PHYSICAL REVIEW A VOLUME 49, NUMBER 6 JUNE 1994

Phase uncertainties of a squeezed state

M. Freyberger and W. Schleich
Abteilung für Quantenphysik, Universität Ulm, D-89069 Ulm, Germany
(Received 20 September 1993)

We investigate various measures of phase uncertainty in their dependence on the average number of photons for the case of a phase squeezed state. We use the following three measures: (i) the geometrical uncertainty $(\delta\varphi_g)^2$, (ii) the dispersion $(\delta\varphi_d)^2$, and (iii) the rotational width $(\delta\varphi_r)^2$. Whereas $(\delta\varphi_g)^2$ and $(\delta\varphi_r)^2$ decay quadratically with the average number of photons $\langle n\rangle$, the dependence of $(\delta\varphi_d)^2$ is only of the form $\ln\langle n\rangle/\langle n\rangle^2$. We give physical pictures for these scaling laws.

PACS number(s): 42.50.Dv, 42.50.Lc

I. INTRODUCTION

Phase optimized states [1] have gained a prominent role in the context of nonclassical states of a single mode of the radiation field. In such a state the phase noise is minimal for a given average number of photons. Various measures of phase noise [2] and their corresponding phase optimized states have been investigated [3,4]. In these cases the phase noise decays proportional to the inverse of the square of the average number of photons [5]. A phase squeezed state is said to show the same behavior [6–9]. In the present paper we consider the phase distribution [10–16] of a phase squeezed state [17] and in particular focus on the following question: What is the phase uncertainty of a phase squeezed state and what is its dependence on the average number of photons?

The paper is organized as follows. In Sec. II we briefly review those properties of a squeezed state which we will use in the remainder of the paper. We define in Sec. III a simple geometrical phase uncertainty $(\delta\varphi_g)^2$ based on the Gaussian Wigner function of a phase squeezed state. When the squeezed Gaussian touches the origin of phase space, we find $(\delta\varphi_g)^2 \sim \langle n\rangle^{-2}$, that is, a decay inversely proportional to the square of the average number of photons. An approach using the Susskind-Glogower phase operator and the dispersion measure $(\delta\varphi_d)^2$ suggests, however, a slower decay as discussed in Sec. IV. We calculate $(\delta\varphi_d)^2$ with the help of a phase distribution in Sec. V and find for the optimal phase squeezed state the dependence $(\delta\varphi_d)^2 \sim \ln\langle n\rangle/\langle n\rangle^2$. This result has also been obtained in Ref. [18]. Here the sum over the photon number probability amplitudes c_m of the squeezed state defining $(\delta\varphi_d)^2$ has been evaluated by first approximating the coefficients c_m and then performing the summation. In contrast to this we obtain this result using two different methods: (i) We first express this sum in an exact way by an integral and then find its asymptotic behavior for a highly displaced and phase squeezed state and (ii) we calculate $(\delta\varphi_d)^2$ using the Wigner phase distribution of a phase squeezed state, that is, the Wigner function expressed in polar coordinates and integrated over the radial variable [19]. For the details of the calculation we refer to the Appendixes A and B. In Sec. VI we show that the logarithmic contribution in $(\delta\varphi_d)^2$ results from the exponential tail of the squeezed Gaussian. We emphasize that the dependence of the phase noise on $\langle n\rangle$ depends crucially on the choice of the measure for phase noise. To bring out this point most clearly we dedicate Sec. VII to a discussion of yet another measure of phase uncertainty [20]: the rotational width $(\delta\varphi_r)^2$. In contrast to $(\delta\varphi_d)^2$ this measure reaches the $\langle n\rangle^{-2}$ dependence for a squeezed vacuum. We summarize our results and conclude with Sec. VIII.

II. A PICO REVIEW OF SQUEEZED STATES

Before we investigate the phase noise of a squeezed state in its dependence on the mean number of photons we briefly review some properties of these states [21–24]. Throughout the paper we concentrate on a squeezed state $|\psi_{sq}\rangle$ defined via its position representation [23]

$$\langle x|\psi_{sq}\rangle = \psi_{sq}(x) = \left(\frac{s}{\pi}\right)^{\frac{1}{4}} \exp\left[-\frac{s}{2}(x-\sqrt{2}\alpha)^2\right] , \quad (1)$$

where the squeezing parameter s and the coherent amplitude α are both positive. We can also represent $|\psi_{sq}\rangle$ in terms of the energy eigenstates $|m\rangle$ of the harmonic oscillator via

$$|\psi_{sq}\rangle = \sum_{m=0}^{\infty} c_m|m\rangle , \quad (2)$$

where the expansion coefficients [24]

$$\begin{aligned} c_m &\equiv \langle m|\psi_{sq}\rangle \\ &= \sqrt{\frac{2\sqrt{s}}{1+s}} \exp\left(-\frac{s\alpha^2}{1+s}\right) \frac{1}{\sqrt{2^m m!}} \left(\frac{s-1}{s+1}\right)^{m/2} \\ &\quad \times H_m\left(\frac{\sqrt{2}s\alpha}{\sqrt{s^2-1}}\right) \end{aligned} \quad (3)$$

involve the Hermite polynomials H_m. We note that these expansion coefficients are always real [25]. They allow us to calculate the average number of photons

$$\langle n\rangle \equiv \sum_{n=0}^{\infty} n|c_n|^2 = \alpha^2 + \frac{(1-s)^2}{4s} , \quad (4)$$

1050-2947/94/49(6)/5056(11)/$06.00

as well as the variance

$$(\delta n)^2 \equiv \langle n^2 \rangle - \langle n \rangle^2 = \frac{\alpha^2}{s} + \frac{(s^2-1)^2}{8s^2}\,. \tag{5}$$

Both quantities depend on the squeezing parameter s and the coherent amplitude α.

We gain more insight into the physical meaning of these parameters by considering the Wigner function [26]

$$P_{sq}^{(W)}(x,p) \equiv \frac{1}{\pi}\int_{-\infty}^{\infty} dy e^{2ipy}\psi_{sq}^*(x+y)\psi_{sq}(x-y) = \frac{1}{\pi}\exp\left[-s(x-\sqrt{2}\alpha)^2 - \frac{p^2}{s}\right] \tag{6}$$

of the squeezed state, that is, the squeezed Gaussian shown in Fig. 1 and its phase space contour lines. In particular we concentrate on the contour line of $\exp(-1/2)$ shown in Fig. 2. This curve is an ellipse with its center on the x axis at $x_c = \sqrt{2}\alpha$. The minor and major axes, that is, the variances $(\delta x)^2$ and $(\delta p)^2$ of the observables x and p [27], read

$$(\delta x)^2 = \frac{1}{2s}\,, \quad (\delta p)^2 = \frac{s}{2}\,. \tag{7}$$

Hence the process of squeezing distributes the fluctuations between position x and momentum p as to maintain the minimum uncertainty product

$$(\delta x)^2(\delta p)^2 = \frac{1}{2s}\frac{s}{2} = \frac{1}{4}\,. \tag{8}$$

For $s = 1$ we find a symmetric distribution of the fluctuations, that is, $\delta x = \delta p$, which is the characteristic of a coherent state. For $s > 1$ the uncertainty in x is reduced at the expense of the uncertainty in p resulting in an *amplitude squeezed state*. The phase properties of such states have been discussed extensively in Ref. [28]. In the present paper we want to concentrate on the case of squeezing in the momentum at the expense of the fluctuations in x, that is, on the condition $0 < s < 1$. The corresponding state is then a *phase squeezed state*, as shown in Fig. 1.

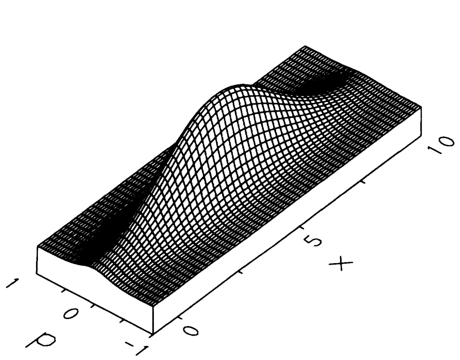

FIG. 1. Wigner function $P_{sq}^{(W)}(x,p)$ in arbitrary units for a phase squeezed state defined by the parameter pair $\alpha = \sqrt{8}$ and $s = 0.1$.

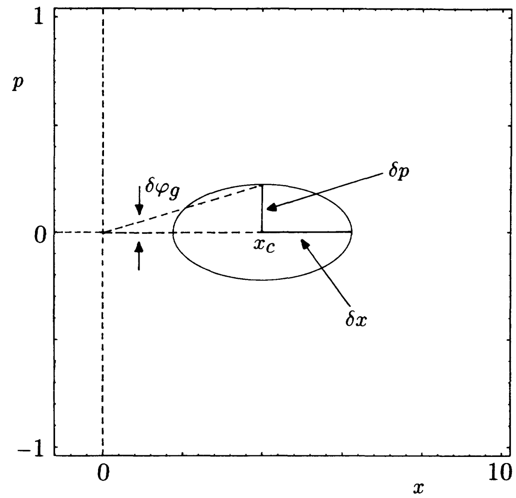

FIG. 2. Phase space contour line of $\exp(-1/2)$ for the Wigner function of Fig. 1, that is, for $\alpha = \sqrt{8}$ and $s = 0.1$. This contour line is an ellipse defined by its center at $x_c = \sqrt{2}\alpha$ and the two axes $\delta x = (2s)^{-1/2}$ and $\delta p = (s/2)^{1/2}$. The geometrical phase uncertainty $(\delta\varphi_g)^2$ is given by the opening angle $(\delta\varphi_g)^2 = (\delta p)^2/x_c^2 = s/(4\alpha^2)$.

III. GEOMETRICAL PHASE UNCERTAINTY

In the present section we use the contour line $\exp(-1/2)$ of the squeezed Gaussian, Eq. (6), to derive a geometrically motivated phase uncertainty for a phase squeezed state. In Fig. 2 we show this contour line for $\alpha = \sqrt{8}$ and $s = 0.1$. In a crude approximation we can estimate the phase uncertainty with the help of the triangle defined by the phase space origin and the minor axis of the ellipse. The angle $\delta\varphi_g$ at the origin suggests a geometrically motivated phase uncertainty

$$(\delta\varphi_g)^2 \equiv \frac{(\delta p)^2}{x_c^2} = \frac{s}{4\alpha^2} = \frac{s^2}{2\xi^2}\,, \tag{9}$$

where $\xi \equiv \sqrt{2s}\alpha$. Moreover this geometrical picture suggests that the best approximation of a phase state—a thin wedge aligned along a radial direction [28], as shown in Fig. 3—is a phase squeezed state with $x_c = \sqrt{2}\alpha = \delta x = (2s)^{-1/2}$, that is, $\xi^2 = 2s\alpha^2 = 1/2$. Such a parameter combination makes the contour line of the state in phase space touch the origin. In this case we find from Eq. (9)

$$(\delta\varphi_g)^2 = s^2\,. \tag{10}$$

It is now interesting to discuss the dependence of $(\delta\varphi_g)^2$ on the average number $\langle n \rangle$ of photons. In the limit of strong phase squeezing, that is, for $0 < s \ll 1$ we find from Eq. (4)

$$\langle n \rangle + \frac{1}{2} = \alpha^2 + \frac{1}{4s} + O(s)\,, \tag{11}$$

which for $\alpha^2 = (4s)^{-1}$ reduces to

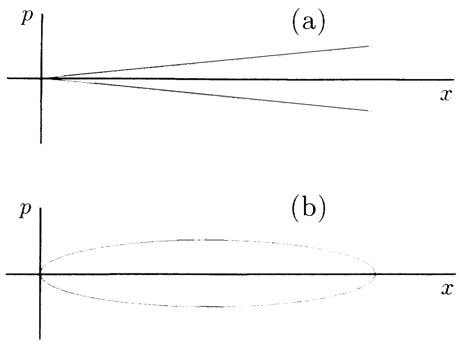

FIG. 3. (a) Elementary representation of a phase state $|\varphi = 0\rangle$ in phase space by a thin wedge aligned along the radial direction $\varphi = 0$. (b) Approximation of a phase state $|\varphi = 0\rangle$ by a highly phase squeezed state whose contour line $\exp(-1/2)$ touches the origin in phase space.

$$\langle n\rangle \approx \frac{1}{2s} . \tag{12}$$

Hence the geometrical phase uncertainty, Eq. (10), reads

$$(\delta\varphi_g)^2 \approx \frac{1}{4\langle n\rangle^2} \tag{13}$$

and decays quadratically with the average number of photons. Note that this behavior is in contrast to that of a coherent state where we find

$$(\delta\varphi_g)^2 \approx \frac{1}{4\langle n\rangle} , \tag{14}$$

that is, a decay linear in $\langle n\rangle$. Therefore a strongly phase squeezed state which touches the origin and thus approximates a phase state enjoys a phase sensitivity $(\delta\varphi_g)^2$ superior to that of a coherent state. However, this geometrical picture does not contain the full truth, as we will show now by various methods.

IV. DISPERSION PHASE UNCERTAINTY: A PHASE OPERATOR APPROACH

So far our estimation of the phase noise in a phase squeezed state has relied on an intuitive geometrical approach. But quantum mechanics forces us to refine this description of phase uncertainty. The analysis presented in this section is based on a concept of phase in quantum mechanics and the associated definition of a measure for phase noise. There are various notions of uncertainty for the phase which suggest themselves [1]. We base the arguments of this section on the periodic measure of the phase *dispersion* [29,30]

$$(\delta\varphi_d)^2 \equiv 1 - |\langle e^{i\varphi}\rangle|^2 . \tag{15}$$

The evaluation of the quantity $\langle e^{i\varphi}\rangle$ depends now on the concept of quantum mechanical phase. When we start from the phase operator [11,12]

$$\widehat{e^{i\varphi}} \equiv \sum_{n=0}^{\infty} |n\rangle\langle n+1| \tag{16}$$

with eigenstates [10–15]

$$|\varphi\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} e^{in\varphi}|n\rangle , \tag{17}$$

the expectation value $\langle\widehat{e^{i\varphi}}\rangle$ reads

$$\langle\psi|\widehat{e^{i\varphi}}|\psi\rangle = \int_{-\pi/2}^{3\pi/2} e^{i\varphi}|\langle\varphi|\psi\rangle|^2 d\varphi = \sum_{n=0}^{\infty} \psi_n^*\psi_{n+1} , \tag{18}$$

where the ψ_n denote the coefficients in the Fock representation

$$|\psi\rangle = \sum_{n=0}^{\infty} \psi_n|n\rangle \tag{19}$$

of a quantum state $|\psi\rangle$.

Based on the phase operator $\widehat{e^{i\varphi}}$, Eq. (16), we now derive an inequality for the dispersion measure $(\delta\varphi_d)^2$. Let us construct the two Hermitian operators [12]

$$\hat{S} = \frac{1}{2i}(\widehat{e^{i\varphi}} - \widehat{e^{i\varphi}}^{\dagger}) \tag{20}$$

and

$$\hat{C} = \frac{1}{2}(\widehat{e^{i\varphi}} + \widehat{e^{i\varphi}}^{\dagger}) \tag{21}$$

for the sine and the cosine of the phase. These operators do not commute with the photon number operator $\hat{n}$ but fulfill the Heisenberg-Robertson inequalities [12]

$$(\delta n)^2(\delta C)^2 \geq \frac{1}{4}\langle\hat{S}\rangle^2 \tag{22a}$$

and

$$(\delta n)^2(\delta S)^2 \geq \frac{1}{4}\langle\hat{C}\rangle^2 , \tag{22b}$$

where $(\delta S)^2 \equiv \langle\hat{S}^2\rangle - \langle\hat{S}\rangle^2$ and $(\delta C)^2 \equiv \langle\hat{C}^2\rangle - \langle\hat{C}\rangle^2$. These inequalities yield for the sum of the variances the inequality

$$\begin{aligned}\langle\hat{S}^2\rangle + \langle\hat{C}^2\rangle - (\langle\hat{S}\rangle^2 + \langle\hat{C}\rangle^2) &\equiv (\delta C)^2 + (\delta S)^2 \\ &\geq \frac{1}{4(\delta n)^2}(\langle\hat{S}\rangle^2 + \langle\hat{C}\rangle^2) .\end{aligned} \tag{23}$$

We introduce into this equation the dispersion, Eq. (15), in the form

$$\begin{aligned}(\delta\varphi_d)^2 &= 1 - \left|\langle\hat{C} + i\hat{S}\rangle\right|^2 \\ &= 1 - (\langle\hat{S}\rangle^2 + \langle\hat{C}\rangle^2)\end{aligned} \tag{24}$$

and arrive at

$$\langle\hat{S}^2\rangle + \langle\hat{C}^2\rangle + (\delta\varphi_d)^2 - 1 \geq \frac{1}{4(\delta n)^2}\left[1 - (\delta\varphi_d)^2\right] , \tag{25}$$

that is,

$$(\delta\varphi_d)^2 \geq 1 - [\langle \hat{S}^2\rangle + \langle \hat{C}^2\rangle]\left[1 + \frac{1}{4(\delta n)^2}\right]^{-1} . \quad (26)$$

This inequality simplifies further by noting the identity [12]

$$\langle \hat{S}^2\rangle + \langle \hat{C}^2\rangle = 1 - \frac{1}{2}|\langle 0|\psi\rangle|^2 , \quad (27)$$

where $|\psi_0|^2 = |\langle 0|\psi\rangle|^2$ is the overlap of the quantum state $|\psi\rangle$ with the vacuum. Hence we find

$$\begin{aligned}(\delta\varphi_d)^2 &\geq \left[\frac{1}{4(\delta n)^2} + \frac{|\psi_0|^2}{2}\right]\left[1 + \frac{1}{4(\delta n)^2}\right]^{-1} \\ &\geq \left[\frac{1}{4(\delta n)^2} + \frac{|\psi_0|^2}{2}\right]\left[1 - \frac{1}{4(\delta n)^2} - \frac{|\psi_0|^2}{2} + \frac{|\psi_0|^2}{2}\right] \\ &\geq (\delta\varphi_d)^2_{est} - \left[(\delta\varphi_d)^2_{est}\right]^2 . \end{aligned} \quad (28)$$

Here we have expanded the right-hand side of the inequality in the case of large number fluctuations $(\delta n)^2 \gg 1$ and have introduced the expression

$$(\delta\varphi_d)^2_{est} \equiv \frac{1}{4(\delta n)^2} + \frac{|\psi_0|^2}{2} . \quad (29)$$

The quantity $(\delta\varphi_d)^2_{est}$ consisting of the sum of the inverse photon number variance and the vacuum overlap defines an estimated lower limit for the phase dispersion measure. In the following we evaluate $(\delta\varphi_d)^2_{est}$ for two examples which will provide a deeper insight into the meaning of this lower limit. We test $(\delta\varphi_d)^2_{est}$ using as a first example the case of a coherent state. In the second example we show that $(\delta\varphi_d)^2_{est}$ of a squeezed state, whose contour line of $\exp(-1/2)$ touches the origin, does *not* obey the $\langle n\rangle^{-2}$ dependence.

The phase dispersion of a coherent state [3]

$$(\delta\varphi_d)^2 = \frac{1}{4\langle n\rangle} + O\left(\frac{1}{\langle n\rangle^2}\right) \quad (30)$$

obeys for large $\langle n\rangle$ the same scaling law as our geometrical uncertainty $(\delta\varphi_g)^2$. The application of the estimation Eq. (29) with $(\delta n)^2 = \alpha^2 = \langle n\rangle$ as predicted by Eq. (4) for $s = 1$ and $|\psi_0|^2 = e^{-\langle n\rangle}$ yields

$$(\delta\varphi_d)^2_{est} = \frac{1}{4\langle n\rangle} + \frac{1}{2}e^{-\langle n\rangle} . \quad (31)$$

We note that $(\delta\varphi_d)^2_{est}$ shows the same $\langle n\rangle$ dependence as $(\delta\varphi_d)^2$ when $\langle n\rangle \gg 1$.

Let us switch now to the phase squeezed state. We insert the fluctuations $(\delta n)^2$ given by Eq. (5) and the vacuum overlap $|\psi_0|^2$ from Eq. (3) into Eq. (29) and arrive at

$$(\delta\varphi_d)^2_{est} \approx \left[\frac{4\alpha^2}{s} + \frac{1}{2s^2}\right]^{-1} + \sqrt{s}\exp\left(-2s\alpha^2\right) , \quad (32)$$

where we have already assumed $s \ll 1$. In order to compare $(\delta\varphi_d)^2_{est}$ to the discussion of $(\delta\varphi_g)^2$ of Sec. III we now specialize to a squeezed state that touches the origin of phase space, that is, for $\xi^2 = 2s\alpha^2 = 1/2$. In this case $(\delta\varphi_d)^2_{est}$ reads

$$(\delta\varphi_d)^2_{est} \approx \frac{2s^2}{3} + \sqrt{\frac{s}{e}} \approx \sqrt{\frac{s}{e}} \approx \frac{1}{\sqrt{2e}}\frac{1}{\sqrt{\langle n\rangle}} , \quad (33)$$

where we have used Eq. (12) for the average number of photons $\langle n\rangle$. We note that the scaling of $(\delta\varphi_d)^2_{est}$ is worse than that of a coherent state and it is governed by the large vacuum contribution $|\psi_0|^2$ which spoils the $\langle n\rangle^{-2}$ scaling of the geometrical phase uncertainty. The expression for $(\delta\varphi_d)^2_{est}$, Eq. (29), clearly indicates that it is not enough to enhance the number fluctuations $(\delta n)^2$ in the phase squeezed state approximating a phase state, that is, to make the squeezed Gaussian [Fig. 3(b)] longer and longer while it still touches the origin. Rather we have to control at least two contributions: the number fluctuations *and* the overlap with the vacuum. Hence the tails of the squeezed Gaussian have to avoid the origin. We have to minimize the vacuum overlap and at the same time maximize the number fluctuations. This implies that the displacement has to be larger than the major axis of the ellipse, that is, $\sqrt{2}\alpha > (2s)^{-1/2}$ or $\xi^2 = 2s\alpha^2 > \frac{1}{2}$. Therefore a meaningful control parameter $\xi \equiv \sqrt{2s}\alpha$ already suggests itself via these considerations, because ξ cannot be a constant any longer as in our first crude approximation. We have to increase ξ while we squeeze the state. Therefore we have to investigate a phase squeezed state with large displacement in order to obtain the optimal $\langle n\rangle$ dependence.

V. DISPERSION PHASE UNCERTAINTY: A PHASE DISTRIBUTION APPROACH

The heuristic arguments of Secs. III and IV have suggested different dependencies of the phase uncertainty of a phase squeezed state on the average number of photons. In the present section we calculate the dispersion

$$\begin{aligned}(\delta\varphi_d)^2 &= 1 - |\langle\widehat{e^{i\varphi}}\rangle|^2 = 1 - \left|\int_{-\pi/2}^{3\pi/2} d\varphi e^{i\varphi}W(\varphi)\right|^2 \\ &= 1 - \left|\sum_{m=0}^{\infty} c^*_m c_{m+1}\right|^2 \end{aligned} \quad (34)$$

for a phase squeezed state $|\psi_{sq}\rangle$ based on the phase distribution [31]

$$W(\varphi) \equiv |\langle\varphi|\psi_{sq}\rangle|^2 = \frac{1}{2\pi}\left|\sum_{m=0}^{\infty} c_m e^{-im\varphi}\right|^2 . \qquad (35)$$

The coefficients c_m, Eq. (3), are the photon number probability amplitudes of the squeezed state.

In Appendix A we express the sum $\langle\widehat{e^{i\varphi}}\rangle = \sum_m c_m^* c_{m+1}$ in terms of an integral. We evaluate this integral in the limit of a phase squeezed state of large displacement, that is, for $\xi \equiv \sqrt{2s}\alpha \gg 1$ and $0 < s \ll 1$ and find for the dispersion

$$(\delta\varphi_d)^2 \approx \frac{s^2}{2\xi^2} + \frac{2}{\sqrt{\pi}\xi}e^{-\xi^2} , \qquad (36)$$

a result that combines two contributions: We recognize in the first term the geometrically estimated phase uncertainty $(\delta\varphi_g)^2$ of Sec. III. Moreover, we find a second term, which, at first sight, seems to be exponentially small. Nevertheless, we will show now that it is this term which creates a $\langle n\rangle$ dependence of the dispersion $(\delta\varphi_d)^2$ different from the $\langle n\rangle^{-2}$ dependence of the geometrical uncertainty $(\delta\varphi_g)^2$.

To this aim we calculate the optimal $\langle n\rangle$ dependence of $(\delta\varphi_d)^2$. This amounts (as we have already seen in Sec. IV) to finding the optimal values of $\xi \gg 1$ and $0 < s \ll 1$ which minimize $(\delta\varphi_d)^2$ under the constraint of a fixed average photon number. When we express $\langle n\rangle$ from Eq. (4) in terms of ξ, that is,

$$\langle n\rangle = \frac{\xi^2}{2s} + \frac{(1-s)^2}{4s} , \qquad (37)$$

which in the limit $\xi \gg 1$ and $0 < s \ll 1$ implies $s \approx \frac{\xi^2}{2\langle n\rangle}$ we find from Eq. (36) the phase dispersion

$$(\delta\varphi_d)^2 \approx \frac{\xi^2}{8\langle n\rangle^2} + \frac{2}{\sqrt{\pi}}\frac{e^{-\xi^2}}{\sqrt{\xi^2}} \qquad (38)$$

as a function of ξ^2. The optimal value ξ_0^2 obeys the equation

$$0 = \frac{d(\delta\varphi_d)^2}{d\xi^2}\Big|_{\xi^2=\xi_0^2} = \frac{1}{8\langle n\rangle^2} - \frac{2e^{-\xi_0^2}}{\sqrt{\pi\xi_0^2}}\left[1 + \frac{1}{2\xi_0^2}\right]$$
$$\approx \frac{1}{8\langle n\rangle^2} - \frac{2e^{-\xi_0^2}}{\sqrt{\pi\xi_0^2}} , \qquad (39)$$

where in the last step we have neglected the term $1/\xi_0^2 \ll 1$. This relation allows us to express the second contribution to $(\delta\varphi_d)^2$ in Eq. (38) in terms of $\langle n\rangle$ and we find

$$(\delta\varphi_d)^2 \approx \frac{1}{8\langle n\rangle^2}(\xi_0^2 + 1) , \qquad (40)$$

which for $\xi_0^2 \gg 1$ reduces to

$$(\delta\varphi_d)^2 \approx \frac{\xi_0^2}{8\langle n\rangle^2} . \qquad (41)$$

In a crude approximation we can solve Eq. (39) for ξ_0^2 by expressing it via

$$\frac{1}{8\langle n\rangle^2} = \frac{2}{\sqrt{\pi}}\exp[-(\xi_0^2 + \tfrac{1}{2}\ln\xi_0^2)] \qquad (42)$$

and neglecting the contribution $\frac{1}{2}\ln\xi_0^2$ compared to ξ_0^2 which yields

$$\xi_0^2 \approx \ln\left(\frac{16\langle n\rangle^2}{\sqrt{\pi}}\right) . \qquad (43)$$

Hence we arrive at [18]

$$(\delta\varphi_d)^2 \approx \frac{\ln\langle n\rangle}{4\langle n\rangle^2} . \qquad (44)$$

Note that this logarithmic dependence is a consequence of the exponential term in $(\delta\varphi_d)^2$. The phase sensitivity of a phase squeezed state of large displacement described via the dispersion shows an improvement compared to the sensitivity of a coherent state, Eq. (30). Nevertheless it does not reach the $\langle n\rangle^{-2}$ scaling law of the geometrical phase uncertainty discussed in Sec. III. Indeed it was shown in Ref. [3] that the $\langle n\rangle^{-2}$ dependence for the dispersion measure can only be obtained by a *phase optimized state*, which is different from a phase squeezed state. However, this does not exclude the $\langle n\rangle^{-2}$ scaling for a squeezed state when we choose a *different measure* for the phase uncertainty. We investigate such a measure in Sec. VII.

We conclude this section by noting that in the limit of $\xi \gg 1$ and $0 < s \ll 1$ the average $\langle e^{i\varphi}\rangle_W$ and with it the dispersion

$$(\delta\varphi_d)^2_W = 1 - |\langle e^{i\varphi}\rangle_W|^2 \qquad (45)$$

calculated via the Wigner function give identical results [32]. To show this we reduce in Appendix B the two-dimensional phase space integration to a one-dimensional integration. The remaining integral enjoys the same asymptotic behavior as the expectation value $\langle\widehat{e^{i\varphi}}\rangle$ obtained in Appendix A. We also give an argument for this asymptotic agreement using the Wigner representation of $\widehat{e^{i\varphi}}$.

VI. DISPERSION PHASE UNCERTAINTY: THE PHYSICS

In the preceding section we have found that the dispersion $(\delta\varphi_d)^2$ consists of two dominant contributions. To gain insight into the physical origin of these two parts, we show in Fig. 4 the phase distribution $W(\varphi)$ for the squeezed state of Fig. 1. We note the dominant peak at the phase angle $\varphi = 0$ in agreement with the phase space picture of Sec. III. However, we also note a small maximum at $\varphi = \pi$ which results from the exponentially decaying tail of the squeezed Gaussian which reaches into the left half of phase space, as apparent in Fig. 1. Recall that the geometrical uncertainty based on the contour line $\exp(-1/2)$ ignores this leakage effect. It is this sec-

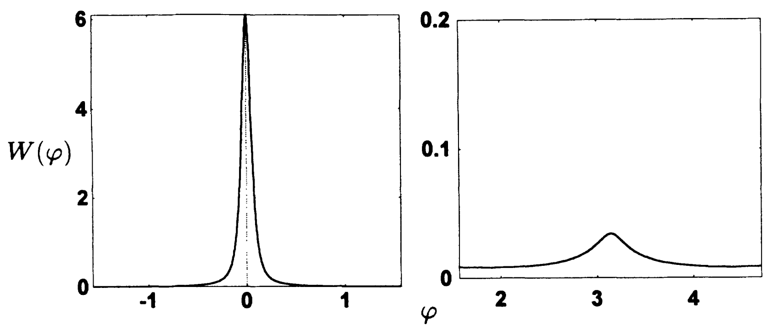

FIG. 4. Phase distribution $W(\varphi)$ of the squeezed state defined by the parameters $\alpha = \sqrt{8}$ and $s = 0.1$. The left figure depicts the region $-\pi/2 \leq \varphi \leq \pi/2$ with the dominant peak at $\varphi = 0$, whereas in the right figure we show the domain $\pi/2 \leq \varphi \leq 3\pi/2$. In order to bring out the much smaller peak at $\varphi = \pi$ we have used two different scales for the vertical axis in the two pictures.

ond peak—and not only the vacuum contribution as in our estimation of Sec. IV—which spoils the $\langle n\rangle^{-2}$ dependence of the phase uncertainty as we shall discuss now.

To demonstrate the influence of the two peaks at $\varphi = 0$ and $\varphi = \pi$ we rewrite the dispersion, Eq. (34), as

$$(\delta\varphi_d)^2 = 1 - \left(\int_{-\pi/2}^{3\pi/2} \cos\varphi W(\varphi) d\varphi\right)^2 , \tag{46}$$

where we have used the symmetry $W(\varphi) = W(-\varphi)$. The trigonometric relation $\cos\varphi = 1 - 2\sin^2(\frac{\varphi}{2})$ and the normalization of W yield

$$(\delta\varphi_d)^2 = 4\left[\left\langle \sin^2\left(\frac{\varphi}{2}\right)\right\rangle - \left\langle \sin^2\left(\frac{\varphi}{2}\right)\right\rangle^2\right] , \tag{47}$$

where

$$\left\langle \sin^2\left(\frac{\varphi}{2}\right)\right\rangle \equiv \int_{-\pi/2}^{3\pi/2} \sin^2\left(\frac{\varphi}{2}\right) W(\varphi) d\varphi . \tag{48}$$

We analyze now the average $\langle \sin^2(\frac{\varphi}{2})\rangle$ and therefore decompose the integral into two integrals with the regions $-\pi/2 \leq \varphi \leq \pi/2$ and $\pi/2 \leq \varphi \leq 3\pi/2$. We apply the trigonometric identity $\sin^2(\frac{\varphi}{2}) = 1 - \cos^2(\frac{\varphi}{2})$ to the second integral and arrive at

$$\begin{aligned}\left\langle \sin^2\left(\frac{\varphi}{2}\right)\right\rangle &= \int_{-\pi/2}^{\pi/2} \sin^2\left(\frac{\varphi}{2}\right) W(\varphi) d\varphi + \int_{\pi/2}^{3\pi/2} W(\varphi) d\varphi - \int_{\pi/2}^{3\pi/2} \cos^2\left(\frac{\varphi}{2}\right) W(\varphi) d\varphi \\ &= \left\langle \sin^2\left(\frac{\varphi}{2}\right)\right\rangle_0 + A_\pi - \int_{\pi/2}^{3\pi/2} \cos^2\left(\frac{\varphi}{2}\right) W(\varphi) d\varphi .\end{aligned} \tag{49}$$

This expression shows that three quantities contribute to the average $\langle \sin^2(\frac{\varphi}{2})\rangle$ and via Eq. (47) to the dispersion: (i) the width $\langle \sin^2(\frac{\varphi}{2})\rangle_0$ of the peak of the phase distribution $W(\varphi)$ at $\varphi = 0$, (ii) the probability A_π caught underneath the peak at $\varphi = \pi$, and (iii) the peak at $\varphi = \pi$ weighted by $\cos^2(\frac{\varphi}{2})$. Note that due to this weight factor—the $\cos^2(\frac{\varphi}{2})$ is small in the neighborhood of the peak at $\varphi = \pi$—the third term is much smaller than the second one and we can neglect it. Hence we have identified the two main contributions to the average

$$\left\langle \sin^2\left(\frac{\varphi}{2}\right)\right\rangle \approx \left\langle \sin^2\left(\frac{\varphi}{2}\right)\right\rangle_0 + A_\pi . \tag{50}$$

Since the dominant peak at $\varphi = 0$ is very narrow we approximate $\langle \sin^2(\frac{\varphi}{2})\rangle_0$ by linearizing the square of the sine function and by approximating the remaining second moment by the geometrical uncertainty $(\delta\varphi_g)^2$. Indeed we find

$$\left\langle \sin^2\left(\frac{\varphi}{2}\right)\right\rangle_0 \approx \frac{1}{4}\langle\varphi^2\rangle \approx \frac{1}{4}(\delta\varphi_g)^2 = \frac{s}{16\alpha^2} . \tag{51}$$

We estimate the area A_π using as the phase distribution

$$W^{(W)}(\varphi) = \int_0^\infty P_{sq}^{(W)}(x = r\cos\varphi, p = r\sin\varphi) r\, dr , \tag{52}$$

that is, the Wigner function, Eq. (6), integrated over the radius rather than the distribution $W(\varphi)$, Eq. (35). Hence we find with the help of $W^{(W)}$

$$\begin{aligned}A_\pi &\approx \int_{\pi/2}^{3\pi/2} W^{(W)}(\varphi) d\varphi \\ &= \int_{\pi/2}^{3\pi/2} d\varphi \int_0^\infty P_{sq}^{(W)}(x = r\cos\varphi, p = r\sin\varphi) r\, dr \\ &= \int_{-\infty}^0 dx \int_{-\infty}^\infty dp P_{sq}^{(W)}(x,p) ,\end{aligned} \tag{53}$$

that is,

$$A_\pi \approx \int_{-\infty}^0 dx |\psi_{sq}(x)|^2 . \tag{54}$$

In the last step we have used the fact that the integration of the Wigner function over momentum provides [26] the probability distribution for the position. This expression for A_π reflects the fact that the squeezed Gaussian extends over the negative x values creating the maximum at $\varphi = \pi$ as already mentioned in the beginning of this section and in Fig. 1. We can express the remaining integration

$$A_\pi \approx \frac{1}{\sqrt{\pi}} \int_0^\infty dx e^{-x^2} - \frac{1}{\sqrt{\pi}} \int_0^\xi dx e^{-x^2}$$
$$\approx \frac{1}{2}\left[1 - \Phi(\xi)\right] \quad (55)$$

in terms of the error function $\Phi(\xi)$ [33], which in the limit of $\xi \equiv \sqrt{2s}\alpha \gg 1$ reduces to

$$A_\pi \approx \frac{1}{2\sqrt{\pi}\xi} e^{-\xi^2} . \quad (56)$$

Hence we find for a displaced phase squeezed state, that is, for $0 < s \ll 1$ and $\xi \gg 1$,

$$\left\langle \sin^2\left(\frac{\varphi}{2}\right)\right\rangle \approx \frac{s^2}{8\xi^2} + \frac{1}{2\sqrt{\pi}\xi} e^{-\xi^2} \ll 1 . \quad (57)$$

Consequently we can neglect the quadratic contribution $\langle \sin^2(\frac{\varphi}{2})\rangle^2$ in the dispersion, Eq. (47), and arrive at

$$(\delta\varphi_d)^2 \approx \frac{s^2}{2\xi^2} + \frac{2}{\sqrt{\pi}\xi} e^{-\xi^2} , \quad (58)$$

which is exactly Eq. (36), the result of the rigorous calculation. We conclude by emphasizing that the dispersion results from two contributions: The first term is the geometrical estimation of Sec. III and the second is the area underneath the peak at $\varphi = \pi$.

VII. THE ROTATIONAL WIDTH

Sections V and VI have shown that it is the global structure of the phase distribution $W(\varphi)$ which determines the phase dispersion: The extremely narrow peak of $W(\varphi)$ at $\varphi = 0$ does not contain the full truth; it is the peak at $\varphi = \pi$ that creates the correct $\ln\langle n\rangle/\langle n\rangle^2$ dependence of $(\delta\varphi_d)^2$. On the other hand, the geometrical phase uncertainty $(\delta\varphi_g)^2$ of Sec. III determines correctly the width of the peak at $\varphi = 0$ and predicts a $\langle n\rangle^{-2}$ scaling law. In this section we discuss a measure for phase uncertainty introduced in Ref. [34], which is not governed by the global structure of the phase distribution but by its *fine structure*. We show that this measure enjoys again a $\langle n\rangle^{-2}$ scaling law. First we present the ideas underlying the definition of this so-called *rotational width* $(\delta\varphi_r)^2$.

A quantum state $|\psi\rangle$ possesses the rotational width $\delta\varphi_r$ when the overlap with its rotated counterpart

$$|\tilde{\psi}\rangle \equiv \exp[-i\hat{n}\delta\varphi_r]|\psi\rangle \quad (59)$$

is a constant smaller than unity. Hence the rotational width is defined implicitely via the relation

$$|\langle\psi|\tilde{\psi}\rangle|^2 = |\langle\psi|\exp[-i\hat{n}\delta\varphi_r]|\psi\rangle|^2 = \beta^2 , \quad (60)$$

where $\beta^2 < 1$ is a constant. We gain insight into this definition when we represent [26] the scalar product

$$|\langle\psi|\tilde{\psi}\rangle|^2 = 2\pi \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp P_{\tilde{\psi}}^{(W)}(x,p) P_{\psi}^{(W)}(x,p) \quad (61)$$

using the Wigner functions $P_{\psi}^{(W)}(x,p)$ and $P_{\tilde{\psi}}^{(W)}(x,p)$ of the state $|\psi\rangle$ and the rotated state $|\tilde{\psi}\rangle$, respectively. The unitary transformation of $|\psi\rangle$ into $|\tilde{\psi}\rangle$ is equivalent to a rotation of the coordinate system in Wigner phase space [35,36] and hence

$$P_{\tilde{\psi}}^{(W)}(x,p) = P_{\psi}^{(W)}(x\cos\delta\varphi_r + p\sin\delta\varphi_r, p\cos\delta\varphi_r - x\sin\delta\varphi_r) . \quad (62)$$

With the help of Eqs. (61) and (62) the implicit definition for $\delta\varphi_r$ reads

$$\beta^2 = 2\pi \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp P_{\psi}^{(W)}(x\cos\delta\varphi_r + p\sin\delta\varphi_r, p\cos\delta\varphi_r - x\sin\delta\varphi_r) P_{\psi}^{(W)}(x,p) . \quad (63)$$

This equation brings out most clearly that it is the partial overlap in phase space between the rotated and the unrotated quantum state which defines $\delta\varphi_r$. The degree of overlap is given by β^2, which is in principle determined by the resolution of an ideal measuring apparatus. For $\beta^2 = 1$, Eq. (63) is just the normalization condition which yields $\delta\varphi_r = 0$. We interpret this definition in phase space with the help of a phase squeezed vacuum state as shown in Fig. 5. The width $\delta\varphi_r$ reflects the angle by which one state is rotated against the other until the overlap, given by the shaded area in Fig. 5(a), reaches $\beta^2 < 1$. When we consider the phase distribution $W(\varphi)$ of a squeezed vacuum, $\delta\varphi_r$ measures the shift of the distribution against its rotated twin, until we can distinguish the different peaks, as shown in Fig. 5(b). This

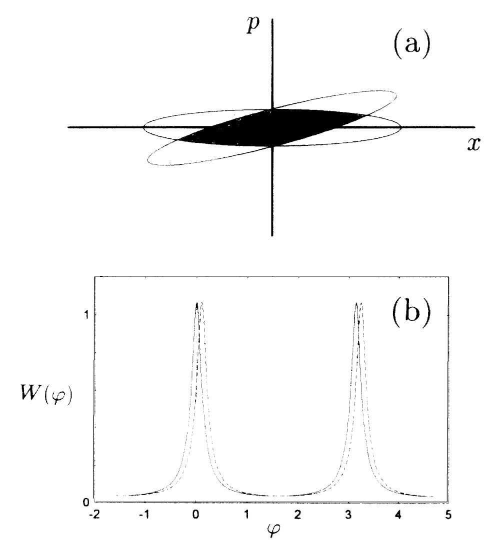

FIG. 5. Illustration of the rotational width $\delta\varphi_r$ for the example of a squeezed vacuum state. (a) Phase space representation of a squeezed vacuum state in combination with its rotated twin. The overlap of the two states given by the shaded area determines $\delta\varphi_r$. (b) The two-peaked phase distribution $W(\varphi)$ of a squeezed vacuum for $s = 0.1$. The solid line corresponds to the unrotated state with peaks located at $\varphi = 0$ and $\varphi = \pi$, whereas the dashed line corresponds to the state rotated by $\delta\varphi_r$. This phase distribution shows peaks located at $\varphi = \delta\varphi_r$ and $\varphi = \pi + \delta\varphi_r$.

picture stresses the fact that $\delta\varphi_r$ is a measure of the *fine structure* of the phase distribution.

We now calculate $\delta\varphi_r$ explicitly for a squeezed state and evaluate the phase space integral

$$\beta^2 = 2\pi \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp P_{sq}^{(W)}(x\cos\delta\varphi_r + p\sin\delta\varphi_r, p\cos\delta\varphi_r - x\sin\delta\varphi_r) P_{sq}^{(W)}(x,p)\,. \tag{64}$$

We concentrate on the case of a squeezed vacuum since in this case we can solve the implicit Eq. (64) for $\delta\varphi_r$ analytically [37]. When we substitute in Eq. (64) the Wigner function $P_{sq}^{(W)}$, Eq. (6), with the displacement $\alpha = 0$, the two-dimensional Gauss integral yields

$$\beta^2 = 2s[4s^2 + (1-s^2)^2 \sin^2(\delta\varphi_r)]^{-\frac{1}{2}}\,, \tag{65}$$

which we solve for $\sin^2(\delta\varphi_r)$ and find

$$\sin^2(\delta\varphi_r) = \left(\frac{1-\beta^4}{\beta^4}\right)\frac{4s^2}{(1-s^2)^2} \approx \left(\frac{1-\beta^4}{\beta^4}\right) 4s^2\,. \tag{66}$$

In the last step we have assumed a highly phase squeezed vacuum, $0 < s \ll 1$. When we linearize $\sin^2(\delta\varphi_r)$ for $s \ll 1$ and rewrite it in terms of the average photon number

$$\langle n\rangle \approx \frac{1}{4s} \tag{67}$$

we arrive at

$$(\delta\varphi_r)^2 \approx \left(\frac{1-\beta^4}{\beta^4}\right)\frac{1}{4\langle n\rangle^2}\,. \tag{68}$$

For this measure we have found the $\langle n\rangle^{-2}$ scaling law for a highly squeezed vacuum state whose phase distribution exhibits two very narrow but equally high peaks at $\varphi = 0$ and $\varphi = \pi$, as shown in Fig. 5. It is not important—as it was in the case of the dispersion measure—to reduce the influence of the peak at $\varphi = \pi$ by displacing the state. This is due to the fact that the rotational width $(\delta\varphi_r)^2$ measures the width of each single peak, that is, the fine structure of $W(\varphi)$. This contrasts the behavior of the dispersion, discussed on Secs. V and VI, which shows an influence of the global structure of $W(\varphi)$.

VIII. CONCLUSIONS

What is the dependence of the phase uncertainty of a phase squeezed state on the average number of photons? This is the central question addressed in this paper. The answer crucially depends on the choice of measure for phase noise: the geometrical phase uncertainty $(\delta\varphi_g)^2$ enjoys for an appropriately aligned phase squeezed state a $\langle n\rangle^{-2}$ dependence. In contrast the dispersion measure $(\delta\varphi_d)^2$ feels the exponential tail of the squeezed Gaussian which gives rise to a logarithmic correction term to $(\delta\varphi_g)^2$. Hence the dispersion measure obeys $(\delta\varphi_d)^2 \approx \ln\langle n\rangle/(4\langle n\rangle^2)$. We regain the $\langle n\rangle^{-2}$ dependence when we consider the rotational width $(\delta\varphi_r)^2$ of a squeezed vacuum. This is quite a remarkable result when we recall that we had to optimize the phase squeezed state to obtain the above mentioned result for $(\delta\varphi_d)^2$. For the rotational width the squeezed vacuum already yields a faster decay. This is due to the fact that $(\delta\varphi_r)^2$ emphasizes the fine structure, that is, the peak structure of the phase distribution rather than its overall behavior as in the case of $(\delta\varphi_d)^2$. This example stresses once more the important role of the appropriate choice of a phase noise measure. But this choice is determined by the experimental setup.

ACKNOWLEDGMENTS

We thank A. Bandilla, M. Collett, B.-G. Englert, and H. Paul for many fruitful discussions.

APPENDIX A: AVERAGE $\langle\widehat{e^{i\varphi}}\rangle$ USING THE LONDON PHASE DISTRIBUTION

In this appendix we express the average

$$\langle\widehat{e^{i\varphi}}\rangle = \sum_{m=0}^{\infty} c_m c_{m+1}\,, \tag{A1}$$

defined by the discrete summation over the photon number probability amplitudes [24]

$$c_m = \sqrt{\frac{2\sqrt{s}}{1+s}} \exp\left(-\frac{s\alpha^2}{1+s}\right) \frac{1}{\sqrt{2^m m!}} \left(\frac{s-1}{s+1}\right)^{m/2} \times H_m\left(\frac{\sqrt{2}s\alpha}{\sqrt{s^2-1}}\right) \tag{A2}$$

by an integral. We then evaluate this integral for the case of a highly phase squeezed state of large displacement, that is, in the limit of $0 < s \ll 1$ and $\xi \equiv \sqrt{2s}\alpha \gg 1$.

When we insert Eq. (A2) into Eq. (A1) and make use of the relation [33]

$$H_{m+1}(x) = 2xH_m(x) - \frac{dH_m(x)}{dx} \tag{A3}$$

we arrive at the compact form

$$\langle\widehat{e^{i\varphi}}\rangle = \frac{\sqrt{2s}}{1+s}\sqrt{c}\exp\left(-\frac{2s\alpha^2}{1+s}\right)\left[2yf(y;c) - \frac{1}{2}\frac{df(y;c)}{dy}\right]\,, \tag{A4}$$

where we have introduced the abbreviations $c \equiv \frac{s-1}{s+1}$ and $y \equiv \frac{\sqrt{2}s\alpha}{\sqrt{s^2-1}}$ and

$$f(y;c) \equiv \sum_{m=0}^{\infty} \frac{c^m H_m^2(y)}{2^m m! \sqrt{m+1}}\,. \tag{A5}$$

We perform this summation by expressing the square root via the relation

$$\frac{1}{\sqrt{m+1}} = \frac{2}{\sqrt{\pi}} \int_0^\infty dt \exp[-(m+1)t^2] \qquad \text{(A6)}$$

and find

$$f(y;c) = \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{e^{-t^2}}{\sqrt{1-\bar{c}^2}} \exp\left(\frac{2\bar{c}}{1+\bar{c}} y^2\right) dt, \qquad \text{(A7)}$$

where we have abbreviated $\bar{c} = ce^{-t^2}$ and used the summation formula [38]

$$\sum_{m=0}^{\infty} \frac{\bar{c}^m H_m^2(y)}{2^m m!} = \frac{1}{\sqrt{1-\bar{c}^2}} \exp\left(\frac{2\bar{c}}{1+\bar{c}} y^2\right). \qquad \text{(A8)}$$

The expression (A7) for f together with its derivative

$$\frac{df}{dy} = \frac{8y}{\sqrt{\pi}} \int_0^\infty dt \frac{\bar{c}}{1+\bar{c}} \frac{e^{-t^2}}{\sqrt{1-\bar{c}^2}} \exp\left(\frac{2\bar{c}}{1+\bar{c}} y^2\right) \qquad \text{(A9)}$$

transform Eq. (A4) into

$$\langle \widehat{e^{i\varphi}} \rangle = \frac{8s\sqrt{s}\alpha}{(1+s)^2\sqrt{\pi}} \int_0^\infty dt \frac{e^{-t^2}}{\sqrt{(1+ce^{-t^2})^3(1-ce^{-t^2})}} \times \exp\left(-\frac{2s\alpha^2}{1+s}\frac{1-e^{-t^2}}{1+ce^{-t^2}}\right). \qquad \text{(A10)}$$

After the substitution

$$\tau = \frac{\sqrt{2s}\alpha}{\sqrt{1+s}} \sqrt{\frac{1-e^{-t^2}}{1+ce^{-t^2}}} \qquad \text{(A11)}$$

we arrive at

$$\langle \widehat{e^{i\varphi}} \rangle = \frac{2}{\sqrt{\pi}} \int_0^{\xi/\sqrt{1+s}} \sqrt{\frac{1}{1-(1-s^2)(\tau/\xi)^2}} \times \sqrt{\frac{2s(\tau/\xi)^2}{\ln\Omega(\tau/\xi)}} \exp(-\tau^2) d\tau \qquad \text{(A12)}$$

where

$$\Omega(\tau/\xi) \equiv \frac{1-(1-s)(\tau/\xi)^2}{1-(1+s)(\tau/\xi)^2}. \qquad \text{(A13)}$$

Note that this representation of the sum, Eq. (A1), as an integral is exact.

We now consider the asymptotics of this integral for $\xi \gg 1$ and $0 < s \ll 1$. Due to the exponential factor $e^{-\tau^2}$ the dominant contribution to the integral arises from values $\tau \lesssim 1$. This allows us to consider τ/ξ as a small quantity and hence

$$\ln\Omega \approx 2s(\tau/\xi)^2(1+(\tau/\xi)^2). \qquad \text{(A14)}$$

Therefore we find to lowest order in $(\tau/\xi)^2$

$$\langle \widehat{e^{i\varphi}} \rangle \approx \frac{2}{\sqrt{\pi}} \int_0^{\xi} \sqrt{\frac{1-(\tau/\xi)^2}{1-(1-s^2)(\tau/\xi)^2}} \exp(-\tau^2) d\tau. \qquad \text{(A15)}$$

Here we have also neglected the s dependence in the upper limit. This expression is identical to that obtained by performing the average $\langle e^{i\varphi} \rangle_W$ using the Wigner function as discussed in Appendix B.

We can estimate the integral by expanding the square root of the integrand which yields

$$\langle \widehat{e^{i\varphi}} \rangle \approx \frac{2}{\sqrt{\pi}} \left(\int_0^{\xi} e^{-\tau^2} d\tau - \frac{s^2}{2\xi^2} \int_0^\infty \tau^2 e^{-\tau^2} d\tau \right). \qquad \text{(A16)}$$

Here we have extended the second integral to infinity, since this term is already of the order $(s/\xi)^2 \ll 1$. Hence

$$\langle \widehat{e^{i\varphi}} \rangle \approx \Phi(\xi) - \frac{s^2}{4\xi^2}, \qquad \text{(A17)}$$

where Φ denotes the error function [33]. Its asymptotic expansion [33]

$$\Phi(\xi) \approx 1 - \frac{1}{\sqrt{\pi}\xi} e^{-\xi^2}, \qquad \text{(A18)}$$

valid for $\xi \gg 1$, finally yields for the dispersion

$$(\delta\varphi_d)^2 \equiv 1 - |\langle \widehat{e^{i\varphi}} \rangle|^2 \approx \frac{s^2}{2\xi^2} + \frac{2}{\sqrt{\pi}\xi} e^{-\xi^2}. \qquad \text{(A19)}$$

We discuss the physical origin of these two contributions in Sec. V.

APPENDIX B: AVERAGE $\langle e^{i\varphi} \rangle_W$ USING THE WIGNER DISTRIBUTION

In this appendix we calculate the average

$$\langle e^{i\varphi} \rangle_W \equiv \int_{-\pi/2}^{3\pi/2} d\varphi e^{i\varphi} W^{(W)}(\varphi) \qquad \text{(B1)}$$

using the phase distribution

$$W^{(W)}(\varphi) = \int_0^\infty P_{sq}^{(W)}(x = r\cos\varphi, p = r\sin\varphi) r\, dr \qquad \text{(B2)}$$

based on the Wigner function $P_{sq}^{(W)}$ of the squeezed state. The phase space integration simplifies when we use Cartesian coordinates. The average $\langle e^{i\varphi} \rangle_W$ then reads

$$\langle e^{i\varphi} \rangle_W = \int_{-\infty}^\infty dx \int_{-\infty}^\infty dp \frac{x+ip}{\sqrt{x^2+p^2}} P_{sq}^{(W)}(x,p) = \frac{1}{\pi} \int_{-\infty}^\infty dx \int_{-\infty}^\infty dp \frac{x}{\sqrt{x^2+p^2}} \times \exp\left[-s(x-\sqrt{2}\alpha)^2 - \frac{p^2}{s}\right]. \qquad \text{(B3)}$$

In the last step we have used the property $P_{sq}^{(W)}(x,p) = P_{sq}^{(W)}(x,-p)$. The integral representation

$$\frac{1}{\sqrt{x^2+p^2}} = \frac{2}{\sqrt{\pi}}\int_0^\infty dt \exp[-(x^2+p^2)t^2] \tag{B4}$$

yields

$$\langle e^{i\varphi}\rangle_W = e^{-2s\alpha^2}\frac{1}{\sqrt{\pi}}\int_0^\infty dt\frac{1}{\sqrt{t^2+1/s}} \times\frac{4}{\sqrt{\pi}}\int_0^\infty dx\,x\exp[-(t^2+s)x^2] \times\sinh(2\sqrt{2}s\alpha x)\,, \tag{B5}$$

where we have already performed the p integration. With the help of

$$\int_0^\infty dx\,xe^{-\beta x^2}\sinh(\gamma x) = \frac{\gamma\sqrt{\pi}}{4}\beta^{-3/2}\exp\left(\frac{\gamma^2}{4\beta}\right) \tag{B6}$$

we find

$$\langle e^{i\varphi}\rangle_W = \frac{2\sqrt{2}\alpha s}{\sqrt{\pi}}\int_0^\infty dt\,(t^2+1/s)^{-\frac{1}{2}}(t^2+s)^{-\frac{3}{2}} \times\exp\left(-\frac{2s\alpha^2t^2}{t^2+s}\right)\,, \tag{B7}$$

which with the substitution

$$\tau \equiv \sqrt{2s}\alpha\frac{t}{\sqrt{t^2+s}} \tag{B8}$$

reads

$$\langle e^{i\varphi}\rangle_W = \frac{2}{\sqrt{\pi}}\int_0^\xi d\tau\sqrt{\frac{1-(\tau/\xi)^2}{1-(1-s^2)(\tau/\xi)^2}}\exp(-\tau^2)\,. \tag{B9}$$

This expression for $\langle e^{i\varphi}\rangle_W$ is exact and is identical to the integral representation of $\langle\widehat{e^{i\varphi}}\rangle$, Eq. (A15), obtained in Appendix A from the London phase distribution for the case of a highly phase squeezed state with large displacement. Hence in this case the expression for the dispersion

$$(\delta\varphi_d)_W^2 = 1 - |\langle e^{i\varphi}\rangle_W|^2, \tag{B10}$$

calculated from the Wigner phase distribution $W^{(W)}$, is also given by Eq. (A19).

We gain more insight into this asymptotic relation between the exact expression Eq. (A10) for the expectation value $\langle\widehat{e^{i\varphi}}\rangle$ of the Susskind-Glogower operator and the result for $\langle e^{i\varphi}\rangle_W$, Eq. (B9), based on the radially integrated Wigner function, when we start from the Wigner representation [36,39]

$$\left[\widehat{e^{i\varphi}}\right]^{(W)}(x,p) \equiv \frac{x+ip}{\sqrt{2\pi}}\int_{-\infty}^\infty dt\cosh^{-2}(t^2/2) \times\exp\left[-(x^2+p^2)\tanh(t^2/2)\right] \tag{B11}$$

of the operator $\widehat{e^{i\varphi}}$. In this representation the average value $\langle\widehat{e^{i\varphi}}\rangle$ is given by the phase space integration

$$\langle\widehat{e^{i\varphi}}\rangle = \int_{-\infty}^\infty dx\int_{-\infty}^\infty dp\left[\widehat{e^{i\varphi}}\right]^{(W)}(x,p)P_{sq}^{(W)}(x,p) \tag{B12}$$

using the Wigner function, Eq. (6), of a squeezed state. Indeed when we perform the resulting Gaussian integrals, we arrive again at Eq. (A10). The relationship between $\langle\widehat{e^{i\varphi}}\rangle$ and $\langle e^{i\varphi}\rangle_W$ comes immediately to light when we consider the phase space integration, Eq. (B12), in the case of our highly displaced squeezed state, that is, for $\xi \gg 1$. Then the integration is concentrated around phase space points $x^2+p^2 \gg 1$ and the Wigner representation, Eq. (B11), simplifies to [39]

$$\left[\widehat{e^{i\varphi}}\right]^{(W)}(x,p) \approx \frac{x+ip}{\sqrt{2\pi}}\int_{-\infty}^\infty dt e^{-(x^2+p^2)t^2/2} = \frac{x+ip}{\sqrt{x^2+p^2}}\,. \tag{B13}$$

We substitute this approximation into Eq. (B12) and obtain

$$\langle\widehat{e^{i\varphi}}\rangle \approx \int_{-\infty}^\infty dx\int_{-\infty}^\infty dp\frac{x+ip}{\sqrt{x^2+p^2}}P_{sq}^{(W)}(x,p) = \langle e^{i\varphi}\rangle_W\,. \tag{B14}$$

Hence it is not surprising that in the limit of $\xi \gg 1$ the integral representations of Eqs. (A15) and (B9) are identical.

[1] For the present state of the phase problem in quantum optics, see the special issue on quantum phase and phase dependent measurements, Phys. Scr. **T48** (1993), edited by W. Schleich and S. M. Barnett.

[2] Measures of uncertainty and the uncertainty principle are discussed in great detail by J. B. M. Uffink, thesis, University of Utrecht, Utrecht, 1990.

[3] Phase optimized states for various measures of phase noise have been investigated. For states minimizing the phase variance, see G. S. Summy and D. T. Pegg, Opt. Commun. **77**, 75 (1990); for the dispersion measure, see A. Bandilla, H. Paul, and H.-H. Ritze, Quantum Opt. **3**,

267 (1991); A. Bandilla, Opt. Commun. **94**, 273 (1992); whereas for the Süssmann measure, see B. Daeubler, Ch. Miller, H. Risken, and L. Schoendorff, Phys. Scr. **T48**, 119 (1993).

[4] For states minimizing various measures of phase noise in a *finite* Hilbert space rather than fixed average number of photons, we refer to I. Bialynicki-Birula, M. Freyberger, and W. Schleich, Phys. Scr. **T48**, 113 (1993).

[5] An exception is the state which minimizes the maximum likelihood for a given average photon number. This state leads to a stronger decay than $\langle n\rangle^{-2}$ as discussed in J. H. Shapiro, S. R. Shepard, and N. Wong, Phys. Rev. Lett. **62**, 2377 (1989); **63**, 2002 (1989). However, see also the discussion in W. P. Schleich, J. P. Dowling, and R. J. Horowicz, Phys. Rev. A **44**, 3365 (1991); S. L. Braunstein, A. S. Lane, and C. M. Caves, Phys. Rev. Lett. **69**, 2153 (1992); A. S. Lane, S. L. Braunstein, and C. M. Caves, Phys. Rev. A **47**, 1667 (1993).

[6] C. M. Caves, Phys. Rev. D **23**, 1693 (1981).

[7] R. S. Bondurant and J. H. Shapiro, Phys. Rev. D **30**, 2548 (1984).

[8] B. Yurke, S. L. McCall, and J. R. Klauder, Phys. Rev. A **33**, 4033 (1986).

[9] V. B. Braginsky and F. Ya. Khalili, Phys. Lett. A **175**, 85 (1993).

[10] F. London, Z. Phys. **37**, 915 (1926); **40**, 193 (1927).

[11] L. Susskind and J. Glogower, Physics **1**, 49 (1964).

[12] P. Carruthers and M. M. Nieto, Rev. Mod. Phys. **40**, 411 (1968).

[13] I. Bialynicki-Birula and Z. Bialynicki-Birula, Phys. Rev. A **14**, 1101 (1976).

[14] I. Bialynicki-Birula and Cao Long Van, Acta Phys. Polon. A **57**, 549 (1980).

[15] D. T. Pegg and S. M. Barnett, Phys. Rev. A **39**, 1665 (1989).

[16] M. M. Nieto, Phys. Scr. **T48**, 5 (1993).

[17] Phase properties of squeezed states have been the topic of papers in the past; see B. C. Sanders, S. M. Barnett, and P. L. Knight, Opt. Commun. **58**, 290 (1986); R. Lynch, J. Opt. Soc. Am. B **4**, 1723 (1987); Hong-Yi Fan and H. R. Zaidi, Opt. Commun. **68**, 143 (1988); N. Grønbech-Jensen, P. L. Christiansen, and P. S. Ramanujam, J. Opt. Soc. Am. B **6**, 2423 (1989); J. A. Vaccaro and D. T. Pegg, Opt. Commun. **70**, 529 (1989); Ts. Gantsog and R. Tanaś, Phys. Lett. A **152**, 251 (1991); D. Cohen, Y. Ben-Areyh, and A. Mann, Opt. Commun. **94**, 227 (1992).

[18] M. J. Collett, Phys. Scr. **T48**, 124 (1993). This result has also been mentioned in Phys. Rev. Lett. **70**, 3400 (1993).

[19] For a treatment of this problem using the Q function and its relation to the present approaches, see A. Bandilla and H.-H. Ritze, Phys. Rev. A **49**, 4447 (1994).

[20] We emphasize that none of the phase uncertainties, discussed in this paper, is based on a Hermitian phase operator. Therefore they are not on the same footing with true quantum mechanical variances such as $(\delta n)^2$.

[21] H. P. Yuen, Phys. Rev. A **13**, 2226 (1976).

[22] D. F. Walls, Nature **306**, 141 (1983).

[23] W. Schleich and J. A. Wheeler, Nature **326**, 574 (1987); J. Opt. Soc. Am. B **4**, 1715 (1987).

[24] W. Schleich, D. F. Walls, and J. A. Wheeler, Phys. Rev. A **38**, 1177 (1988).

[25] We find from Eq. (3) that $c_m \sim c^{m/2}H_m(y)$, where $c = \frac{s-1}{s+1}$ and $y = \frac{\sqrt{2}s\alpha}{\sqrt{s^2-1}}$. In the case of $0 < s < 1$ we find $c_m \sim i^m \tilde{c}^{m/2} H_m(-i\tilde{y})$, where, $\tilde{c} = \frac{1-s}{1+s}$ and $\tilde{y} = \frac{\sqrt{2}s\alpha}{\sqrt{1-s^2}}$. The relation $H_m^*(-i\tilde{y}) = (-1)^m H_m(-i\tilde{y})$ immediately yields $c_m^* = c_m$, that is, the coefficients c_m are real for any $s > 0$.

[26] M. Hillery, R. F. O'Connell, M. O. Scully, and E. P. Wigner, Phys. Rep. **106**, 212 (1984).

[27] Since in the present paper we deal with variances and the width $(\Delta y)^2$ of a Gaussian distribution is related to its variance $(\delta y)^2$ via $(\delta y)^2 = \langle y^2\rangle = \frac{1}{\sqrt{\pi}\Delta y}\int_{-\infty}^{\infty} y^2 \exp\left[-\left(\frac{y}{\Delta y}\right)^2\right] dy = \frac{1}{2}(\Delta y)^2$, the minor and major axes of the elliptical contour line $\exp(-1/2)$ of the squeezed Gaussian directly give the variances. Note that earlier work (Refs. [23] and [24]) is based on the contour line $\exp(-1)$.

[28] For a discussion of the phase distribution of an *amplitude* squeezed state, see W. P. Schleich, R. J. Horowicz, and S. Varro, Phys. Rev. A **40**, 7405 (1989).

[29] A. Bandilla and H. Paul, Ann. Phys. (Leipzig) **23**, 323 (1969).

[30] H. Paul, Fortschr. Phys. **22**, 657 (1974).

[31] Note, however, that this phase distribution is based on the states $|\varphi\rangle$, Eq. (17), which are not orthogonal. Hence this phase distribution is not on the same footing with the usual probability densities in quantum mechanics, which are based on the orthogonal eigenstates of an observable.

[32] For comparison to a dispersion that is derived from the Q function, see Ref. [19].

[33] I. S. Gradsteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic, New York, 1965).

[34] J. B. M. Uffink and J. Hilgevoord, Found. Phys. **15**, 925 (1985); see also J. B. M. Uffink, Phys. Lett. **108A**, 59 (1985).

[35] A. Luis and L. L. Sánchez-Soto, J. Phys. A **24**, 2083 (1991).

[36] B.-G. Englert, J. Phys. A **22**, 625 (1989).

[37] We can also perform the phase space integration, Eq. (64), for the general case with a displaced squeezed state. Moreover, in this case the implicit relation for $\delta\varphi_r$ is more complicated and cannot be solved analytically. Note that this general case does not add anything new to the principal idea of this paragraph.

[38] W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics* (Springer, New York, 1966).

[39] J. Bergou and B.-G. Englert, Ann. Phys. (N.Y.) **209**, 479 (1991).

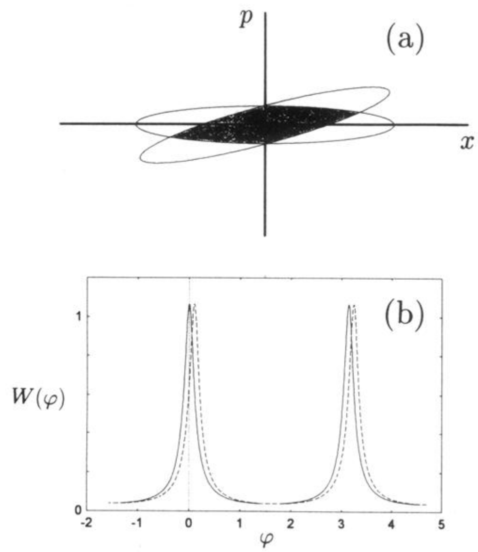

FIG. 5. Illustration of the rotational width $\delta\varphi_r$ for the example of a squeezed vacuum state. (a) Phase space representation of a squeezed vacuum state in combination with its rotated twin. The overlap of the two states given by the shaded area determines $\delta\varphi_r$. (b) The two-peaked phase distribution $W(\varphi)$ of a squeezed vacuum for $s = 0.1$. The solid line corresponds to the unrotated state with peaks located at $\varphi = 0$ and $\varphi = \pi$, whereas the dashed line corresponds to the state rotated by $\delta\varphi_r$. This phase distribution shows peaks located at $\varphi = \delta\varphi_r$ and $\varphi = \pi + \delta\varphi_r$.