Photon distribution in two-mode squeezed coherent states with complex displacement and squeeze parameters

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Two-mode squeezed coherent states with complex squeeze and displacement parameters are studied, taking advantage of the SU(2) dynamical symmetry underlying two-mode systems. Expressions for the photon distributions in such states are derived using an SU(2) identity and the fact that the two-mode squeeze operator can be viewed as a rotated version of the product of reciprocal single-mode squeeze operators. An important $U(1)\times U(1)$ invariance of this photon distribution is established. As a consequence, the three phases of the complex squeeze and displacement parameters enter the photon distribution through just one $U(1)\times U(1)$ invariant combination. An associated Gouy effect is noted. Numerical examples of two-mode photon distributions are shown, and interesting new features demonstrated. Second-order coherence properties and their nonclassical nature are briefly studied.

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I. INTRODUCTION

Photon distribution in nonclassical states of light has been studied by several authors [1]. Interest in such studies was triggered, in part at least, by the work of Schleich and Wheeler [2], who showed that photon distribution in the squeezed coherent state of a single-mode system has an oscillatory behavior. They further suggested that such an oscillatory behavior of the photon distribution can be taken as signature of the nonclassical nature of the state involved.

More recently, Dutta *et al.* [3] made a more systematic study of the single-mode squeezed coherent state with *complex* squeeze and displacement parameters, and showed that in some range of the complex parameters the oscillations in the photon distribution exhibit collapses and revivals somewhat similar to the one familiar in the Jaynes-Cummings model [4]; the former is in the photon number domain rather than in the time domain. It should be emphasized that this new beat behavior in the photon distribution is not shared by squeezed coherent states with real squeeze and displacement parameters studied by other authors.

In an interesting recent work, Caves *et al.* [5] studied the two-mode squeezed coherent state, and demonstrated some interesting features of the photon distribution in such states for real values of the two-mode squeeze and displacement parameters. We wish to note in particular the qualitatively different distributions obtained for parallel ($\alpha_1=\alpha_2$) and antiparallel ($\alpha_1=-\alpha_2$) values of the real displacement parameters, shown respectively in Figs. 1(b) and 2(b) of their work. The striking difference in the two cases tempts one to ask: How does the photon distribution interpolate between these extreme ends? Motivated by this question, and by the work of Dutta *et al.* [3], which demonstrated the phase sensitivity of the photon distribution in the single-mode case, we study in this paper the photon distribution in two-mode squeezed coherent states with *complex* squeeze and displacement parameters.

The content of this paper is organized as follows. In Sec. II we develop an expression for the photon number distribution in an arbitrary two-mode squeezed coherent state with complex squeeze and displacement parameters. The analysis of Caves *et al.* is based on normal ordering techniques. Our approach is symmetry based: we exploit the SU(2) dynamical symmetry underlying two-mode systems. This allows us to view the two-mode squeeze operator as a rotated version of product of reciprocal single-mode squeezings. Thus the probability amplitude for the photon distribution becomes a linear combination of product of the well known single-mode Yuen matrix elements [6] given in terms of Hermite polynomials, the coefficients of the linear combination being determined by the matrix elements of a particular SU(2) rotation. Finally, an identity relating associated Laguerre polynomial helps us to write the probability amplitude in terms of a single associated Laguerre polynomial. It is of interest to note that this identity itself is an immediate consequence of the SU(2) structure. Conformity of our final result with that of Caves *et al.* is noted.

In Sec. III we bring out the fact that this two-mode photon distribution possesses a $U(1)\times U(1)$ invariance property. As a consequence, even though our problem has three phases (each one arising from the two displacement parameters and the third from the squeeze parameter), the photon distribution depends only on one $U(1)\times U(1)$-invariant linear combination χ of these phases. We bring out also a Gouy phase [7] in the manner in which this invariant χ influences the argument of the associated Laguerre polynomial.

Some examples of photon distribution are studied numerically in Sec. IV. Our principal aim is to bring out the sensitivity of the photon distribution to the U(1)

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$\times$U(1) invariant χ. It will be seen that while our results are in conformity with the results of Caves *et al.* for those values of χ that correspond to their studies, there are new interesting features for other values.

In Sec. V we study some properties which turn out to be invariant to the phases. Second-order coherence functions are briefly considered in Sec. VI, and it is shown that they exhibit nonclassical behavior in some range of χ. And we conclude with some final remarks in Sec. VII.

II. PHOTON DISTRIBUTION

The general two-mode squeezed coherent state is unitarily related to $|\text{vac}\rangle=|0,0\rangle$, the ground state of the two-mode system described by annihilation operators a and b, in the following familiar manner:

$$\begin{aligned}
&|z;\alpha_1,\alpha_2\rangle=D(\alpha_1,\alpha_2)S(z)|0,0\rangle\ ,\\
&S(z)=\exp(z^*ab-za^\dagger b^\dagger)\ ,\quad D(\alpha_1,\alpha_2)=D(\alpha_1)D(\alpha_2)\ ,\\
&D(\alpha_1)=\exp(\alpha_1 a^\dagger-\alpha_1^* a)\ ,\quad D(\alpha_2)=\exp(\alpha_2 b^\dagger-\alpha_2^* b)\ .
\end{aligned}\tag{1}$$

Here z is a complex two-mode squeeze parameter and α_1, and α_2 are complex displacement (coherent excitation) parameters. Detailed analysis of two-mode squeezed coherent states has been made by several authors [8]. In the above definition we have allowed, following Caves *et al.* [5], the squeeze operator to act on vacuum and then displace the resulting two-mode squeezed vacuum. Sometimes it will be more convenient to order these operations the other way in the definition of the squeezed coherent state. Both definitions are equivalent, and we have the following identity:

$$\begin{aligned}
&|z;\alpha_1,\alpha_2\rangle=S(z)D(\tilde\alpha_1,\tilde\alpha_2)|0,0\rangle\ ,\\
&\tilde\alpha_1=\alpha_1\mu+\alpha_2^*\nu\ ,\quad \tilde\alpha_2=\alpha_2\mu+\alpha_1^*\nu\ ,\\
&z=re^{2i\phi}\ ,\quad \mu=\cosh r\ ,\quad \nu=e^{2i\phi}\sinh r\ .
\end{aligned}\tag{2}$$

The photon distribution $P(n_1,n_2)$ in the two-mode squeezed coherent state $|z;\alpha_1,\alpha_2\rangle$ is given by

$$\begin{aligned}
P(n_1,n_2)&=|c(n_1,n_2)|^2\ ,\\
c(n_1,n_2)&=\langle n_1,n_2|z;\alpha_1,\alpha_2\rangle\\
&=\langle n_1,n_2|S(z)D(\tilde\alpha_1,\tilde\alpha_2)|0,0\rangle\ ,
\end{aligned}\tag{3}$$

where $|n_1,n_2\rangle$ are the familiar Fock states of the two-mode systems. We will compute $P(n_1,n_2)$ in several steps.

As the first step, we exploit the dynamical SU(2) symmetry underlying the two-mode system. Two-boson realization of the SU(2) symmetry is originally due to Schwinger [9] and has more recently played an important role in quantum optics [10, 11]. The basis of all these applications is the easily verified fact that the Hermitian operators J_1,J_2,J_3 defined through

$$J_1=\frac{a^\dagger b+b^\dagger a}{2}\ ,\quad J_2=-i\frac{a^\dagger b-b^\dagger a}{2}\ ,\quad J_3=\frac{a^\dagger a-b^\dagger b}{2}\tag{4}$$

satisfy the SU(2) algebra $[J_k,J_l]=i\epsilon_{klm}J_m$. This fact becomes obvious if one notes that $J_k=\frac{1}{2}\xi^\dagger\sigma_k\xi$, where ξ is a two-element column vector with entries a, b, and σ_k being the Pauli matrices. With the help of these SU(2) generators, we can write our two-mode squeeze operator $S(z)$ as

$$S(z)=\exp\left[-i\frac{\pi}{2}J_2\right]S_a(z)S_b(-z)\exp\left[i\frac{\pi}{2}J_2\right]\ ,\tag{5}$$

where $S_a(z)$, $S_b(-z)$ are the single-mode squeeze operators

$$\begin{aligned}
S_a(z)&=\exp[\tfrac{1}{2}(z^*a^2-za^{\dagger 2})]\ ,\\
S_b(-z)&=\exp[-\tfrac{1}{2}(z^*b^2-zb^{\dagger 2})]\ .
\end{aligned}\tag{6}$$

Since $\exp[-i(\pi/2)J_2]$ produces $\pi/4$ rotation in the mode space, the important identity (5) shows that our two-mode squeeze operator $S(z)$ is indeed a rotated version of product of single-mode squeeze operators producing *reciprocal* squeezing.

When the identity (5) is used in (3) we obtain

$$\begin{aligned}
c(n_1,n_2)=\langle n_1,n_2|&e^{-i(\pi/2)J_2}S_a(z)S_b(-z)\\
&\times e^{i(\pi/2)J_2}D(\tilde\alpha_1,\tilde\alpha_2)|0,0\rangle\ .
\end{aligned}\tag{7}$$

Since

$$\exp\left[i\frac{\pi}{2}J_2\right]D(\tilde\alpha_1,\tilde\alpha_2)\exp\left[-i\frac{\pi}{2}J_2\right]=D\left[\frac{\tilde\alpha_1+\tilde\alpha_2}{\sqrt2},\frac{\tilde\alpha_2-\tilde\alpha_1}{\sqrt2}\right]\ ,\tag{8}$$

and since $\exp[i(\pi/2)J_2]$ acts as identity operator on $|0,0\rangle$, we have the useful relation

$$e^{i(\pi/2)J_2}D(\tilde\alpha_1,\tilde\alpha_2)|0,0\rangle=\left|\frac{\tilde\alpha_1+\tilde\alpha_2}{\sqrt2},\frac{\tilde\alpha_2-\tilde\alpha_1}{\sqrt2}\right\rangle\ .\tag{9}$$

This allows us to rewrite (7) as

$$c(n_1,n_2)=\sum_{n_1',n_2'}\langle n_1,n_2|e^{-i(\pi/2)J_2}|n_1',n_2'\rangle\left\langle n_1',n_2'|S_a(z)S_b(-z)|\frac{\tilde\alpha_1+\tilde\alpha_2}{\sqrt2},\frac{\tilde\alpha_2-\tilde\alpha_1}{\sqrt2}\right\rangle\ .\tag{10}$$

As the next step, we recognize that the matrix elements entering (10) are well known from other contexts. The expression $\langle n_1,n_2|e^{-i(\pi/2)J_2}|n_1',n_2'\rangle$ are the Wigner matrix elements [12] familiar from the quantum theory of angular momentum:

$$\langle n_1,n_2|e^{-i(\pi/2)J_2}|n_1',n_2'\rangle=d^j_{mm'}\left[\frac{\pi}{2}\right]=d^j_{m'm}\left[-\frac{\pi}{2}\right]\ ,\tag{11}$$

where

$$j=(n_1'+n_2')/2=(n_1+n_2)/2,\quad m=(n_1-n_2)/2,\quad m'=(n_1'-n_2')/2\ ;$$

$$d^j_{m'm}\left[-\frac{\pi}{2}\right]=(-1)^{m'-m}d^j_{m'm}\left[\frac{\pi}{2}\right]$$
$$=(-1)^{m'-m}2^{-j}\sum_{\mu}(-1)^{\mu-m'+m}\frac{[(j+m')!(j-m')!(j+m)!(j-m)!]^{1/2}}{(j-m'-\mu)!(j+m+\mu)!\mu!(m'-m+\mu)!}\ . \tag{12}$$

The other expression in (10) is product of the Yuen matrix elements of single-mode squeeze operators between coherent states and Fock states [6,3],

$$\left\langle n_1',n_2'|S_a(z)S_b(-z)|\frac{\tilde\alpha_1+\tilde\alpha_2}{\sqrt2},\frac{\tilde\alpha_2-\tilde\alpha_1}{\sqrt2}\right\rangle=\left\langle n_1'|S_a(z)|\frac{\tilde\alpha_1+\tilde\alpha_2}{\sqrt2}\right\rangle\left\langle n_2'|S_b(-z)|\frac{\tilde\alpha_2-\tilde\alpha_1}{\sqrt2}\right\rangle,$$
$$\left\langle n_1'|S_a(z)|\frac{\tilde\alpha_1+\tilde\alpha_2}{\sqrt2}\right\rangle=(n_1'!\mu)^{-1/2}\left[\frac{\nu}{2\mu}\right]^{n_1'/2}H_n[(\tilde\alpha_1+\tilde\alpha_2)(4\mu\nu)^{-1/2}]\exp\left[-\tfrac14|(\tilde\alpha_1+\tilde\alpha_2)|^2+\frac{\nu^*}{4\mu}(\tilde\alpha_1+\tilde\alpha_2)^2\right]. \tag{13}$$

And $\langle n_2'|S_b(-z)|1/\sqrt2(\tilde\alpha_2-\tilde\alpha_1)\rangle$ has an expression similar to (13) with $(\tilde\alpha_1+\tilde\alpha_2)$ replaced by $(\tilde\alpha_2-\tilde\alpha_1)$ and ν by $-\nu$.

For our final step, we need the important identity [11,13]

$$\sum_{m'=-j}^{+j}\Omega^j_{m'm}[2^{2j}(j+m')!(j-m')!]^{-1/2}H_{j+m'}(x)H_{j-m'}(y)$$
$$=\exp\{-i\frac{\pi}{2}[2(j-|m|)-(j-m)]\}\left[\frac{(j-|m|)!}{(j+|m|)!}\right]^{1/2}(x^2+y^2)^{2|m|}L^{2|m|}_{j-|m|}(x^2+y^2)e^{2im\theta}\ , \tag{14}$$

$$\Omega^j_{m'm}=i^{m'-m}d^j_{m'm}\left[\frac{\pi}{2}\right]. \tag{15}$$

The above identity has played an important role in constructing the normal-mode spectrum of the twisted Gaussian Schell model beam in classical optics [11]. Its derivation is straightforward and brings out the power of viewing SU(2) described in [4] as the dynamical symmetry of the two-dimensional isotropic oscillator in the x-y plane. Eigenstates of such an oscillator can be constructed by either diagonalizing J_3, in which case the eigenstates will be products of Hermite polynomials in x and y, or by diagonalizing J_2, which generates rotations in the x-y plane in which case the eigenstates will be the rotationally covariant associated Laguerre polynomials in x^2+y^2. The identity (15) is a consequence of the fact that J_2 and J_3 are related through conjugation by $\exp[-i(\pi/2)J_1]$. In fact, $\Omega^j_{m'm}$ are the matrix elements of $\exp[-i(\pi/2)J_1]$ and the factor $i^{m'-m}$ in the relationship (15), arising from the fact that

$$\exp\left[-i\frac{\pi}{2}J_1\right]=\exp\left[i\frac{\pi}{2}J_3\right]\exp\left[-i\frac{\pi}{2}J_2\right]\times\exp\left[-i\frac{\pi}{2}J_3\right]. \tag{16}$$

It is important to appreciate that the identity (14) connecting the Hermite polynomials and the associated Laguerre polynomials is valid not only for real x,y but also for complex values of x,y with ρ,θ, defined, in either case, through $x+iy=\rho e^{i\theta}$ so that $\rho^2=x^2+y^2$, $e^{2i\theta}=(x^2-y^2+2ixy)/(x^2+y^2)$.

Using the expressions (12) and (13) in (10) and making use of the identity (14), we have our final expression,

$$c(n_1,n_2)\equiv c(j+m,j-m)$$
$$=\exp\left[-i\frac{\pi}{2}(j-|m|)\right]\left[\frac{(j-|m|)!}{(j+|m|)!}\right]^{1/2}[\tilde\alpha_1\tilde\alpha_2/(\mu\nu)]^{|m|}\mu^{-1}(\nu/\mu)^j$$
$$\times L^{2|m|}_{j-|m|}\left[\frac{\tilde\alpha_1\tilde\alpha_2}{\mu\nu}\right]\left[\frac{\tilde\alpha_1}{\tilde\alpha_2}\right]^m\exp\left[\frac{-|\tilde\alpha_1|^2-|\tilde\alpha_2|^2}{2}\right]\exp\left[\frac{\nu^*\tilde\alpha_1\tilde\alpha_2}{\mu}\right]. \tag{17}$$

The double sum over n_1', n_2' in (10) reduced to a single sum over $m'=(n_1'-n_2')/2$ owing to the fact that the rotation matrix element $\langle n_1,n_2|e^{-i\pi/2J_2}|n_1',n_2'\rangle$ in (10) is nonzero only when $n_1+n_2=n_1'+n_2'$, thus enabling us to use the identity (15).

To relate our final expression to that of Caves *et al.*, we note that $p=j-|m|$ is the smaller of n_1, n_2 and $q=j+|m|$ is the larger. Thus substituting for μ, ν from

(2), we can rewrite (17) as

$$c(n_1,n_2)=(-1)^p\left[\frac{p!}{q!}\right]^{1/2}\tilde{\alpha}_1^{n_1-p}\tilde{\alpha}_2^{n_2-p}\frac{(\tanh r)^p}{\cosh r}(e^{2i\phi})^p \times L_p^{q-p}\left[\frac{2\tilde{\alpha}_1\tilde{\alpha}_2}{\sinh 2r}e^{-2i\phi}\right] \times\exp\left[\frac{-(\alpha_1^*\tilde{\alpha}_1+\alpha_2^*\tilde{\alpha}_2)}{2\cosh r}\right] . \qquad (18)$$

It is seen that (18), for real values of the parameters, indeed reproduces Eq. (2.14) of Caves *et al.*, since their $\mu_j=\tilde{\alpha}_j/\cosh r$.

III. U(1)×U(1) INVARIANCE AND GUOY EFFECT

We have three complex parameters in the problem. These are $z=|z|e^{2i\phi}$, $\alpha_1=|\alpha_1|e^{i\theta_1}$, and $\alpha_2=|\alpha_2|e^{i\theta_2}$. However, symmetry considerations should convince one that the phases θ_1, θ_2, ϕ will not enter the photon distribution independently. To see this, let us write $c(n_1,n_2)$ in more detail as

$$c(n_1,n_2;z;\alpha_1,\alpha_2)=\langle n_1,n_2|z;\alpha_1,\alpha_2\rangle . \qquad (19)$$

Now note that

$$\exp(i\zeta_1 a^\dagger a)\exp(i\zeta_2 b^\dagger b)|z;\alpha_1,\alpha_2\rangle =|ze^{i(\zeta_1+\zeta_2)};\alpha_1 e^{i\zeta_1},\alpha_2 e^{i\zeta_2}\rangle .$$

Projecting onto the Fock state $|n_1,n_2\rangle$ we have

$$c(n_1,n_2;ze^{i(\zeta_1+\zeta_2)};\alpha_1 e^{i\zeta_1},\alpha_2 e^{i\zeta_2}) =e^{i(n_1\zeta_1+n_2\zeta_2)}c(n_1,n_2;z;\alpha_1,\alpha_2) .$$

We see that under the U(1)×U(1) transformations generated by $\exp(i\zeta_1 a^\dagger a)$, $\exp(i\zeta_2 b^\dagger b)$, the probability amplitude $c(n_1,n_2;z;\alpha_1,\alpha_2)$ defined in (18) changes only by a phase. Since the photon distribution is given by the square of the absolute value of this amplitude, we see that it has U(1)×U(1) invariance:

$$P(n_1,n_2;ze^{i(\zeta_1+\zeta_2)},\alpha_1 e^{i\zeta_1},\alpha_2 e^{i\zeta_2})=P(n_1,n_2;z;\alpha_1,\alpha_2) . \qquad (20)$$

This U(1)×U(1) invariance is analogous to the U(1) invariance in the single-mode case [3] and implies that our photon distribution will depend on the three phases θ_1, θ_2 and ϕ only through the U(1)×U(1)-invariant combination $\theta_1+\theta_2-2\phi$.

It may be instructive to verify that the photon distribution described by the probability amplitude given in (18) indeed possesses this U(1)×U(1) invariance. To this end, note that under the transformation

$$\alpha_j\to\alpha_j e^{i\zeta_j} , \quad z\to ze^{i(\zeta_1+\zeta_2)} , \qquad (21)$$

we have $\theta_j\to\theta_j+\zeta_j$, $\nu\to\nu e^{i(\zeta_1+\zeta_2)}$, and $2\phi\to 2\phi+\zeta_1+\zeta_2$. Further, it is clear from (2) that $\tilde{\alpha}_j\to\tilde{\alpha}_j e^{i\zeta_j}$ under (21). Thus,

$$\tilde{\alpha}_1^{n_1-p}\tilde{\alpha}_2^{n_2-p}(e^{2i\phi})^p\to\tilde{\alpha}_1^{n_1-p}\tilde{\alpha}_2^{n_2-p}(e^{2i\phi})^p e^{i(n_1\zeta_1+n_2\zeta_2)} .$$

To complete the verification we will now show that the argument of the associated Laguerre polynominal as well as the exponent in the last factor in (18) are functions of only the U(1)×U(1) invariant combination $\chi=\theta_1+\theta_2-2\phi$. From (2) connecting the α's to the $\tilde{\alpha}$'s we have

$$\alpha_1^*\tilde{\alpha}_1+\alpha_2^*\tilde{\alpha}_2=(|\alpha_1|^2+|\alpha_2|^2)\cosh r+2|\alpha_1\alpha_2|\sinh r\cos\chi -i2|\alpha_1\alpha_2|\sinh r\sin\chi . \qquad (22)$$

We further deduce from (2),

$$\tilde{\alpha}_1\tilde{\alpha}_2 e^{-2i\phi}=\tfrac{1}{2}(|\alpha_1|^2+|\alpha_2|^2)\sinh 2r +2|\alpha_1\alpha_2|(\cosh 2r\cos\chi+i\sin\chi) . \qquad (23)$$

Thus, the expression (17) has the behavior required by (19) under the U(1)×U(1) transformation (21), showing explicitly that our photon distribution is indeed U(1)×U(1) invariant. Having appreciated this fact, we switch for brevity to use of $P(n_1,n_2)$, rather than $P(n_1,n_2;z;\alpha_1,\alpha_2)$, to denote the photon distribution.

Our analysis in the foregoing paragraphs shows that there are only two ways in which χ, the U(1)×U(1) invariant combination of the phases of α_1, α_2, and z, enters the photon number distribution: through the exponent as in (22), and through the associated Laguerre polynomial as in (23). The former one is independent of n_1,n_2 and hence contributes to the overall amplitude of the distribution. That is, it just ensures the fact that $P(n_1,n_2)$

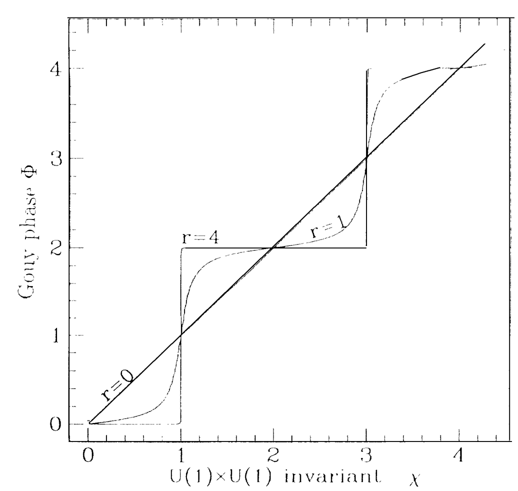

FIG. 1. The Gouy phase Φ for different values of the U(1)×U(1) invariant phase χ. Both Φ and χ are in units of π and $\alpha_1=\alpha_2=7$.

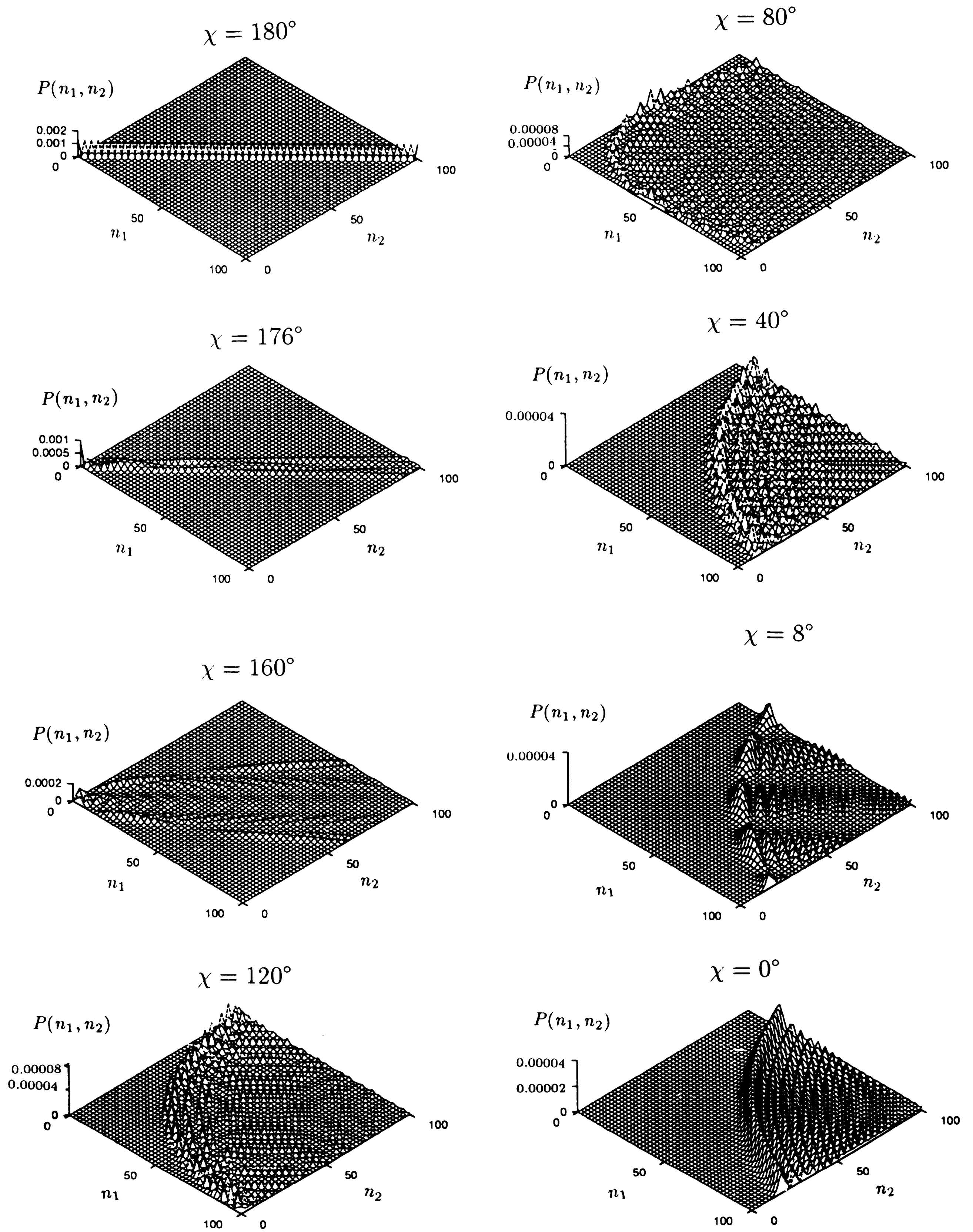

FIG. 2. Photon number distribution $P(n_1,n_2)$ as a function of n_1 and n_2 for $\alpha_1=\alpha_2=7.00$ and $r=4.00$. The distribution is concentrated along the diagonal for $\chi=180°$. As χ decreases, oscillations along and perpendicular to the diagonal pick up and saturate around $\chi=120°$. Thereafter, there is a gradual collapse of the oscillations perpendicular to the diagonal which evolves with decreasing χ towards the parabola like ripple structure at $\chi=0°$.

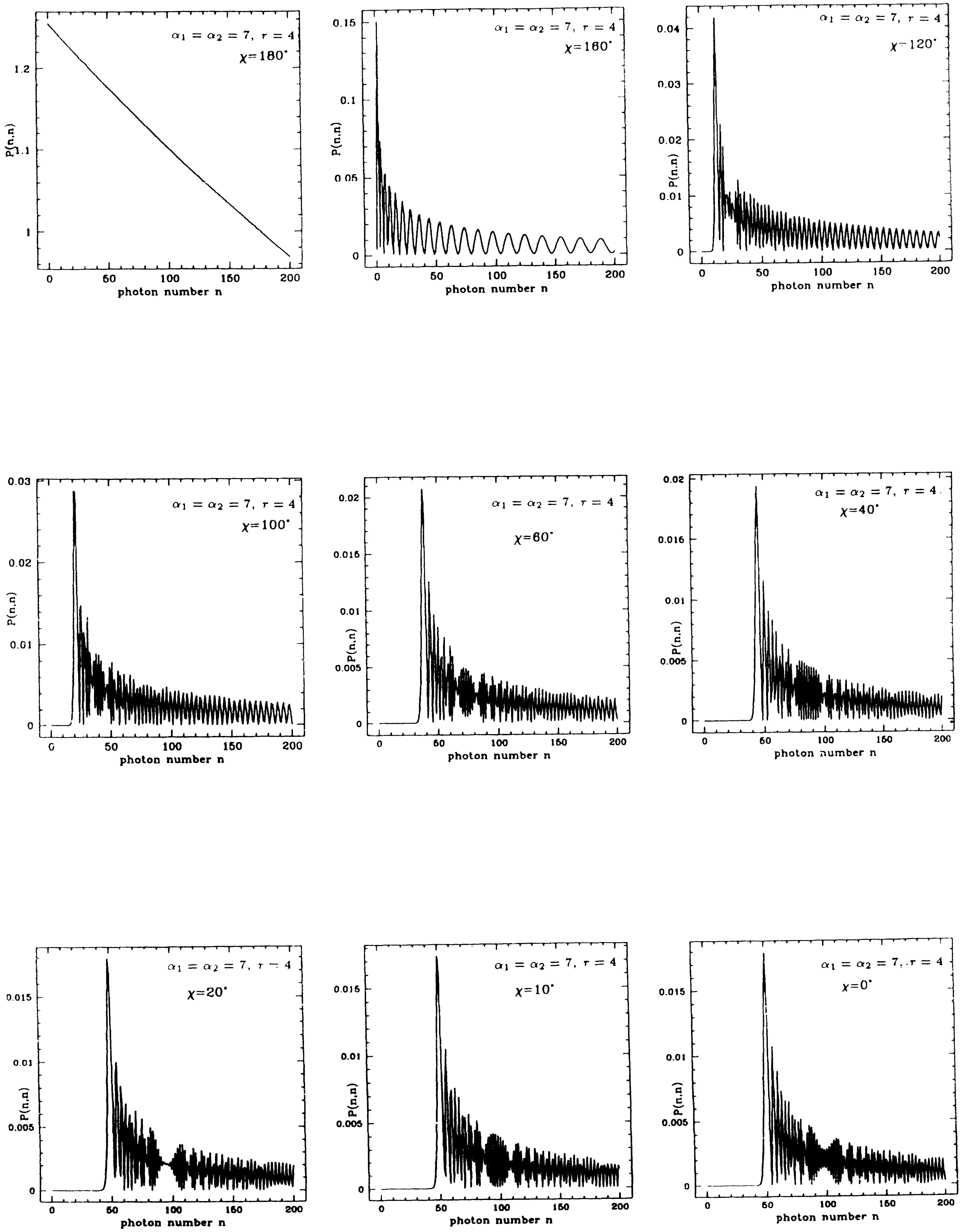

FIG. 3. Diagonal photon number distribution $P(n,n)$ as a function of n with $\alpha_1 = \alpha_2 = 7$ and $r = 4$. Collapses and revivals are seen in the distribution, and these are reminiscent of the ones in the Jaynes-Cummings model. Figure shows $P(n,n)$ in units of 10^{-3}.

summed over n_1,n_2 is normalized to unity. Thus it need not be pursued any further. The role of χ in the latter however, is nontrivial.

We will see in the next section that the dependence of the argument of the associated Laguerre polynominal on χ leads to a sensitive χ dependence of $P(n_1,n_2)$. But here we wish to note the interesting manner in which the phase of the argument of the associated Laguerre polynomial depends on χ. To this end let Φ be the phase of the argument of the associated Laguerre polynomial in (18):

$$L_p^{q-p}\left[\frac{2\tilde{\alpha}_1\tilde{\alpha}_2}{\sinh 2r}e^{-2i\phi}\right]=L_p^{q-p}\left[\frac{2|\tilde{\alpha}_1\tilde{\alpha}_2|}{\sinh 2r}e^{i\Phi}\right]. \quad (24)$$

From (23) we see that

$$\Phi=\arctan\left[\frac{|\alpha_1\alpha_2|\sin\chi}{\frac{1}{2}(|\alpha_1|^2+|\alpha_2|^2)\sinh 2r+|\alpha_1\alpha_2|\cosh 2r\cos\chi}\right]. \quad (25)$$

We show in Fig. 1. the behavior of Φ as a function of χ for the case $|\alpha_1|=|\alpha_2|$. It is seen that while Φ is linear in χ for $r=0$, with increasing value of the squeeze parameter r, Φ becomes a highly nonlinear function of χ. This is the Gouy effect for two-mode squeezed coherent states. The Gouy effect for (focused) light beams has been known for a long time, [14,15] and recently the Gouy effect for single-mode squeezed light has also been studied [16].

We note in passing that if either α_1 or α_2 equals zero, then the argument of the associated Laguerre polynomial becomes real positive irrespective of the phase of the squeeze parameter z.

IV. EXAMPLES OF PHOTON DISTRIBUTIONS

We have given in (18) the probability amplitude $c(n_1,n_2)$ for the two-mode squeezed coherent state; the square of the absolute value of this expression gives the photon distribution $P(n_1,n_2)$. We are primarily interested in the effect of the phases θ_1, θ_2, 2ϕ of the complex parameters α_1, α_2, z. We have already shown that these phases enter the photon distribution only through the argument of the associated Laguerre polynomial, and that also in the $U(1)\times U(1)$ invariant combination $\chi=\theta_1+\theta_2-2\phi$. We give in Fig. 2 the distribution $P(n_1,n_2)$ for fixed $|\alpha_1|=|\alpha_2|$ and fixed r, and selected values of χ in the range $0\leq\chi<\pi$.

It should be appreciated that the effective range of χ, as far as $P(n_1,n_2)$ in (18) is concerned, is $0\leq\chi\leq\pi$ rather than the full $0\leq\chi<2\pi$. This comes about from the fact that $P(n_1,n_2)$ is invariant under $\chi\rightarrow 2\pi-\chi$.

It is easy to see that Fig. 1(b) and Fig. 2(b) of Caves *et al.* correspond to $\chi=0$ and π, respectively. And for these values of χ our results in Fig. 2 are clearly in agreement with theirs. But from $\chi=0$ to $\chi=\pi$ the distribution "evolves" in an interesting manner. As χ is increased from zero, the ripple perpendicular to the diagonal starts breaking. With increasing value of χ these breaks increase in number, the period parallel to the diagonal increases, and the distribution pulls itself towards $(n_1,n_2)=(0,0)$. With further increase, the strength of the distribution falls rapidly as one moves away from the diagonal so that when $\chi=180°$ is reached, one is left with essentially a diagonal distribution. Thus, our Fig. 2 gives insight into the manner in which the photon distribution interpolates between the two extreme limits studied in [5].

To gain further understanding of the photon distribution, we probe the diagonal distribution $P(n,n)$ in some detail. In Fig. 3 we present $P(n,n)$ for the same values of $|\alpha_1|=|\alpha_2|$ and r as in Fig. 2, and for various values of χ. Collapses and revivals in the oscillation may be noticed. This result is reminiscent of the findings of Dutta *et al.* for the single-mode case. The major departure from the single-mode case is that in the present case the collapses and revivals are persistent for a wider range of the parameter χ. In particular, they survive even in the limit $\chi=0°$.

It may be noticed that the oscillations in $P(n,n)$ are most rapid at $\chi=0°$; and the period of oscillation steadily increases as χ goes to the limit 180°, where the diagonal distribution becomes essentially a constant. The region near $\chi=180°$ is further explored in Fig. 4.

It is of interest to analyze the photon distribution in n_1 for fixed n_2. This corresponds to state reduction, which has received considerable interest recently [17]. In Fig. 5, we show $P(n_1)\equiv P(n_1,n_2)$ for constant n_2 (i.e., the distribution as a function of n_1 for fixed n_2) for the same values of parameters $|\alpha_1|=|\alpha_2|$ and r, as in Figs. 2–4 and for selected values of χ. Again, collapses and revivals can be noticed. But the structure of this phenomenon is now quite different from the diagonal case and much richer.

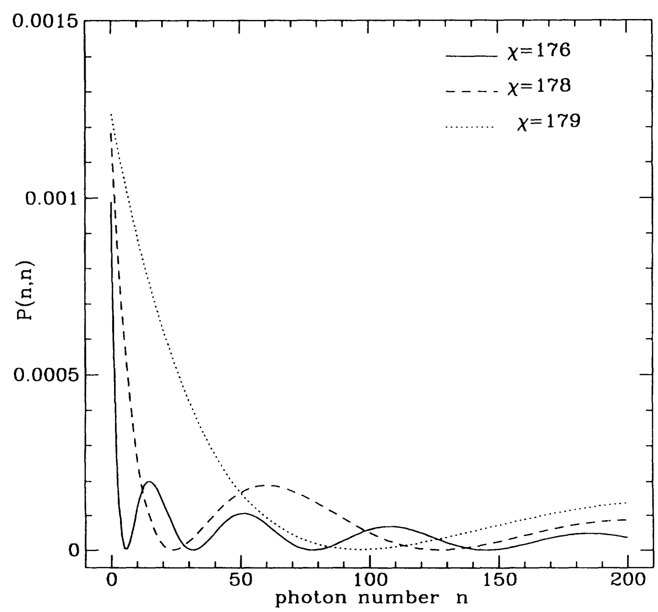

FIG. 4. A closer look at the diagonal distribution in the region around $\chi=180°$. It can be seen that the amplitude and the period of the oscillations decrease as χ decreases.

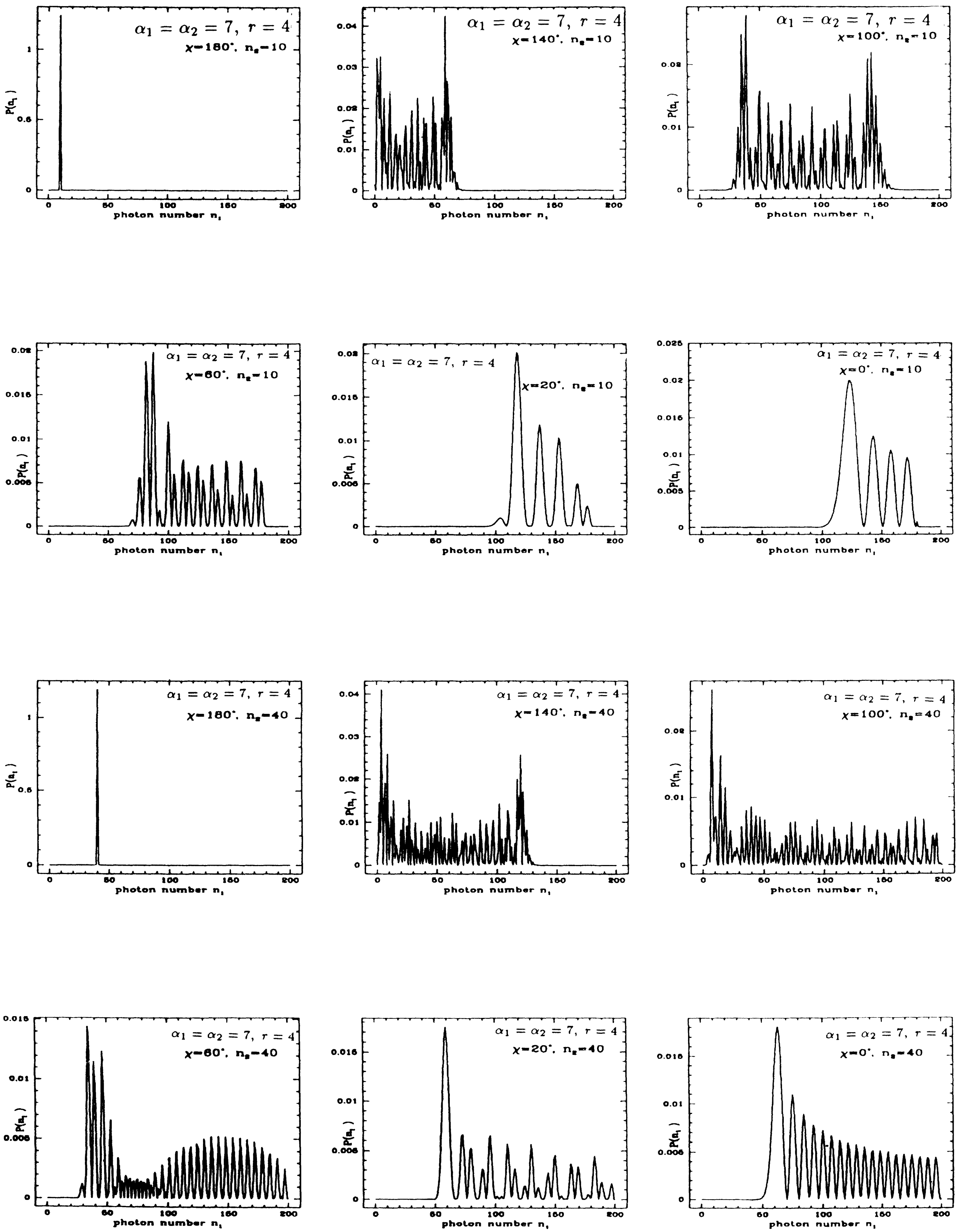

FIG. 5. Off-diagonal distribution $P(n_1) \equiv P(n_1, n_2)$ for fixed values of n_2. There are collapses and revivals similar to the diagonal case in Fig. 3, but the structure is richer. Figure shows $P(n_1)$ in units of 10^{-3} and the parameter values are $\alpha_1 = \alpha_2 = 7$, $r = 4$.

V. PROPERTIES INSENSITIVE TO PHASE

We have shown in the preceding two sections that the photon distributions in the two-mode squeezed coherent state $|z;\alpha_1,\alpha_2\rangle$ is quite sensitive to the $U(1)\times U(1)$-invariant combination of the phases of the squeeze and displacement parameters. But there are properties of this state which are insensitive to the phases of these operators. We present in this section examples of two such properties.

The first such property we consider is the total energy E in the state $|z;\alpha_1,\alpha_2\rangle$. This is given by the expectation value of $(a^\dagger a+b^\dagger b)$. The computation is straightforward:

$$E=\langle z;\alpha_1,\alpha_2|(a^\dagger a+b^\dagger b)|z;\alpha_1,\alpha_2\rangle$$
$$=\langle \mathrm{vac}|S^\dagger(z)D^\dagger(\alpha_1,\alpha_2)(a^\dagger a+b^\dagger b)D(\alpha_1,\alpha_2)S(z)|\mathrm{vac}\rangle \ . \tag{26}$$

The contribution from the $a^\dagger a$ term is

$$\langle \mathrm{vac}|S^\dagger(z)(a^\dagger+\alpha_1^*)(a+\alpha_1)S(z)|\mathrm{vac}\rangle$$
$$=|\alpha_1|^2+\langle \mathrm{vac}|S^\dagger(z)a^\dagger aS(z)|\mathrm{vac}\rangle$$
$$=|\alpha_1|^2+\sinh^2 r \ , \tag{27}$$

where we make use of the fact,

$$S^\dagger(z)aS(z)=a\cosh r-b^\dagger e^{2i\phi}\sinh r \ . \tag{28}$$

The expression (27) is similar to the one in the single-mode case, but the $\sinh^2 r$ term comes from the expectation value of $b^\dagger b$. It is easy to see that the contribution from the $b^\dagger b$ term in (27) equals $|\alpha_2|^2+\sinh^2 r$. Hence, the energy of the state $|z;\alpha_1,\alpha_2\rangle$ is [5].

$$E=|\alpha_1|^2+|\alpha_2|^2+2\sinh^2 r \ . \tag{29}$$

Thus the total-energy content of $|z;\alpha_1,\alpha_2\rangle$ is insensitive even to the invariant combination χ, even though the photon distribution itself is phase sensitive! That is, changing the value of χ simply redistributes the photons in the various two-mode Fock states without changing the total number of photons.

The next quantity we consider is the reduced density operator for mode 1. Caves *et al.* compute this through the P distribution. Our computation is based on the equivalent two-mode Wigner distribution $W(\xi_1,\xi_2)$. The advantage of the Wigner distribution over the P distribution arises from the fact that squeezing transformations simply act as linear transformations on the arguments of this distribution. Displacement operators act as rigid translations as in the P distribution case. Using these facts and the fact that the Wigner distribution for $|\mathrm{vac}\rangle$ is given by $W(\xi_1,\xi_2)=4/\pi^2\exp[-2(|\xi_1|^2+|\xi_2|^2)]$, the Wigner distribution for the state $|z;\alpha_1,\alpha_2\rangle$ is easily computed to be

$$W(\xi_1,\xi_2)=\frac{4}{\pi^2}\exp\{-2[(|\xi_1-\alpha_1|^2+|\xi_2-\alpha_2|^2)\cosh 2r$$
$$+\sinh 2r(\xi_1-\alpha_1)(\xi_2-\alpha_2)e^{-2i\phi}$$
$$+(\xi_1^*-\alpha_1^*)(\xi_2^*-\alpha_2^*)e^{2i\phi}]\} \ . \tag{30}$$

While the two-mode squeezed coherent state $|z;\alpha_1,\alpha_2\rangle$ has such a nice Gaussian Wigner distribution, it is well known that this state, being nonclassical, has no P-distribution function in the familiar sense of the term function.

The single-mode Wigner distribution corresponding to the reduced density operator for mode 1 is now obtained by taking the marginal $\int d^2\xi_2 W(\xi_1,\xi_2)$, where we have

$$W(\xi_1)=\int d^2\xi_2 W(\xi_1,\xi_2)$$
$$=\frac{2}{\pi\cosh 2r}\exp\left[-\frac{2|\xi_1-\alpha_1|^2}{\cosh 2r}\right] , \tag{31}$$

which corresponds to a displaced, but not squeezed, thermal state. We see that the phase of the squeeze operator does not enter this reduced Wigner distribution. In fact, the P distribution corresponding to (31) can be written down by inspection. We have

$$P(\xi_1)=\frac{1}{\pi\sinh^2 r}\exp\left[-\frac{2|\xi_1-\alpha_1|^2}{\sinh^2 r}\right] , \tag{32}$$

which coincides with the result of Caves *et al.*, consistent with its insensitiveness to phase.

VI. SECOND-ORDER COHERENCE FUNCTION

In the last section, we considered examples of properties of $|z;\alpha_1,\alpha_2\rangle$ which are insensitive to the phase of the squeeze parameter. We now turn briefly to some coherence properties which turn out to be sensitive to the phase.

We consider the Glauber coherence functions $g_{ab}^{(2)}(0)$ and $g_p^{(2)}(0)$. These are defined through

$$g_{ab}^{(2)}(0)=1+\frac{\langle \hat{n}_a\hat{n}_b\rangle-\langle \hat{n}_a\rangle\langle \hat{n}_b\rangle}{\langle \hat{n}_a\rangle\langle \hat{n}_b\rangle} , \tag{33}$$

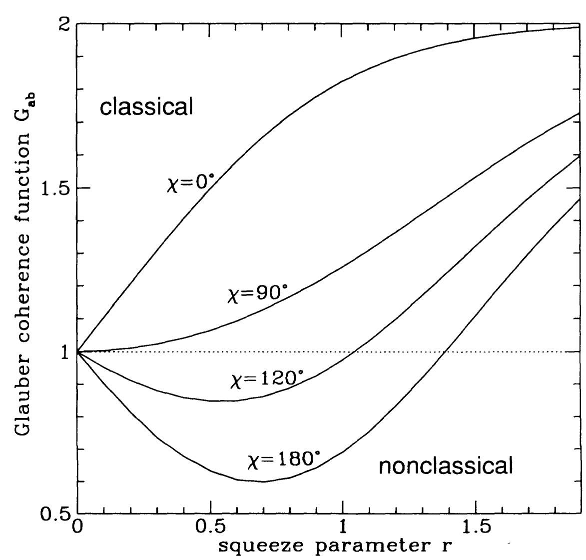

FIG. 6. The Glauber coherence function G_{ab} as a function of the squeeze parameter r. Nonclassical behavior is seen for $\chi>90°$. Here, $\alpha_1=1$ and $\alpha_2=2$.

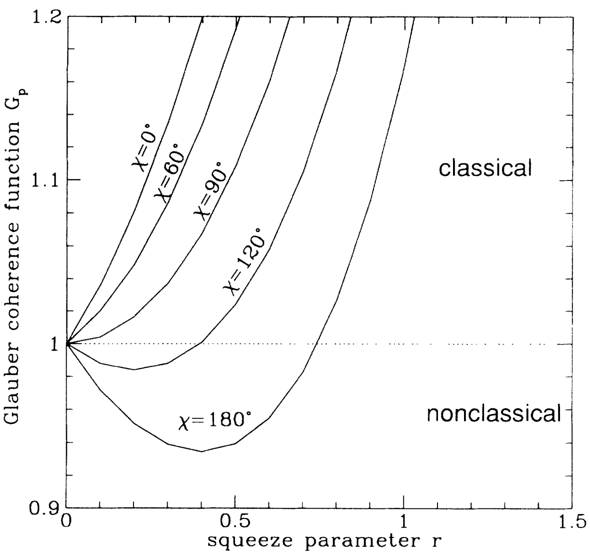

FIG. 7. The Glauber coherence function G_p as a function of the squeeze parameter r. Nonclassical behavior is seen for $\chi > 90°$. Here $\alpha_1=1$ and $\alpha_2=2$.

$$g_p^{(2)}(0)=1+\frac{\langle[\Delta(\hat{n}_a+\hat{n}_b)]^2\rangle-\langle(\hat{n}_a+\hat{n}_b)\rangle}{\langle(\hat{n}_a+\hat{n}_b)\rangle} . \tag{34}$$

Motivation for these definitions can be found in Gilles and Knight [18] and Agarwal [19]. Since these functions depend only on the photon distribution, it is clear that they can depend on the phases of z, α_1, α_2 atmost through the $U(1)\times U(1)$ invariant combination χ.

Classical values of these functions are bounded from below by unity. It is seen from Figs. 6 and 7 that these coherence functions take nonclassical values for some range of values of r whenever $\chi > 90°$.

VII. CONCLUDING REMARKS

We have studied the photon distribution in two-mode squeezed coherent states with complex squeeze and displacement parameters. The entire analysis was guided, often explicitly and sometimes implicitly, by an appreciation of the SU(2) structure underlying two-mode systems as a dynamical symmetry. Thus, realization of the fact that the two-mode squeeze operator is essentially a product of two correlated (in fact, reciprocal) single-mode squeeze operators allowed us to write the probability amplitude for photon distribution using the well known Yuen results for the matrix elements in the single-mode case and the matrix elements of a particular SU(2) rotation. Finally, the SU(2) identity given in (14) enabled us to write the photon distribution in the compact close form (18).

The $U(1)\times U(1)$ invariance of the photon distribution helped in simplifying the analysis, particularly in respect of numerical studies. That is, even though there were three phases in the problem to begin with, it turned out that there is only one nontrivial phase [the $U(1)\times U(1)$-invariant linear combination χ] which we have to consider as far as photon distribution is concerned. Our numerical analysis concentrated on the effect of this phase on various properties.

In all our examples we have taken $|\alpha_1|=|\alpha_2|$. To keep the length of the paper within reasonable limits, we have not included numerical examples for the case $|\alpha_1|\neq|\alpha_2|$. Nevertheless, it is useful to conclude with some general observations on this issue.

The defining relations (2) can be written as

$$\tilde{\alpha}_1=e^{i\theta_1}(|\alpha_1|\mu+|\alpha_2|\nu|e^{-i\chi}) ,$$
$$\tilde{\alpha}_2=e^{i\theta_2}(|\alpha_2|\mu+|\alpha_1|\nu|e^{-i\chi}) . \tag{35}$$

It is now transparent that $|\alpha_1|=|\alpha_2|$ implies $|\tilde{\alpha}_1|=|\tilde{\alpha}_2|$. It may further be noted that q,p in (18) are invariant under interchange of n_1 and n_2. Thus, it follows from (18) that $P(n_1,n_2)=P(n_2,n_1)$ whenever $|\alpha_1|=|\alpha_2|$. That is, the photon distribution is invariant under reflection about the diagonal $n_1=n_2$. This property is manifest in Fig. 2.

If $|\alpha_1|\neq|\alpha_2|$, then $P(n_1,n_2)$ will be expected to become asymmetric with respect to the diagonal. From (18) we see that the only source of asymmetry in n_1,n_2 is the factor $|\tilde{\alpha}_1|^{2(n_1-p)}|\tilde{\alpha}_2|^{2(n_2-p)}$. Since

$$\frac{|\tilde{\alpha}_1|^2}{|\tilde{\alpha}_2|^2}=\frac{1-(|\alpha_2|^2-|\alpha_1|^2)/A}{1+(|\alpha_2|^2-|\alpha_1|^2)/A} ,$$
$$A=(|\alpha_1|^2+|\alpha_2|^2)\cosh 2r+|\alpha_1\alpha_2|\cos\chi\sinh 2r , \tag{36}$$

as can be seen from (35), one will expect the asymmetry to become less and less prominent with increasing value of the squeeze parameter $r=|z|$.

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