a^{\dagger}]}{{\rm Tr}_a\{\exp[-\lambda t a a^{\dagger}](a^{\dagger})^m W_{\rm in} a^m\}}.\end{aligned} \tag{5.10}$$

It is seen that (5.9) and (5.10) are equivalent to the elementary probability distribution and the state of photons in the continuous measurement of photon number with quantum counter [16], where one photon is registered by the photon counter at each of the times $t_{k_m} > t_{k_{m-1}} > \cdots > t_{k_1}$ and no photon is registered in the rest of the measurement interval. Therefore, the measurement using parametric amplifiers and photodetectors subject to projection postulate yields the results equivalent to those obtained in the continuous measurement of photon number with quantum counter described by the quantum Markov process.

Next, we consider measurement with four-wave mixer, instead of beam splitter, and photodetector with which the measurement obeys the projection postulate. The setup is the same as that shown in Fig. 3, except for replacing parametric amplifier with four-wave mixer. We assume here that one mode of the four-wave mixer is highly excited and treated classically and that its Hamiltonian is given by

$$H = \omega_A a^\dagger a + \omega_B b^\dagger b + \lambda_0 a^\dagger a(b\epsilon^* e^{i\omega t} + b^\dagger \epsilon e^{-i\omega t}), \quad (5.11)$$

where $(a, a^\dagger)$ and $(b, b^\dagger)$ are the signal and reference modes and $\epsilon e^{-i\omega t}$ stands for the classical field [26]. If we assume $\omega_B = \omega$ and ϵ to be real, we obtain the time-translation generator of states in the interaction representation,

$$H_{\text{int}} = \lambda a^\dagger a(b + b^\dagger), \quad (5.12)$$

where we set $\lambda = \lambda_0 \epsilon$.

When the input state of the reference mode is a vacuum state, the output state $\hat{W}_{\text{out}}$ of the four-wave mixer is given by

$$\hat{W}_{\text{out}} = V(W_{\text{in}} \otimes W_{\text{vacuum}})V^\dagger, \quad (5.13a)$$

$$V = \exp[-i\tau\lambda a^\dagger a(b + b^\dagger)], \quad (5.13b)$$

where τ is the transit time of the photon passing through the four-wave mixer. When the photodetector for the output of the reference mode exhibits an m-photon state, the non-normalized output state of the signal mode becomes

$$\begin{aligned} W_{\text{out}}(m) &= \langle m|\hat{W}_{\text{out}}|m\rangle \\ &= \frac{1}{m!} g^m (a^\dagger a)^m \exp[-\tfrac{1}{2}g(a^\dagger a)^2] W_{\text{in}} \\ &\quad \times \exp[-\tfrac{1}{2}g(a^\dagger a)^2](a^\dagger a)^m, \end{aligned} \quad (5.14)$$

where we set $g = \tau\lambda^2$ and $|m\rangle$ is a number eigenstate of the reference mode. Thus it is found from the result given in Ref. [17] that this state $W_{\text{out}}(m)$ is identical to that obtained by the continuous quantum nondemolition measurement of photon number described by the quantum Markov process. When $\tilde{\lambda}$ represents the coupling between the photon and the detector and T is a measurement time in which m photons are counted, then we have the relation $g = \tilde{\lambda}T$. The probability $\tilde{P}(m)$ of the photodetector showing an m-photon state is given by

$$\tilde{P}(m) = \frac{1}{m!} g^m \text{Tr}_a \left\{\exp[-g(a^\dagger a)^2](a^\dagger a)^{2m} W_{\text{in}}\right\}, \quad (5.15)$$

which is equal to that obtained in the continuous measurement. Therefore, it is found that the measurement using the four-wave mixer and the photodetector gives the same results as those obtained in the continuous quantum nondemolition measurement described by the quantum Markov process.

VI. SUMMARY

We have investigated the photon statistics in the output state of the signal mode of a lossless beam splitter under the condition that the photodetector for the output state of the reference mode exhibits the m-photon state, and we have found the oscillatory behavior of the photon statistics for the even and odd coherent state inputs. We have shown that the measurements using beam splitters and photodetectors in the setups shown by Figs. 1 and 2 yield results equivalent to those obtained in the continuous measurement of photon number described by the quantum Markov process. We have also shown that the measurements with parametric amplifiers and photodetectors shown in Figs. 3 and 4 give the results equivalent to those obtained in the continuous measurement of photon number with quantum counter. Furthermore, we have found that the measurement using the four-wave mixer and the photodetector yields the equivalent results given by the continuous quantum nondemolition measurement of photon number described by the quantum Markov process. Here it should be noted that the measurements performed by the photodetectors in all the setups that we proposed obey the projection postulate, while the continuous measurements of the photon number do not obey it. Therefore, it is found from our results that the continuous measurements of photon number described by the quantum Markov process can be simulated by combining the first kind of measurement with appropriate optical devices.

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