

Atomic states with spectroscopic squeezing

G. S. Agarwal

School of Physics, University of Hyderabad, Hyderabad 500 134, India

R. R. Puri

Theoretical Physics Division, Bhabha Atomic Research Centre, Bombay 400 085, India

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The spectroscopic squeezing characteristics of the angular-momentum state $\exp(\theta S_z)\exp[-(i\pi/2)S_y]|S, m\rangle$ are calculated. The parameter $\sqrt{2S}\Delta S_x/|\langle S_z \rangle|$ is shown to be less than or equal to 1 and takes the asymptotic value $(1+S)^{-1/2}$ as $\theta \rightarrow 0$ if $m \approx 0$. This small value makes those states especially useful in applications where quantum noise reduction is required. Also, the spin-spin correlations of those states are nonzero and negative.

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The importance of the atomic squeezed states in spectroscopic work is beginning to be recognized [1]. Consider the Heisenberg uncertainty relation involving the uncertainties in the angular-momentum operators \mathbf{S}

$$\Delta S_x \Delta S_y \geq \frac{1}{2} |\langle S_z \rangle|, \tag{1}$$

where uncertainties like ΔS_x are defined in the usual manner. In view of this it is possible to define the squeezed atomic states as those for which, say,

$$(\Delta S_x)^2 < \frac{1}{2} |\langle S_z \rangle|. \tag{2}$$

There can be many states satisfying the squeezing condition (2). Wódkiewicz and Eberly [2] considered a class of atomic coherent states for which condition (2) is satisfied and for which equality sign in (1) holds. The squeezed atomic states which are of interest, in this paper, are defined by the solution of the eigenvalue problem

$$[\exp(\theta)S_+ + \exp(-\theta)S_-]|\psi\rangle = \lambda|\psi\rangle, \quad \theta \text{ real}. \tag{3}$$

These include in the *special* case the state $\exp[\beta/2(S^+ - S^-)]|S, -S\rangle$ considered by Wódkiewicz and Eberly. These states lead to the equality sign in (1) and satisfy the condition (2). The angular momentum states satisfying (3) have been constructed [3] and their possible production mechanisms have been discussed [4]. Note the similarity of the eigenvalue Eq. (3) defining the squeezed atomic states to the corresponding problem for bosons [5]. Note that the squeezing condition (2) defined from the usual considerations of the uncertainty principle may have to be modified in the light of a given experiment to be performed [1,6]. For example, Wineland *et al.* [1] have found in the context of Ramsey spectroscopy that one would like to have the parameter ξ_R defined by

$$\begin{aligned} \xi_R &= \sqrt{2S} (\Delta S_x) / |\langle S_z \rangle| \\ &= \sqrt{(S/|\langle S_z \rangle|)} (\sqrt{2}\Delta S_x / \sqrt{|\langle S_z \rangle|}) \end{aligned} \tag{4}$$

less than 1:

$$\xi_R < 1. \tag{5}$$

Although $\sqrt{2}\Delta S_x / \sqrt{|\langle S_z \rangle|}$ in (4) is less than one by virtue of (2) but $S/|\langle S_z \rangle| \geq 1$. Hence, the states satisfying (2) do not necessarily satisfy (5). Wineland *et al.* [1] showed that the states satisfying (5) can be produced by an effective interaction of the form $(S_+ a + a^+ S_-)$, where the oscillator mode is in a squeezed state. In our earlier work [4] we considered a similar interaction except that our model involved interaction with a broadband of squeezed oscillator modes. We showed that such a model produces the atomic squeezed states satisfying the condition (2). In view of the similarities of the two models and in view of the spectroscopic importance of the atomic squeezed states, it is desirable to find out whether the states (3) would be suitable candidates for the spectroscopic applications, i.e., whether they satisfy condition (5) as well. That is the objective of the present paper. We calculate the parameter ξ_R for the states (3) and show that it is indeed bounded by unity from above. We give numerical results for ξ_R as a function of the parameter θ and show that ξ_R can become much smaller than 1. We will refer to ξ_R as the spectroscopic squeezing parameter.

Squeezed states satisfying the equality in the Heisenberg uncertainty relation (1) for a collection of N two-level atoms are given by [3]

$$\begin{aligned} |\psi_m\rangle &= A_m \exp(\theta S_z) \exp(-i\pi S_y/2) |S, m\rangle, \\ m &= -S, -S+1, \dots, S, \end{aligned} \tag{6}$$

where A_m is the normalization constant and \mathbf{S} represents the angular-momentum operator corresponding to the spin value $S = N/2$. For simplicity we let $\theta > 0$. The state $|S, m\rangle$, as usual, is the simultaneous eigenstate of S^2 and S_z with eigenvalues $S(S+1)$ and m , respectively. As shown in Ref. 3, the state $|\psi_m\rangle$ defined by (6) is the solution of the eigenvalue problem (3) with $\lambda = 2m$ so that (3) can be rewritten as

$$[S_x \cosh(\theta) + i \sinh(\theta) S_y] |\psi_m\rangle = m |\psi_m\rangle. \tag{7}$$

Note that $|\psi_m\rangle$ is an eigenstate of a non-self-adjoint operator but the eigenvalues are real. The phase space representations of these states are discussed by Agarwal, Dowling, and Schleich [7]. A class of the squeezed spin states involving unitary transformation on coherent spin states has been proposed by Kitagawa and Ueda [6]. From (7) it is clear that

$$\langle \psi_m | S_x | \psi_m \rangle = m / \cosh(\theta), \quad \langle \psi_m | S_y | \psi_m \rangle = 0. \quad (8)$$

To determine $\langle S_x^2 \rangle$ and $\langle S_y^2 \rangle$ we use (7) in the forms

$$\begin{aligned} \langle \psi_m | [S_x \cosh(\theta) - i \sinh(\theta) S_y] \\ \times [S_x \cosh(\theta) + i \sinh(\theta) S_y] | \psi_m \rangle = m^2 \langle \psi_m | \psi_m \rangle, \end{aligned} \quad (9)$$

$$\langle \psi_m | [S_x \cosh(\theta) + i \sinh(\theta) S_y]^2 | \psi_m \rangle = m^2 \langle \psi_m | \psi_m \rangle,$$

and arrive at the following mean value equations

$$\begin{aligned} \cosh^2(\theta) \langle S_x^2 \rangle - \sinh^2(\theta) \langle S_y^2 \rangle &= m^2, \\ \langle S_x S_y \rangle + \langle S_y S_x \rangle &= 0, \end{aligned} \quad (10)$$

$$\begin{aligned} \cosh^2(\theta) \langle S_x^2 \rangle + \sinh^2(\theta) \langle S_y^2 \rangle \\ - \sinh(\theta) \cosh(\theta) \langle S_z \rangle = m^2. \end{aligned}$$

On combining Eqs. (8) and (10) we find the uncertainty in

$$\begin{aligned} \langle S_z \rangle &= A_m^2 \langle S, m | \exp(i\pi S_y / 2) \exp(\theta S_z) S_z \exp(\theta S_z) \exp(-i\pi S_y / 2) | S, m \rangle \\ &= A_m^2 \frac{d \langle S, m | \exp(i\pi S_y / 2) \exp(2\theta S_z) \exp(-i\pi S_y / 2) | S, m \rangle}{d(2\theta)} \\ &= A_m^2 d(A_m^{-2}) / d(2\theta) \equiv d(\ln(A_m^{-2})) / d(2\theta), \end{aligned} \quad (16)$$

and hence

$$\xi_R^{-2} = \frac{1}{S \tanh(\theta)} \frac{d(\ln(D))}{d(2\theta)}, \quad D \equiv A_m^{-2}. \quad (17)$$

To determine D we proceed as follows:

$$\begin{aligned} D &= \langle S, m | \exp(i\pi S_y / 2) \exp(2\theta S_z) \exp(-i\pi S_y / 2) | S, m \rangle \\ &= \langle S, m | \exp(-2\theta S_x) | S, m \rangle, \end{aligned} \quad (18)$$

where we have used the property that the rotation operator $\exp(i\pi S_y / 2)$ represents a rotation by $\pi/2$ about the y direction. On using the disentangling theorem [9]

$$\begin{aligned} \exp[-\theta(S_+ + S_-)] &= \exp(-\tau S_+) (1 - \tau^2)^S z \exp(-\tau S_-), \\ \tau &\equiv \tanh(\theta), \end{aligned} \quad (19)$$

Eq. (18) gives

$$\begin{aligned} D &= \langle S, m | \exp(-\tau S_+) (1 - \tau^2)^S z \exp(-\tau S_-) | S, m \rangle \\ &= \sum_{p=0}^{\infty} \frac{(\tau)^{2p} (1 - \tau^2)^{m-p}}{(p!)^2} \langle S, m | (S_+)^p (S_-)^p | S, m \rangle \\ &= \sum_{p=0}^{S+m} \frac{(\tau)^{2p} (1 - \tau^2)^{m-p}}{(p!)^2} \frac{(S+m)!(S-m+p)!}{(S+m-p)!(S-m)!}. \end{aligned} \quad (20)$$

We now prove that $\xi_R \leq 1$ for all S, m , and θ . To that

S_x to be given by

$$\Delta S_x^2 = \tanh(\theta) \langle S_z \rangle / 2. \quad (11)$$

Note that for $\theta > 0$, $1 > \tanh(\theta) > 0$, $\langle S_x \rangle$ should be positive and hence

$$\Delta S_x^2 < \langle S_z \rangle / 2. \quad (12)$$

Thus, the component S_x exhibits squeezing. We also find that

$$\Delta S_y^2 = \langle S_z \rangle \coth(\theta) / 2, \quad (13)$$

which, along with (11), leads to

$$\Delta S_x^2 \Delta S_y^2 = \langle S_z \rangle^2 / 4. \quad (14)$$

The state $|\psi_m\rangle$ is thus a minimum-uncertainty state.

We next examine the value of the spectroscopic squeezing parameter ξ_R defined by (4) for the system in the state [8] $|\psi_m\rangle$. On using (11) in (4) we get

$$\xi_R^{-2} = |\langle S_z \rangle| / S \tanh(\theta). \quad (15)$$

We thus need to know the value of $\langle S_z \rangle$ in the state $|\psi_m\rangle$ which we now compute. On using (6), we get

end, define $x = \tau^2 \equiv \tanh^2(\theta)$ so that Eqs. (17) and (20) read

$$\xi_R^{-2} = \frac{(1-x)}{S} \frac{d(\ln(D))}{dx}, \quad (21)$$

$$\begin{aligned} D &= \sum_{p=0}^{S+m} \frac{(S+m)!(S-m+p)! x^p (1-x)^{m-p}}{(S+m-p)!(S-m)!(p!)^2} \\ &\equiv (1-x)^m F(-S-m, S-m+1; 1; -x/(1-x)), \end{aligned} \quad (22)$$

where $F(a, b; c; z)$ is the hypergeometric function. Using the relation [10]

$$F(a, b; c; z) = (1-z)^{-a} F(a, c-b; c; -z/(1-z)), \quad (23)$$

the expression (22) for D reduces to

$$\begin{aligned} D &= (1-x)^{-S} F(-S-m, -S+m; 1; x), \\ &\equiv (1-x)^{-S} \sum_{p=0}^{S-|m|} \frac{(S+m)!(S-m)! x^p}{(S+m-p)!(S-m-p)!(p!)^2}. \end{aligned} \quad (24)$$

Finally, substituting Eq. (24) in Eq. (21) we get

$$\begin{aligned} \xi_R^{-2} &= 1 + \frac{(1-x)^{-S+1}}{DS} \\ &\times \sum_{p=0}^{S-|m|} \frac{(S+m)!(S-m)! p x^{p-1}}{(S-m-p)!(S+m-p)!(p!)^2}. \end{aligned} \quad (25)$$

Since the second term in (25) is positive semidefinite, it follows that $\xi_R \leq 1$.

Note that ξ_R is symmetric under $m \rightarrow -m$, i.e., ξ_R has the same value in the states $|\psi_m\rangle$ and $|\psi_{-m}\rangle$. Also, it is clear from (25) that

$$\xi_R = 1, \quad m = \pm S, \tag{26}$$

whereas $\xi_R < 1$ if $m \neq \pm S$. Analytic expressions for ξ_R can also be obtained in certain limiting cases. In the limit $\theta \rightarrow 0$, i.e., $x \rightarrow 0$, it is straightforward to show that

$$\xi_R^{-2} \rightarrow 1 + \frac{(S+m)(S-m)}{S} \quad \text{as } \theta \rightarrow 0. \tag{27}$$

In particular,

$$\xi_R \rightarrow (1+S)^{-1/2} \quad \text{for } m=0, \quad \theta \rightarrow 0. \tag{28}$$

Hence, the spectroscopic squeezing parameter ξ_R is of the order of $1/\sqrt{S}$ for large N and for $m \approx 0$. The state $|\psi_0\rangle$ [Eq. (6)] should, therefore, be of practical interest in the context of the Ramsey spectroscopy.

For $\theta \rightarrow \infty$, i.e., $x \rightarrow 1$, on the other hand, it follows from (25) that

$$\xi_R \rightarrow 1 \quad \text{as } \theta \rightarrow \infty. \tag{29}$$

We show by means of numerical calculations that ξ_R , in fact, increases monotonically with θ .

The results of numerical evaluation of ξ_R are presented in Figs. 1 and 2. In Fig. 1, we have plotted ξ_R as a function of θ for a fixed $N \equiv 2S = 20$ and for $m = 0, \pm 4, \pm 6, \pm 8$. Clearly, ξ_R increases monotonically from the limiting value (27) as $\theta \rightarrow 0$ to 1 as $\theta \rightarrow \infty$. Also, ξ_R increases with increasing $|m|$ for a given value of θ . In Fig. 2, we have plotted ξ_R as a function of θ for a fixed $m = 0$ and for $N = 2, 4, 10, 20$. It is seen that the higher the number N of atoms the lower the value of ξ_R is for a given θ . The figures show that ξ_R indeed may become very small thereby making the atomic squeezed states very useful for the spectroscopic applications where a reduction in noise is required.

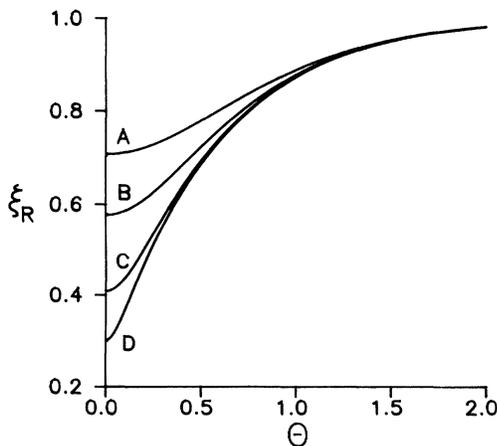


FIG. 1. The spectroscopic squeezing parameter ξ_R as a function of θ for $N=20$. The uppermost curve is for $m = \pm 8$ followed by the curves for $m = \pm 6, \pm 4$, and 0, respectively.

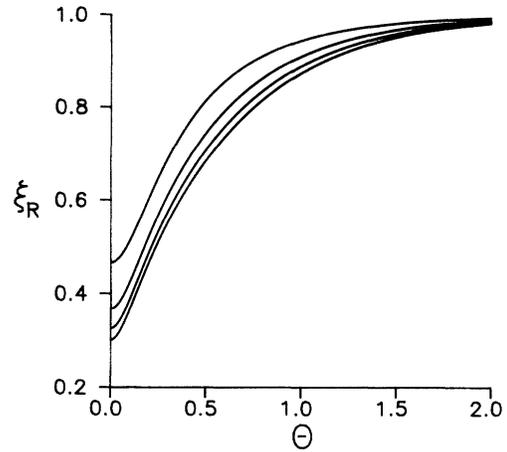


FIG. 2. The spectroscopic squeezing parameter ξ_R as a function of θ for $m=0$. The uppermost curve is for $N=2$ followed by the curves for $N=4, 10$, and 20 respectively.

Let us now discuss the experimental conditions required for the generation of the states (6) and their relationship with the parameter θ . It has been shown in Ref. [4] that a system of N two-level atoms interacting with a broadband squeezed bath has the state (6) as its steady state if N is even. A squeezed bath is constituted, for example, by the radiation from a cavity containing a non-linear medium undergoing degenerate two-photon down conversion on pumping by an external field [11]. That radiation is multimode with pairwise correlation between the modes separated in frequency by the frequency of the pump. The longest correlation time τ_c between the modes is $(\chi - |\epsilon|)^{-1}$, where $|\epsilon|$ is the parametric amplification factor and 2χ is the rate of leakage of the photons from the cavity with the condition $\chi \geq |\epsilon|$ required for stability. If τ_c is much shorter than the time of decay of the atoms with which the field interacts then it can be considered as a broadband field. The parameter θ in (6) is then given in terms of the parameters of the parametric amplifier by the relation

$$\exp(2\theta) = 4\chi|\epsilon|(\chi^2 + |\epsilon|^2) / [4\chi^2|\epsilon|^2 + (\chi^2 + |\epsilon|^2)^2]. \tag{30}$$

Clearly the value $\theta=0$, for which there is maximum spectroscopic squeezing, is obtained for $\chi = |\epsilon|$. That is also the value for which the bath attains optimal squeezing [11] (which as is well known corresponds to 50% squeezing inside the cavity and much larger squeezing outside).

The spectroscopic squeezing in $|\psi_m\rangle$ is associated with the finiteness of the correlation

$$C_{\mu\nu}^{ij} = \langle S_{\mu}^i S_{\nu}^j \rangle - \langle S_{\mu}^i \rangle \langle S_{\nu}^j \rangle, \tag{31}$$

between the μ th and the ν th components ($\mu, \nu = x, y, z$) of the spin vectors of the i th and the j th atoms. The correlation function vanishes for a state which is a product of single-atom states. We evaluate C_{xx}^{ij} for the squeezed atomic state and show that the spins in that state are correlated whenever $\xi_R < 1$. There are no spin-spin correlations if $\xi_R = 1$.

Note that the state $|\psi_m\rangle$ is obtained, as is seen from its definition (6), by a collective atomic transformation on

the collective atomic state $|S, m\rangle$. However, the transformation operator in (6) is evidently a product of the single-atom operators but the state $|S, m\rangle$, and consequently $|\psi_m\rangle$, cannot be expressed as a product of the single-atom states unless $m = \pm S$. Hence, the spin-spin correlations in state $|\psi_m\rangle$ can be nonzero if $m \neq \pm S$ and those are the values of m for which $\xi_R < 1$. On the other hand, $|S, \pm S\rangle$ and hence $|\psi_{\pm S}\rangle$ is a product of the single-atom states. Hence, the spin-spin correlations are absent if $m = \pm S$ and, for those values of m , $\xi_R = 1$. We evaluate C_{xx}^{ij} and show that the spins in the squeezed atomic state $|\psi\rangle$ are indeed correlated.

Since the state $|\psi_m\rangle$ is a collective atomic state, the evaluation of $C_{\mu\nu}^{ij}$ in that the state involves finding the mean values of the single-atom operators in collective atomic states. The problem of the evaluation of those mean values can be reduced to the evaluation of the mean values of collective atomic operators by invoking the fact that the collective atomic state $|\psi_m\rangle$ is symmetric under the exchange of the atoms so that

$$\begin{aligned} \langle S_\mu^i \rangle &= \langle S_\mu \rangle / N, \\ \langle S_\mu^i S_\nu^j \rangle &= \left[\langle S_\mu S_\nu \rangle - \sum_i \langle S_\mu^i S_\nu^i \rangle \right] / [N(N-1)]. \end{aligned} \quad (32)$$

Since S_μ^i are spin- $\frac{1}{2}$ operators therefore the product $S_\mu^i S_\nu^i$ can always be reduced to a single spin- $\frac{1}{2}$ operator. Hence, the term $\sum_i S_\mu^i S_\nu^i$, appearing in (32) is also reducible to a collective atomic operator.

Let us now evaluate the spin-spin correlation function C_{xx}^{ij} . Using Eqs. (31) and (32) it can be shown that

$$C_{xx}^{ij} = \left[\Delta S_x^2 - \frac{N}{4} + \frac{\langle S_x \rangle^2}{N} \right] / [N(N-1)]. \quad (33)$$

On using the expression (8) for $\langle S_x \rangle$; Eq. (11) to express ΔS_x^2 in terms of $\langle S_z \rangle$, and Eq. (15) to write $\langle S_z \rangle$ in terms of ξ_R , we get

$$C_{xx}^{ij} = \frac{N}{4} [x \xi_R^{-2} - 1 + 4m^2(1-x)/N^2] / [N(N-1)]. \quad (34)$$

Substituting the expression (25) for ξ_R in (34) it can be shown that

$$\begin{aligned} C_{xx}^{ij} &= \frac{-N(1-x)}{4D} \\ &\times \sum_{p=0}^{S-|m|} \frac{(S+m)!(S-m)!x^p[1-2p/N-4m^2/N^2]}{(S-m-p)!(S+m-p)!(p!)^2N(N-1)}. \end{aligned} \quad (35)$$

Since $1-2p/N-4m^2/N^2 > 0$ for $0 < p < S-|m|$ for $|m| < S$, it follows that $C_{xx}^{ij} < 0$, i.e., the spin-spin correlations in the state $|\psi_m\rangle$ ($m \neq \pm S$) are finite and negative. For $|m| = S$, on the other hand, it is clear from (35) that $C_{xx}^{ij} = 0$.

In conclusion, we have shown that the spectroscopic squeezing parameter ξ_R is always less than or equal to one for the atomic squeezed states and that ξ_R may, in fact, become very small in some cases thereby making these states very useful in spectroscopic applications. We have also demonstrated that, for $\xi_R < 1$, the spin-spin correlations are finite and negative which may also be considered a signature of the nonclassical character of the state.

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