Tunneling in the presence of driving in a cavity that contains a Kerr medium and is parametrically pumped

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Driving is applied to an optical system consisting of a Kerr medium combined with a second-order parametric process driven by a pump field which is sinusoidally modulated in amplitude. The coherent tunneling in this system is found to be both enhanced and suppressed by this, depending on the frequency of the driving chosen. A variation of time-independent perturbation theory is applied to this system and found to successfully describe this behavior.

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I. INTRODUCTION

In a previous paper [1] we have considered the behavior of a system, which was called the Cassinian oscillator, consisting of a Kerr medium combined with a pumped parametric amplifier. This system was found to exhibit tunneling between the two wells centered around the classical fixed points. In this paper we will examine the behavior of the system when it is driven by sinusoidally varying the strength of the parametric pump. This causes the motion of the classical counterpart of the system to become chaotic. It will be shown that the tunneling frequency of the quantum system can be either enhanced or suppressed by this driving. This result, found by numerically modeling the quantum system, is also predicted by applying a form of perturbation theory similar to standard time-independent perturbation theory to the problem of calculating the Floquet states of the system given the unperturbed states.

The coherent tunneling of a quantum system across classically forbidden barriers has long been seen as a striking example of the differences between classical and quantum mechanics. On the other hand classical systems have been found often to exhibit a pseudorandom motion called chaos for which there is no obvious quantum counterpart. Much recent work has been done attempting to find behavior which distinguishes between quantum systems whose classical counterparts are chaotic and those whose are not. The effect of driving on the behavior of a tunneling system has previously been considered by a number of authors [2–6]. Most of this previous work has modeled the system numerically and much of the analytic study used two-state systems [5]. As will be seen, two-state systems do not display the effect of resonances with higher levels which is found in this paper. An exception to this is [6] which describes a generalization of time-independent perturbation theory for systems whose perturbations are in the form of periodic driving. This work had been previously done by Sambe [7] and others [8,9]. It is this method which is used here as one way to model the system.

The rest of this paper is set out as follows. In the next section the model Hamiltonian will be presented and the main results of our previous paper [1] will be reviewed. The behavior of the tunneling frequency in the low-driving-frequency limit will then be found. Then the methods used to find both the classical and quantum motions of the system numerically will be introduced. Next, perturbation theory will be applied to both the classical and quantum models. Both classical and quantum results obtained by these methods are then presented and contrasted. In the last section the results of this investigation will be reviewed and an attempt made to interpret them. Future paths will also be discussed.

II. MODEL

The Hamiltonian of the model (in the interaction picture) is

$$H = \frac{\hbar\chi}{2}(a^\dagger a)^2 - \frac{\hbar\kappa(1+\epsilon\cos\Omega t)}{2}(a^2 + a^{\dagger 2}), \qquad (1)$$

where a is the annihilation operator for the cavity field, χ is proportional to the third-order nonlinear susceptibility, κ is proportional to the product of the amplitude of the parametric pump field and the second-order susceptibility, ϵ is the strength of the perturbation, and Ω is the frequency of the driving. This is identical to the Hamiltonian of [1] when ϵ is equal to zero. In that paper the equivalent classical equations of motion were shown to be

$$\dot{X}_1 = 4(X_1^2 + X_2^2)X_2 + 4\mu^2 X_2, \qquad (2)$$

$$\dot{X}_2 = -4(X_1^2 + X_2^2)X_1 + 4\mu^2 X_1, \qquad (3)$$

with X_1 and X_2 commuting numbers which replace the quantum operators defined by $X_1 = \frac{1}{2}(a + a^\dagger)$ and $X_2 = \frac{-i}{2}(a - a^\dagger)$. The dot represents differentiation with respect to the scaled time $\tau = t\chi/4$ and we have defined $\mu^2 = \frac{\kappa}{\chi}$. The trajectories of the system through phase space were calculated and shown to be ovals of Cassini with foci on the X_1 axis at $\pm\mu$, and the frequencies of oscillation of the system were shown to be given by

$$\omega(E) = \frac{4\pi\beta^2}{F(\pi, \frac{\mu^2}{\beta^2})} \text{ if } E > 0, \tag{4}$$

$$\omega(E) = \frac{4\pi\beta^2}{F(2\theta_c, \frac{\mu^2}{\beta^2})} \text{ if } E < 0. \tag{5}$$

Here, $\beta^4 = \mu^4 + E$, E is the total energy of the system, and the critical angle θ_c is defined by

$$\cos 2\theta_c = \pm\frac{\sqrt{|E|}}{\mu^2}, \tag{6}$$

with F an elliptic function of the second type [10]. The quantum system's energy levels were numerically calculated by diagonalizing the Hamiltonian matrix in the number basis, and it was shown that the first two energy levels are localized at the classical fixed points and are parity eigenstates of opposite parity. A state initially localized at one of these fixed points, for example a coherent state [11], will tunnel across to the other well and then back again with a frequency given by the unperturbed tunnel splitting $\Delta_0 = \omega_1 - \omega_0$. The subscripts refer to the number of the eigenstates, which are arranged in ascending order of energy. This assumes that the localized state is mainly representable by a superposition of the lowest two eigenstates; however, for a coherent state this assumption is found to be a good one. This tunnel splitting is approximately given by

$$\Delta_0 = \chi \exp\left(-2\left|\frac{\kappa}{\chi}\right|\right). \tag{7}$$

III. DYNAMICS

A. Classical dynamics

The equations of motion still hold in the driven system if the κ in μ is replaced by $\kappa(1 + \epsilon\cos\Omega t)$. These equations can then be numerically solved. The results of this are then presented as stroboscopic phase portraits (Fig. 1). As well as these numerical simulations, the system was also subjected to classical perturbation theory, which revealed the expected locations of the resonant islands. This procedure is treated in most of the relevant literature [12].

B. Low-frequency limit

In the low-frequency limit we assume that the value of $\kappa(1 + \epsilon\cos\Omega t)$ varies slowly enough that the system can be assumed to be described by the eigenstates of the undriven system with κ having the value of $\kappa(1 + \epsilon\cos\Omega t)$. The overlap of an arbitrary state with an energy eigenstate is assumed to be constant as the frequencies and compositions of these states are slowly varied. This is the adiabatic approximation [13]. It is also assumed that the tunneling frequency is still much larger then the driving frequency. Now consider the evolution of a coherent state

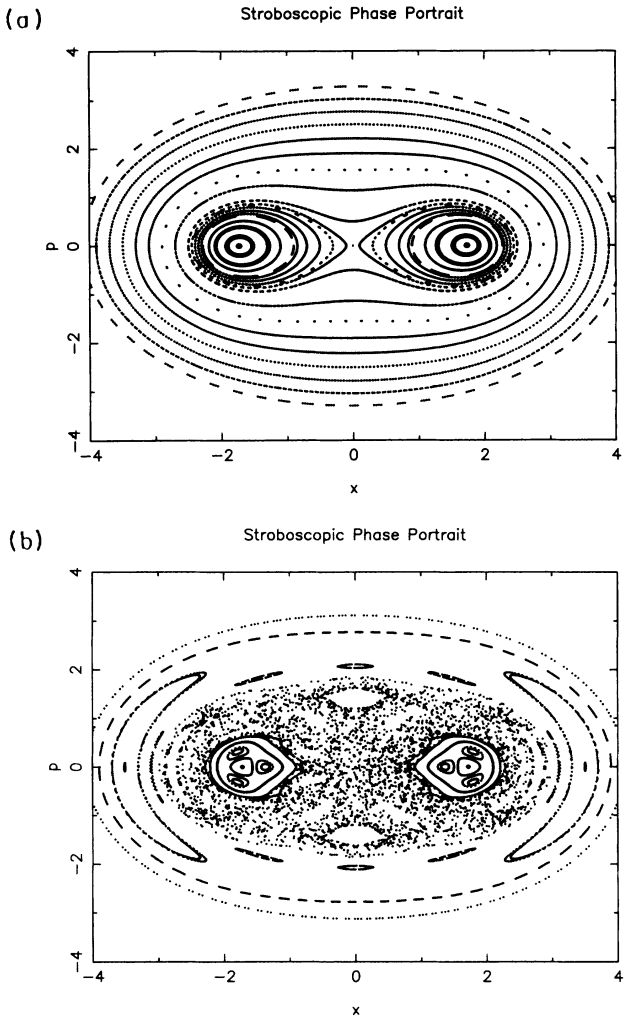

FIG. 1. Phase-space diagrams for the driven system with $\kappa = 12$, $\chi = 4$, $\Omega = 70.0$ (dimensionless units) and ϵ equal to (a) 0.0 (b) 0.1.

which starts off centered at one of the wells. It will start off composed of only the two lowest-energy eigenstates and as it evolves will retain this composition. After one period of the driving, the system will have returned to its original state except for the phase changes of the energy eigenstates. The difference between these phase changes for the two tunneling states will play a role exactly equivalent to the phase change difference due to free evolution in the unperturbed system, and this can be used to find the frequency of the tunneling over an integer number of periods. This frequency is called the perturbed tunnel splitting Δ_p. As the system evolves for a time dt at an instant t the phase of the mth eigenstate will change by $-i\omega_m(\kappa(t), \chi)dt$ so over a time of one period $T = \frac{2\pi}{\Omega}$ each eigenstate will change its phase by

$$-i\int_0^T \omega(\kappa(t), \chi)dt. \tag{8}$$

The system's state after one period is then given by

$$|\Psi(T)\rangle = e^{-i\int_0^T \omega_1(\kappa(t),\chi)dt}|\omega_1\rangle + e^{-i\int_0^T \omega_0(\kappa(t),\chi)dt}|\omega_0\rangle, \tag{9}$$

if the two-state approximation is made. As for undriven tunneling one of the phases is taken outside the sum

$$|\Psi(T)\rangle = e^{-i\int_0^T \omega_1(\kappa(t),\chi)dt}\Big[|\omega_1\rangle + e^{i\int_0^T \omega_0(\kappa(t),\chi)dt - i\int_0^T \omega_0(\kappa(t),\chi)dt}|\omega_0\rangle\Big]. \tag{10}$$

So the total change of relative phase, the equivalent of $-i\Delta_0 t$ in the undriven system, will be

$$-i\Delta_p T = -i\int_0^T \Delta_0(\kappa(t),\chi)dt, \tag{11}$$

with Δ_0 the tunnel splitting given by Eq. (7) and Δ_p the perturbed tunnel splitting. The perturbed tunnel splitting then evaluates to

$$\Delta_0 = \chi e^{-\frac{2\kappa}{\chi}} I_0\left(\frac{2\kappa\epsilon}{\chi}\right), \tag{12}$$

where I_0 is a modified Bessel function of the first kind. The behavior of this equation will be described in Sec. IV where it will also be compared to the prediction given by perturbation theory.

C. The Floquet states

The Floquet theorem states that when the Hamiltonian of a system is time dependent but periodic then the solutions of the Schrödinger equation are of the form

$$|\Upsilon_m(t)\rangle = e^{-i\omega_m t}|v_m(t)\rangle, \tag{13}$$

where $|v_m(t)\rangle = |v_m(t+T)\rangle$ and ω_m is called the quasifrequency. This is similar to the case in which there is a periodic variation of the potential in position space. These Floquet states are useful as evolution of one of them by one period leaves them in the original state multiplied by a phase factor. Thus only the form of the Floquet states at $t = 0$ is needed to simulate evolution in fixed steps of a period. The quasifrequencies ω_m then play the part of eigenfrequencies and the level splitting can be written as $\Delta_s = \omega_1 - \omega_0$.

The Floquet states at $t = 0$ are eigenstates of the evolution operator over one period. This operator will be called the Floquet operator F. It can be calculated numerically as follows. First $F|\Psi(0)\rangle = |\Psi(T)\rangle$ so $\langle m|F|n\rangle = \langle m|n(T)\rangle$; thus, by using a routine which solves the time-dependent Schrödinger equation over one period with a number state as initial conditions, each row of the numerical representation of the Floquet operator may be found. Then the Floquet states are found as the eigenstates of this operator. The quasienergies are not found directly, rather $e^{-i\omega_m T}$ is found. From this the frequencies can be found modulus the driving frequency. This method is used rather than the matrix continued fraction technique of [2] because setting up the calculation on a computer is more straightforward and so less error prone. However, it has the limitation that the time required increases as the period. This is why only driving frequencies of greater then 50 are used numerically.

D. Perturbation theory

The quantum motion was also examined by applying a perturbative method to the problem of calculating the Floquet states analytically. The details of this method have been previously published in [6]. Here we will repeat the development done in that paper but will use the above model.

First the Hamiltonian Eq. (1) is written in the form of an unperturbed system plus a small driving perturbation.

$$H(t) = \frac{\hbar\chi}{2}(a^\dagger a)^2 - \frac{\hbar\kappa}{2}(a^2 + a^{\dagger 2}) - \frac{\hbar\epsilon\kappa}{2}(\cos\Omega t)\,(a^2 + a^{\dagger 2}), \tag{14}$$

$$= H_0 + \epsilon V \cos\Omega t. \tag{15}$$

So

$$H_0 = \frac{\hbar\chi}{2}(a^\dagger a)^2 - \frac{\hbar\kappa}{2}(a^2 + a^{\dagger 2}) \tag{16}$$

and

$$V = \frac{\hbar\kappa}{2}(a^2 + a^{\dagger 2}). \tag{17}$$

Denote the Floquet states by $|\Upsilon_m(t)\rangle$. Then $|v_m(t)\rangle = e^{i\omega_m t}|\Upsilon_m(t)\rangle$ is a state with period T. This will equal the Floquet state at $t = 0$, which is the goal of this calculation. Substituting this into the Schrödinger equation for $|\Upsilon_m(t)\rangle$ gives the following equation for $|v_m(t)\rangle$.

$$\hbar\omega_m|v_m\rangle = -i\hbar\frac{d}{dt}|v_m\rangle + H(t)|v_m\rangle. \tag{18}$$

$|v_m(t)\rangle$ can be expanded as a Fourier series as it is periodic.

$$|v_m(t)\rangle = \sum_{ln} e^{i\Omega l t} v_{ml}|v_m\rangle. \tag{19}$$

The $e^{i\Omega l t}$ can be seen as the t representation of a set of states of period T in a one-dimensional subspace. They can then be written as $|l\rangle$. These states are eigenstates of the operator $-i\hbar\frac{d}{dt}$.

$$-i\hbar\frac{d}{dt}|l\rangle = -\Omega l\hbar. \tag{20}$$

In this way the Schrödinger equation for $|v_m\rangle$ will appear as an eigenvalue equation. Again, this is similar to the study of a periodic lattice in space. Now expand $|v_m\rangle$ around $|\omega_m^{(0)}\rangle$, the unperturbed states, as a series in ϵ and do the same with ω_m. Substituting this into the Schrödinger equation Eq. (18) and equating equal orders of ϵ gives

$$\text{0th order}: \quad \hbar\omega_m^{(0)}|\omega_m^{(0)}\rangle|0\rangle = K_0|\omega_m^{(0)}\rangle|0\rangle, \tag{21}$$

1st order :

$$\hbar\omega_m^{(1)}|\omega_m^{(0)}\rangle|0\rangle + \hbar\omega_m^{(0)}|v_m^{(1)}\rangle = K_0|v^{(1)}\rangle + (\cos\Omega t)V|\omega_m^{(0)}\rangle|0\rangle, \tag{22}$$

2nd order :

$$\hbar\omega_m^{(0)}|v_m^{(2)}\rangle + \hbar\omega_m^{(1)}|v_m^{(1)}\rangle + \hbar\omega_m^{(2)}|\omega_m^{(0)}\rangle|0\rangle = K_0|v_m^{(2)}\rangle + (\cos\Omega t)V|v_m^{(1)}\rangle. \tag{23}$$

Here K_0 is $H_0 - i\hbar\frac{d}{dt}$ and $\omega_m^{(0)}$ denotes the zeroth-order correction (which is in that case the unperturbed frequency) to the mth frequency, and similarly for other superscripts. The zeroth-order equation is just the original eigenvalue equation for the unperturbed system. The first-order equation when both sides are multiplied by $\langle\omega_m^{(0)}|\langle 0|$ will give for $\omega_m^{(1)}$

$$\hbar\omega_m^{(1)} = \langle 0|\cos\omega t|0\rangle\langle\omega_m^{(0)}|V|\omega_m^{(0)}\rangle. \tag{24}$$

However by using the t representation for the $|l\rangle$ states it is found that

$$\langle l|\cos\Omega t|k\rangle = \tfrac{1}{2}(\delta_{l,k+1} + \delta_{l,k-1}). \tag{25}$$

So in general $\omega_m^{(1)}$ is zero. To find the second-order frequency correction it is necessary to find the first-order correction to the $|v_m\rangle$ states. This is found in terms of its expansion in terms of the unperturbed states,

$$|v_m^{(1)}\rangle = \sum_{l,n} v_{mln}^{(1)}|\omega_n^{(0)}\rangle|l\rangle. \tag{26}$$

This is substituted into the first-order equation Eq. (21) and both sides are multiplied by $\langle\omega_n|\langle k|$. After using Eq. (25) this gives

$$|v_m^{(1)}\rangle = \frac{1}{\hbar}\sum_n \langle\omega_n^{(0)}|V|\omega_m^{(0)}\rangle\left[\frac{\omega_m^{(0)} - \omega_n^{(0)}}{(\omega_m^{(0)} - \omega_n^{(0)})^2 - \Omega^2}\right]|\omega_n\rangle. \tag{27}$$

The second-order frequency correction is now found from the second order equation Eq. (22). Similarly to what was done for the first equation, both sides are multiplied by $\langle\omega_m^{(0)}|\langle 0|$ and then the first-order correction to the state is substituted in. This gives

$$\omega_m^{(2)} = \frac{\kappa^2}{8}\sum_n \frac{|\langle\omega_m^{(0)}|(a^2 + a^{\dagger 2})|\omega_n^{(0)}\rangle|^2(\omega_m^{(0)} - \omega_n^{(0)})}{(\omega_m^{(0)} - \omega_n^{(0)})^2 - \Omega^2}. \tag{28}$$

By using the first-order corrections for the states and the second-order corrections for the frequency the tunneling behavior of the system for small perturbations is predicted analytically from the unperturbed states, which are in this case found numerically. The results can then be compared with those found by a direct numerical solution to the Schrödinger equation.

IV. RESULTS

All of the results were generated with $\kappa = 12$ and $\chi = 4$ in arbitrary units. Most of the dynamics are dependent on the ratio $\frac{\kappa}{\chi}$, which leads to the relative structure of the energy eigenvalues and eigenvectors. The only exception to this is varying the value of χ with the above ratio held constant, which causes a directly proportional change in the eigenfrequencies. The values chosen above are convenient as the eigenvalue spectrum has only two states below the separatrix and the tunneling times are conveniently short. The undriven tunneling frequency is 0.1937 and the two tunneling levels are approximately 11 below the separatrix frequency as measured in arbitrary units. As the above figures have no units associated with them and the ratio of $\frac{\kappa}{\chi}$ is dimensionless, the above discussion is only intended to give some idea of the relative sizes of the system properties. However, $\frac{\kappa}{\chi}$ determines one squared radius of the fixed points and thus the average field intensity for states localized at these fixed points. So our choice of constants means that the fixed points correspond to a photon number of approximately 3. This increases linearly with pump amplitude.

A. Classical results

The classical phase-space portraits of the system were calculated numerically by solving the classical equations of motion Eq. (2) by the Runge-Kutta method, the results of which were plotted as series of points for various initial conditions at intervals of the period of the driving force, which are Poincaré sections. These pictures reveal that inside the separatrix resonances of all orders occur and for the frequencies that are plotted the third- and fourth-order points are visible. A chaotic area forms around the separatrix and moves inward around these points, which form stable islands in this sea. For the parameter range used so far the size of these islands is much smaller then the minimum size of a coherent state and so such effects as tunneling between these islands have not been studied. Outside the separatrix only resonances of even periods are visible, as is predicted by the application of classical perturbation theory.

B. Quantum results

The most comprehensive numerical results about the quantum behavior were those gained by calculating the Floquet states and the quasifrequencies. The numerical integration of the Schrödinger equation was used as a test

on the validity of these results. Hence the results gained by that method add nothing to what the Floquet states reveal and so are not discussed any further here.

The first thing of interest found is that in general, for small values of ϵ, it is still a good approximation that only two of the Floquet states are included in the makeup of a coherent state. This is shown numerically in Fig. 2 where the magnitude of the overlap of a coherent state with the two Floquet states having the highest overlap with this state is plotted against Ω. It can be seen from this that for most values of Ω the first two states make up over 95% of of the coherent state. This figure only drops at certain values of Ω where a third state begins to make a contribution to the mix. These points are where a multiple of Ω is equal to the frequency difference between one of the tunneling states and a higher-energy Floquet state. These resonances are frequencies where perturbation theory breaks down and the motion of the system at driving frequencies above the resonance frequency is very different from that below. The values are only calculated down to a frequency of 50 dimensionless frequency units, as the time required to calculate the Floquet states rises in direct proportion to the period and the calculation time required to extend the curve below this cutoff is too great.

Although perturbation theory breaks down very near a resonance due to the denominators of the form $(\omega_m^{(0)} - \omega_n^{(0)})^2 - \Omega^2$ becoming zero, it can still explain the above behavior. The perturbed energy eigenfunctions are of the form $|v_m\rangle = |\omega_m^{(0)}\rangle + |v_m^{(1)}\rangle$ with the first-order corrections $|v_m^{(1)}\rangle$ given by Eq. (27). Most terms in this sum will be small as $\langle\omega_n^{(0)}|V|\omega_m^{(0)}\rangle$ will only be large for the $m = n$ state. So the first-order-corrected states are usually unchanged from the unperturbed states and so the results of the above paragraph, that the two-state approximation is valid away from resonances, hold. However, if the driving frequency is near a resonance, with $\omega_m - \omega_n$ nearly equal to Ω for some n, and m equals 1 or 0, then the mth corrected state will have a large contribution from the nth state. Then a coherent state can no longer be modeled as simply the sum of the first two Floquet states but must include a large contribution from a third, the nth state. Then there will be three frequencies in the tunneling rate, not only the difference between the 0th quasifrequency and the 1st but also the differences between these two and the nth. This result was also found in [2].

An important feature which needs to be pointed out here is that the value of $\langle\omega_n^{(0)}|V|\omega_m^{(0)}\rangle$ will be zero if the parity of $|\omega_m^{(0)}\rangle$ is opposite to that of $|\omega_n^{(0)}\rangle$. Then because the two tunneling states have opposite parity only one will show the effects of a resonance for any value of Ω, even though they will both be near a resonant frequency with an upper state at almost the same driving frequency. One of the consequences of this is that in the above case only three states are of importance rather then four, even though the difference between the resonances associated with the zeroth state and the first state is not resolvable on Fig. 2. Another important effect of this will be seen when the behavior of the system with changing ϵ is considered.

To look at how the tunneling rate changes with varying ϵ the second derivative of the tunneling rate, evaluated at $\epsilon = 0$, as a function of ϵ is plotted versus Ω in Fig. 3. There are in fact two curves on this graph; in one the results are calculated numerically and in the other are the results predicted by perturbation theory. The almost identical nature of these predictions shows that perturbation theory is a successful device for analytically calculating and understanding the tunneling behavior of driven systems.

The numerical line was obtained by letting $\Delta_p = (\omega_1^{(0)} - \omega_0^{(0)}) + \frac{1}{2}\frac{d^2\Delta_p}{d\epsilon^2}\epsilon^2$, and by calculating two values of Δ_p, one at zero and the other at a small ϵ value, the second derivative was found. Although this method does assume the relation given by perturbation theory it still seems that this assumption would not have given both

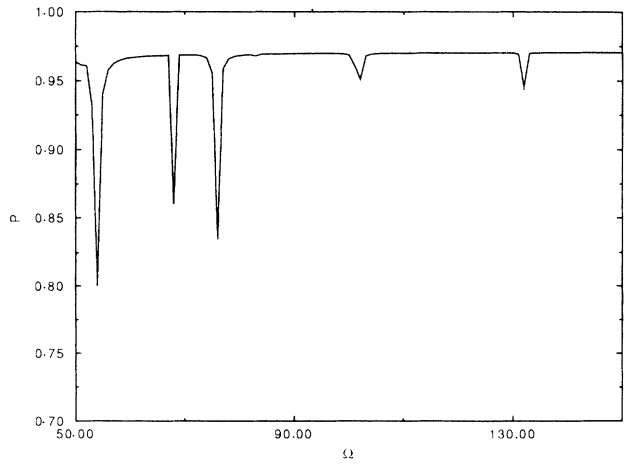

FIG. 2. Fraction of the overlap with a coherent state centered on one of the fixed points made up by the first two (tunneling) states as a function of Ω. The other parameters are as in Fig. 1.

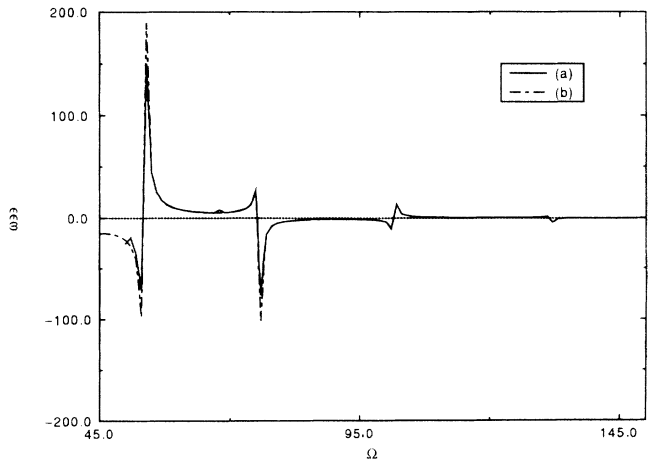

FIG. 3. Second derivative of the tunneling frequency with respect to ϵ evaluated at $\epsilon = 0$ plotted as a function of Ω. The other parameters are as in Fig. 1.

results in such close agreement unless it was justified. When the line is above the zero axis the second derivative is positive. As the first derivative is zero at $\epsilon = 0$, a positive second derivative means that the tunneling frequency will increase with increasing driving strength. When the second derivative is negative the tunneling frequency will decrease with increased driving. In this case the tunneling can be totally suppressed by sufficiently powerful driving. Greater driving strength would lead to renewed tunneling with the phase changing in the reverse direction. At the resonances the value of the second derivative goes to infinity, as would be expected from the denominator in Eq. (28). The reason for the change of sign at the resonances is that the actual tunneling frequency is given by

$$\Delta_s = (\omega_1^{(0)} - \omega_0^{(0)}) + \epsilon^2(\omega_1^{(2)} - \omega_0^{(2)}). \tag{29}$$

Now when a resonance is approached one of the second-order corrections will go to infinity, while the one for the state with opposite parity will remain small, as the term $\langle\omega_n^{(0)}|V|\omega_m^{(0)}\rangle$ with the state with which it is in resonance is exactly zero due to parity considerations. The sign of the correction will be positive before the resonance as $(\omega_m^{(0)} - \omega_n^{(0)})^2$ is greater then Ω^2 before the resonance, but is negative after the resonance. The value of the second derivative will therefore go to infinity at the resonance and then come in from the infinity of opposite sign on the resonance's far side. At the next resonance, the other frequency will be the one going to infinity, and as the multiplying sign of this is reversed it will go to the same sign infinity as the derivative has just come from. So the general form of the tunneling behavior can be understood in terms of the above perturbation theory.

The equation for the low-frequency limit, Eq. (12), can also be compared to the results of perturbation theory. The modified Bessel function I_0 can be expanded in the series

$$I_0(x) = 1 + (\tfrac{1}{2}x)^2 + \frac{(\frac{1}{2}x)^4}{4} + \cdots. \tag{30}$$

The first two terms of this expansion correspond to the ϵ and ϵ^2 terms in the perturbative expansion. When this is combined with Eq. (12) it gives

$$\Delta_0 = \chi e^{\frac{-2\kappa}{\chi}} + \chi e^{\frac{-2\kappa}{\chi}} \left(\frac{\kappa\epsilon}{\chi}\right)^2. \tag{31}$$

The first term is simply the unperturbed tunneling frequency. The second term is the equivalent of the ϵ^2 term in the perturbative series. This has the same κ^2 dependence as the second term above; however, confirming the other parts of this would require an analytic expression for the unperturbed states, which we do not have. Nevertheless the absence of a first-order power of ϵ does give further justification of the method used to calculate numerically the second derivative of the tunneling rate.

V. CONCLUSIONS

It has been shown above that the tunneling behavior of the quantum optical system examined here can, for small driving forces, be explained and predicted by the use of a perturbation theory. This allows some degree of analytical understanding of both this system and of the systems studied by others. Our results show that tunneling can be either enhanced or suppressed for suitable values of the driving frequency. Nevertheless, the system here was examined for values in the deep quantum regime, while to attain the goal of studying the chaotic effects of the system will require moving to large photon numbers. It appears that any classical effects on a scale smaller than $\hbar$ will be averaged out in the quantum system.

One interesting idea is that the enhancement of the tunneling frequency due to the effect of driving can be used in experiments to observe this tunneling. The major problem in devising experiments to observe coherent tunneling in optical systems is that the low values of nonlinearities available means that any tunneling will have very low frequencies and therefore the states will have their coherences destroyed by dissipation before any tunneling occurs. However the enhancement of tunneling rates possible when near resonances could result in this kind of experiment being possible. We are currently investigating the effect of dissipation in this context.

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