Time evolution of harmonic oscillators with time-dependent parameters: A step-function approximation

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The time evolution of a quantum harmonic oscillator with a series of sudden jumps of the mass or the frequency is determined in the form of a recursion relation. An approximating method is developed for determining the time evolution of harmonic oscillators with arbitrary derivable functions of the frequency or the mass. The approximate solution is shown to tend to the analytical one in the limiting case. As a demonstration of the approximating method, the solution of the problem of damped oscillation in the square of the oscillator frequency is presented.

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I. INTRODUCTION

The problem of the one-dimensional harmonic oscillator with time-dependent parameters has received considerable attention in recent years [1–12]. This model proved to be an effective description of various phenomena and is widely used in different fields of physics. In quantum optics the time-dependent mass can describe an external influence on the quantized electromagnetic field, e.g., in a decaying or driven Fabry-Pérot cavity [2]. General nonadiabatic changes in the oscillator frequency or mass have been shown to lead to the generation of squeezed states if the oscillator was originally in a coherent state [3]. Another area where time-dependent oscillator parameters are introduced is ion trapping. The new experimental results made it possible to create simple, one-particle systems by using a very rapidly changing electric field [13]. One kind of ion trap, called the Paul trap, can be described by a one-dimensional harmonic oscillator with changing frequency for each rectangular coordinate [14,15].

There are various methods to determine the time evolution of time-dependent harmonic oscillators, such as the time-dependent canonical transformation method applied by Colegrave and Abdalla [4], the path-integral approach applied by Gerry [5], the evolution operator method introduced by Cheng and Fung [6], the use of the Wei-Norman–type procedures exploited by Dattoli, Richetta, and Torre [7], or the direct integration of equations of motion used by Agarwal and Kumar [8]. The analytical solutions for time-dependent mass or frequency are known only for certain functions of parameters.

The limiting case of the very rapidly changing (i.e., the sudden jump) in the oscillator frequency was first considered by Janszky and Yushin [9] and also studied by other authors [10]. A more detailed discussion of the possibility of generating highly squeezed states by a series of sudden jumps between two values of the oscillator frequency or mass was recently studied by Janszky, Adam, and Földesi [11,12].

In this paper we determine the time evolution of time-dependent harmonic oscillators with steplike functions of parameters in the form of a recursion relation. Using the recursion relation we develop an approximating method to determine the time evolution of the time-dependent scaled creation and annihilation operators in the Heisenberg picture for any continuous, derivable functions of parameters. We show that the approximate solutions tend to the exact ones in the limiting case. We compare the approximate and exact analytical solutions in the case of linear sweep of the restoring force (i.e., the square of the frequency changes linearly in time) and show how the approximate results tend to the exact solution. As an example we solve the problem of the damped oscillation of the restoring force. We discuss the possible cases of undercritical, critical, and overcritical damping.

The structure of our paper is the following. In Sec. II we review the treatment of arbitrary sudden changes of the Hamiltonian and consider one sudden jump in the frequency or mass. In Sec. III, we give exact results for the steplike time dependency of the mass frequency in the form of a recursion relation. In Sec. IV we define an approximating method for determining the time evolution of an oscillator with arbitrary continuous frequency or mass functions and show that the approximate solution tends to the exact one. In Sec. V we consider the analytically solvable case of the linear sweep of the restoring force and give further examples for the usage of the approximation in the case of the damped oscillation of the restoring force. In Sec. VI we summarize the results and conclusions.

II. QUANTUM MECHANICS OF SUDDEN CHANGES

The problem of a time-dependent Hamiltonian in the limiting case of a sudden change is well known in the literature. We shortly summarize here the most important results. The general case is when the Hamiltonian changes in time continuously from a constant initial value $\hat{H}_0$ at time t_0 to another constant final value $\hat{H}_1$ at time t_1 during a period $T=t_1-t_0$. We introduce a new variable

$$s=(t-t_0)/T \tag{1}$$

and denote

$$\hat{H}(s)=\hat{H}(t_0+sT) . \quad (2)$$

Thus we have a continuous $\hat{H}(s)$ function and

$$\hat{H}(0)=\hat{H}_0 , \quad \hat{H}(1)=\hat{H}_1 . \quad (3)$$

The time-evolution operator of the system can also be considered as a function of s,

$$\hat{U}(t,t_0)=\hat{U}_T(s) . \quad (4)$$

The sudden change is the limiting case when $T\to 0$. In this case the dynamic state of the system remains unchanged

$$\lim_{T\to 0} \hat{U}_T(1)=1 . \quad (5)$$

This important result is independent of which parameters are used and how they have been changed in the Hamiltonian, as we did not have any conditions for the initial $\hat{H}_0$ and final $\hat{H}_1$ [16].

It will be useful at several later points to compare the two familiar kinds of quantum-mechanical descriptions, the Schrödinger and the Heisenberg pictures. We have to make a clear distinction between operators being time independent in the Schrödinger picture (such as coordinate and momentum operators) and operators defined to be time dependent even in the Schrödinger picture. In the Schrödinger picture we have for the wave function

$$\Psi(t,q)=\hat{U}(t,t_0)\Psi(t_0,q) , \quad (6)$$

thus, after the sudden change,

$$\Psi(t_1,q)=\hat{U}(t_1,t_0)\Psi(t_0,q)=\hat{U}_T(1)\Psi(t_0,q) . \quad (7)$$

In the limiting case the wave function remains unchanged

$$\lim_{t_1\to t_0} \Psi(t_1,q)=\lim_{T\to 0} \hat{U}_T(1)\Psi(t_0,q)=\Psi(t_0,q) . \quad (8)$$

The final state of the system for the old Hamiltonian $\hat{H}_0$ will be the initial state for the new Hamiltonian $\hat{H}_1$. In the Heisenberg picture the same quantum-mechanical process is described by constant wave function and time-dependent operators. An operator that is time independent in the Schrödinger picture will change in time in the Heisenberg picture

$$f_H(t)=\hat{U}^{-1}(t,t_0)f_S\hat{U}(t,t_0) . \quad (9)$$

We choose the initial moment, when the two pictures coincide, at $t=t_0$. A similar equation is valid for operators which are time dependent in the Schrödinger picture

$$g_H(t)=\hat{U}^{-1}(t,t_0)g_S(t)\hat{U}(t,t_0) . \quad (10)$$

Equations (8)–(10) determine the connection between the value of an operator after a sudden jump in the Heisenberg and the Schrödinger picture. For an operator being time independent in the Schrödinger picture we have

$$\lim_{t_1\to t_0} \hat{f}_H(t_1)=\hat{f}_H(t_0)=\hat{f}_S . \quad (11)$$

For a time-dependent operator in the Schrödinger picture we have

$$\lim_{t_1\to t_0} \hat{g}_H(t_1)=\lim_{t_1\to t_0} \hat{U}^{-1}(t_1,t_0)\hat{g}_s(t_1)\hat{U}(t_1,t_0)=\hat{g}_s(t_1) . \quad (12)$$

With these preliminaries completed we shall now discuss special cases of sudden changes in greater detail.

The two important parameters of a quantum harmonic oscillator are the frequency and the mass. Both of them can vary in time, modeling different physical phenomena. A harmonic oscillator with time-dependent frequency and mass is described by the Hamiltonian

$$\hat{H}=\frac{1}{2M(t)}\hat{p}^2+\frac{M(t)\omega^2(t)}{2}\hat{q}^2 \quad (\hbar=1) , \quad (13)$$

where $\hat{q}$ and $\hat{p}$ are the momentum and coordinate operators. It is convenient to define the time-dependent scaled creation and annihilation operators

$$\hat{a}=\tfrac{1}{2}[\sqrt{2M(t)\omega(t)}\hat{q}+i\sqrt{2/M(t)\omega(t)}\hat{p}] , \quad (14a)$$

$$\hat{a}^\dagger=\tfrac{1}{2}[\sqrt{2M(t)\omega(t)}\hat{q}-i\sqrt{2/M(t)\omega(t)}\hat{p}] . \quad (14b)$$

Squeezing properties of a harmonic oscillator are defined by the variance of quadrature operators (i.e., the real and imaginary part of $\hat{a}$ and $\hat{a}^\dagger$: $\hat{a}=\hat{X}_+ +i\hat{X}_-$, and $\hat{a}^\dagger=\hat{X}_+ -i\hat{X}_-$). The harmonic oscillator is squeezed if the variance of one of the quadratures is less than $\frac{1}{4}$. The variances of the time-dependent scaled quadratures, which originate from the time-dependent scaled $\hat{a}$ and $\hat{a}^\dagger$, show the current squeezing properties of a harmonic oscillator with changing parameters. First we consider a sudden frequency change at $t=0$:

$$\omega(t)=\begin{cases}\omega_0 , & t<0\\ \omega_1 , & t>0 ,\end{cases} \quad M(t)=M . \quad (15)$$

To determine the behavior of the system we have to solve the equations of motion at every time. It is enough to find the time evolution of the creation and annihilation operators as all physical quantities describing the system can be expressed with $\hat{a}$ and $\hat{a}^\dagger$. When $t<0$ we have a single harmonic-oscillator problem with the solutions

$$\hat{a}(t<0)=\hat{a}_{\text{in}}\exp[-i\omega_0(t-t_{\text{in}})] , \quad (16a)$$

$$\hat{a}^\dagger(t<0)=\hat{a}^\dagger_{\text{in}}\exp[i\omega_0(t-t_{\text{in}})] \quad (16b)$$

in the Heisenberg picture, where $\hat{a}_{\text{in}}$ and $\hat{a}^\dagger_{\text{in}}$ belong to a certain initial condition at $t=t_{\text{in}}$. When $t>0$ we also have a harmonic oscillator, but with changed parameters. We find the same kind of solution for the rescaled $\hat{a}$ and $\hat{a}^\dagger$. The initial conditions are now determined by the values of $\hat{a}$ and $\hat{a}^\dagger$, just after the jump. Thus we have

$$\hat{a}(t>0)=\hat{a}(+0)\exp[-i\omega_1 t] , \quad (17a)$$

$$\hat{a}^\dagger(t>0)=\hat{a}^\dagger(+0)\exp[i\omega_1 t] . \quad (17b)$$

We know that $\hat{q}$ and $\hat{p}$ are time-independent operators in the Schrödinger picture; therefore with the use of Eq. (11) we have

$$\hat{q}(+0)=\hat{q}(-0) , \; \hat{p}(+0)=\hat{p}(-0) \quad (18)$$

in the Heisenberg picture, which means that the coordinate and momentum operators remain unchanged just after the jump. From Eq. (18) and the definitions of $\hat{a}$ and $\hat{a}^\dagger$ [Eq. (14)] we find

$$\hat{a}(+0)=\frac{\omega_1+\omega_0}{2\sqrt{\omega_1\omega_0}}\hat{a}(-0)+\frac{\omega_1-\omega_0}{2\sqrt{\omega_1\omega_0}}\hat{a}^\dagger(-0)\ , \tag{19a}$$

$$\hat{a}^\dagger(+0)=\frac{\omega_1+\omega_0}{2\sqrt{\omega_1\omega_0}}\hat{a}^\dagger(-0)+\frac{\omega_1-\omega_0}{2\sqrt{\omega_1\omega_0}}\hat{a}(-0)\ ,$$

$$|u|^2-|v|^2=1\ . \tag{19b}$$

The values of the time-dependent scaled operators after the jump are expressed with their values before the jump through a Bogoliubov transformation [9]. Equations (16), (17), and (19) describe the system, if we know the initial state vector (which remains time independent in the Heisenberg picture). The operators $\hat{q}$ and $\hat{p}$ can be expressed with $\hat{a}$ and $\hat{a}^\dagger$; thus $\langle\Delta\hat{q}^2\rangle$ and $\langle\Delta\hat{p}^2\rangle$ can also be determined. Let us see the behavior of $\langle\Delta\hat{q}^2\rangle$ and $\langle\Delta\hat{p}^2\rangle$ before and after a frequency jump, if the oscillator was initially in its ground state

$$\langle\Delta\hat{q}^2(t<0)\rangle=\frac{1}{2M\omega_0}\ ,\quad \langle\Delta\hat{p}^2(t<0)\rangle=\frac{M\omega_0}{2}\ , \tag{20}$$

$$\langle\Delta\hat{q}^2(t>0)\rangle=\frac{1}{4M\omega_1}\left\{\frac{\omega_1}{\omega_0}[1+\cos(2\omega_1 t)]+\frac{\omega_0}{\omega_1}[1-\cos(2\omega_1 t)]\right\}\ , \tag{21a}$$

$$\langle\Delta\hat{p}^2(t>0)\rangle=\frac{M\omega_1}{4}\left\{\frac{\omega_0}{\omega_1}[1+\cos(2\omega_1 t)]+\frac{\omega_1}{\omega_0}[1-\cos(2\omega_1 t)]\right\}\ . \tag{21b}$$

Equation (21) shows that after the sudden jump $\langle\Delta\hat{q}^2\rangle$ and $\langle\Delta\hat{p}^2\rangle$ will not change immediately, though they will oscillator later in time. The oscillator, however, will be squeezed just after the jump. This fact is indicated by the variance of the quadrature operators. The variance of $\hat{X}_+$ and $\hat{X}_-$ are

$$\langle\Delta\hat{X}_+^2(t<0)\rangle=\langle\Delta\hat{X}_-^2(t<0)\rangle=\tfrac{1}{4}\ , \tag{22}$$

$$\langle\Delta\hat{X}_+^2(t>0)\rangle=\frac{1}{8}\left[\frac{\omega_1}{\omega_0}[1+\cos(2\omega_1 t)]+\frac{\omega_0}{\omega_1}[1-\cos(2\omega_1 t)]\right]\ , \tag{23a}$$

$$\langle\Delta\hat{X}_-^2(t>0)\rangle=\frac{1}{8}\left[\frac{\omega_0}{\omega_1}[1+\cos(2\omega_1 t)]+\frac{\omega_1}{\omega_0}[1-\cos(2\omega_1 t)]\right]\ . \tag{23b}$$

The case of the sudden mass jump is very similar mathematically to the above-discussed case of the frequency jump. Now we have

$$M(t)=\begin{cases}M_0\ , & t<0\\ M_1\ , & t\geq 0\ ,\end{cases}\quad \omega(t)=\omega\ . \tag{24}$$

Using the same method as for the changing frequency, we have for $\hat{a}$ and $\hat{a}^\dagger$

$$\hat{a}(t>0)=\left[\frac{M_1+M_0}{2\sqrt{M_1M_0}}\hat{a}(-0)+\frac{M_1-M_0}{2\sqrt{M_1M_0}}\hat{a}^\dagger(-0)\right]\times\exp(-i\omega t)\ , \tag{25a}$$

$$\hat{a}^\dagger(t>0)=\left[\frac{M_1+M_0}{2\sqrt{M_1M_0}}\hat{a}^\dagger(-0)+\frac{M_1-M_0}{2\sqrt{M_1M_0}}\hat{a}(-0)\right]\times\exp(i\omega t)\ . \tag{25b}$$

The similarity in the expression of $\hat{a}$ and $\hat{a}^\dagger$ in the two cases of frequency and mass change leads to similar behavior in the variance of the coordinate, momentum, and quadrature operators [12].

III. STEPLIKE TIME DEPENDENCY

In the previous section we summarized the time evolution of a harmonic oscillator with time-dependent frequency and mass in the case of one sudden change at $t=0$. In this section we discuss the problem of an arbitrary steplike time dependency of the oscillator parameters. We consider first an oscillator with changing frequency. Let $\omega(t)$ be a steplike function

$$\omega(t)=\begin{cases}\omega_0\ , & t<t_0=0\\ \omega_1\ , & t_0\leq t<t_1\\ \omega_2\ , & t_1\leq t<t_2\\ \vdots & \\ \omega_i\ , & t_{i-1}\leq t\leq t_{i+1}\\ \vdots & \end{cases} \tag{26}$$

After the first jump $\hat{a}$ and $\hat{a}^\dagger$ can be expressed with a linear combination of the initial $\hat{a}(-0)=\hat{a}_0$ and $\hat{a}^\dagger(-0)=\hat{a}_0^\dagger$, as it was shown in Sec. II. Similarly, after the second jump, $\hat{a}(t_1+0)$ and $\hat{a}^\dagger(t_1+0)$ are linear combinations of $\hat{a}(t_1-0)$ and $\hat{a}^\dagger(t_1-0)$. Taking into account that between the jumps the time evolution is described by Eqs. (17), we find that $\hat{a}(t_1+0)$ and $\hat{a}(t_1+0)$ and $\hat{a}^\dagger(t_1+0)$ are linear combinations of $\hat{a}_0$ and $\hat{a}_0^\dagger$. Hence, following this procedure, we can express the time dependent $\hat{a}$ and $\hat{a}^\dagger$ at every later moment with a linear combination of the initial $\hat{a}_0$ and $\hat{a}_0^\dagger$

$$\hat{a}(t)=u(t)\hat{a}_0+v(t)\hat{a}_0^\dagger\ , \tag{27a}$$

$$\hat{a}^\dagger(t)=v^*(t)\hat{a}_0+u^*(t)\hat{a}_0^\dagger\ ,\quad |u|^2-|v|^2=1\ . \tag{27b}$$

The problem is to determine the complex functions $u(t)$ and $v(t)$. Equations (17) and (19) can be generalized for more than one step, deriving this way a recursion relation for the values of $u(t)$ and $v(t)$ just after jumps. Denoting for simplicity $u_i=u(t_i+0)$, and $v_i=v(t_i+0)$, we have for u_i and v_i $(i=1,2,\ldots)$

$$u_i=(A_i u_{i-1}+B_i v_{i-1}^*)\cos[\omega_i(t_i-t_{i-1})]-i(A_i u_{i-1}-B_i v_{i-1}t^*)\sin[\omega_i(t_i-t_{i-1})]\ , \tag{28a}$$

$$v_i=(A_iv_{i-1}+B_iu^*_{i-1})\cos[\omega_i(t_i-t_{i-1})] -i(A_iv_{i-1}-B_iu^*_{i-1})\sin[\omega_i(t_i-t_{i-1})] \ , \tag{28b}$$

where A_i and B_i are constants

$$A_i=\frac{\omega_{i+1}+\omega_i}{2\sqrt{\omega_{i+1}\omega_i}} \ , \quad B_i=\frac{\omega_{i+1}-\omega_i}{2\sqrt{\omega_{i+1}\omega_i}} \ . \tag{28c}$$

Between two jumps a simple harmonic motion takes place with the following time dependency:

$$u(t)=u_i\exp[-i\omega_{i+1}(t-t_i)] \ , \tag{29a}$$

$$v(t)=v_i\exp[i\omega_{i+1}(t-t_i)] \ , \quad t_i<t<t_{i+1} \ . \tag{29b}$$

The initial conditions for u_i and v_i are

$$u_0=A_0=\frac{\omega_1+\omega_0}{2\sqrt{\omega_1\omega_0}} \ , \quad v_0=B_0=\frac{\omega_1-\omega_0}{2\sqrt{\omega_1\omega_0}} \tag{30}$$

which are the values of $u(t)$ and $v(t)$ after the first jump, in agreement with Eq. (19) of Sec. II. We thus have the solutions of equations of motion for an arbitrary steplike $\omega(t)$ function in the form of a recursion relation.

Similar equations can be obtained if the oscillator mass is given by a steplike function. We assume for $M(t)$

$$M(t)=\begin{cases} M_0 \ , & t<t_0=0 \\ M_1 \ , & t_0\le t<t_1 \\ M_2 \ , & t_1\le t<t_2 \\ \vdots & \\ M_i \ , & t_{i-1}\le t<t_{i+1} \\ \vdots & \end{cases} \tag{31}$$

The time evolution of $\hat{a}$ and $\hat{a}^\dagger$ is of the form of Eq. (27) and we will find for $u(t)$ and $v(t)$

$$u_i=(A_iu_{i-1}+B_iv^*_{i-1})\cos[\omega(t_i-t_{i-1})] -i(A_iu_{i-1}-B_iv^*_{i-1})\sin[\omega(t_i-t_{i-1})] \ , \tag{32a}$$

$$v_i=(A_iv_{i-1}+B_iu^*_{i-1})\cos[\omega(t_i-t_{i-1})] -i(A_iv_{i-1}-B_iu^*_{i-1})\sin[\omega(t_i-t_{i-1})] \ , \tag{32b}$$

where

$$A_i=\frac{M_{i+1}+M_i}{2\sqrt{M_{i+1}M_i}} \ , \quad B_i=\frac{M_{i+1}-M_i}{2\sqrt{M_{i+1}M_i}} \ . \tag{32c}$$

Between two jumps

$$u(t)=u_i\exp[-i\omega(t-t_i)] \ , \tag{33a}$$

$$v(t)=v_i\exp[i\omega(t-t_i)] \quad \text{for } t_i<t<t_{i+1} \ . \tag{33b}$$

Thus we have determined the time evolution of the time-dependent scaled creation and annihilation operators of the harmonic oscillator for arbitrary steplike time functions of the frequency or mass.

IV. APPROXIMATION OF CONTINUOUS CHANGES BY STEPS

The Hamiltonian of the harmonic oscillator [Eq. (13)] can be written in terms of $\hat{a}$ and $\hat{a}^\dagger$

$$\hat{H}=\omega(t)[\hat{a}^\dagger(t)\hat{a}(t)+\tfrac{1}{2}] \ . \tag{34}$$

We emphasize here that, as it follows from their definition [Eq. (14)], $\hat{a}$ and $\hat{a}^\dagger$ explicitly depend on time. The equation of motion for $\hat{a}$ and $\hat{a}^\dagger$ in the Heisenberg picture are

$$\frac{d\hat{a}(t)}{dt}=\frac{\partial\hat{a}(t)}{\partial t}+i[\hat{H},\hat{a}] \ , \tag{35a}$$

$$\frac{d\hat{a}^\dagger(t)}{dt}=\frac{\partial\hat{a}^+(t)}{\partial t}+i[\hat{H},\hat{a}^\dagger] \ . \tag{35b}$$

Let us assume an arbitrary $\omega(t)$ frequency function (we consider only derivable functions) and constant mass. Substituting the definitions of $\hat{a}$ and $\hat{a}^\dagger$ we have

$$\frac{d\hat{a}(t)}{dt}=-i\omega(t)\hat{a}(t)+\frac{1}{2\omega(t)}\frac{d\omega(t)}{dt}\hat{a}^\dagger \ , \tag{36a}$$

$$\frac{d\hat{a}^\dagger(t)}{dt}=-i\omega(t)\hat{a}^\dagger(t)+\frac{1}{2\omega(t)}\frac{d\omega(t)}{dt}\hat{a} \ . \tag{36b}$$

The solutions of these equations can be written in the form of a time-dependent linear combination of the initial values of $\hat{a}_0$ and $\hat{a}_0^\dagger$

$$\hat{a}(t)=u(t)\hat{a}_0+v(t)\hat{a}_0^\dagger \ , \tag{37a}$$

$$\hat{a}^\dagger(t)=u^*(t)\hat{a}_0^\dagger+v^*(t)\hat{a}_0 \ , \quad |u|^2-|v|^2=1 \ . \tag{37b}$$

Substituting Eq. (37) into Eq. (36) we find for $u(t)$ and $v(t)$

$$\frac{du(t)}{dt}=-i\omega(t)u(t)+\frac{1}{2\omega(t)}\frac{d\omega(t)}{dt}v^*(t) \ , \tag{38a}$$

$$\frac{dv(t)}{dt}=-i\omega(t)v(t)+\frac{1}{2\omega(t)}\frac{d\omega(t)}{dt}u^*(t) \ . \tag{38b}$$

To determine $u(t)$ and $v(t)$ we apply the results of Sec. III. We approximate the $\omega(t)$ function by a steplike function. The idea is similar to the one used in mathematical analysis in the introduction of integrals. Let us consider an arbitrary derivable $\omega(t)$ function and assume that the initial state of the system is given at the $t_0=0$ time. We approximate $\omega(t)$ by the following steplike function:

$$\omega_{\text{appr}}(t)=\begin{cases} \omega_1=\omega(t_0) \ , & t_0\le t<t_1 \\ \omega_2=\omega(t_1) \ , & t_1\le t<t_2 \\ \vdots & \\ \omega=\omega(t_{i-1}) \ , & t_{i-1}\le t<t_i \\ \vdots & \end{cases} \tag{39}$$

The time evolution of an oscillator with $\omega_{\text{appr}}(t)$ frequency can be obtained exactly. We have the $\hat{a}_{\text{appr}}(t)$ and $\hat{a}^\dagger_{\text{appr}}(t)$ solutions in the form

$$\hat{a}_{\text{appr}}(t)=u_{\text{appr}}(t)\hat{a}_0+v_{\text{appr}}(t)\hat{a}_0^\dagger \ , \tag{40a}$$

$$\hat{a}^\dagger_{\text{appr}}(t)=u^*_{\text{appr}}(t)\hat{a}_0^\dagger+v^*_{\text{appr}}(t)\hat{a}_0 \ , \tag{40b}$$

where $u_{\text{appr}}(t)$ and $v_{\text{appr}}(t)$ are determined by the recursion relations of Eqs. (28) and (29).

The approximating $\omega_{\text{appr}}(t)$ tends to the exact $\omega(t)$ if

$t_i - t_{i-1} \to 0$ for every i. We also expect that choosing smaller and smaller steps our approximating solutions $u_{\rm appr}(t)$ and $v_{\rm appr}(t)$ tend to the exact ones. Let us examine the recursion relations of Eq. (28) in the limiting case of $t_i \to t_{i-1}$. We can do the following substitutions:

$$\lim_{t_i \to t_{i-1}} \omega_{i+1} = \lim_{t_i \to t_{i-1}} \omega_i = \omega(t_i) ,$$
$$\lim_{t_i \to t_{i-1}} \frac{\omega_{i+1} - \omega_i}{t_i - t_{i-1}} = \left.\frac{d\omega(t)}{dt}\right|_{t_i} , \tag{41}$$

$$\lim_{t_i \to t_{i-1}} \frac{u_i - u_{i-1}}{t_i - t_{i-1}} = \left.\frac{du(t)}{dt}\right|_{t_i} ,$$
$$\lim_{t_i \to t_{i-1}} \frac{v_i - v_{i-1}}{t_i - t_{i-1}} = \left.\frac{dv(t)}{dt}\right|_{t_i} . \tag{42}$$

Linearizing the sine and cosine functions

$$\cos[\omega_i(t_i - t_{i-1})] \approx 1 , \tag{43a}$$

$$\sin[\omega_i(t_i - t_{i-1})] \approx \omega(t) dt , \tag{43b}$$

substituting Eqs. (41) and (42), and keeping only the first order terms in dt, we obtain different equations from Eq. (28)

$$\frac{du_{\rm appr}(t)}{dt} = -i\omega(t) u_{\rm appr}(t) + \frac{1}{2\omega(t)} \frac{d\omega(t)}{dt} v^*_{\rm appr}(t) , \tag{44a}$$

$$\frac{dv_{\rm appr}(t)}{dt} = -i\omega(t) v_{\rm appr}(t) + \frac{1}{2\omega(t)} \frac{d\omega(t)}{dt} u^*_{\rm appr}(t) . \tag{44b}$$

Comparing Eqs. (38) with Eq. (44) we find that $u_{\rm appr}(t)$ and $v_{\rm appr}(t)$ satisfy the exact equations of motion. Hence we have

$$\lim_{t_i \to t_{i-1}} \hat{a}_{\rm appr}(t) = \hat{a}(t) ,$$
$$\lim_{t_i \to t_{i-1}} \hat{a}^\dagger_{\rm appr}(t) = \hat{a}^\dagger(t) . \tag{45}$$

Thus we have proved that the approximate solution tends to the exact one in the limiting case.

The conditions of the applicability of the approximation follow from Eqs. (41)–(43). The condition for linearizing the sine and cosine functions is

$$\omega_i(t_i - t_{i-1}) \ll 1 \tag{46}$$

and for Eq. (41)

$$\left.\frac{d^2\omega}{dt^2}\right|_{t_i} (t_i - t_{i-1}) \ll \left.\frac{d\omega}{dt}\right|_{t_i} . \tag{47}$$

When using the approximation we have to choose the steps small enough to satisfy the conditions of Eqs. (46) and (47).

Similarly to the case of changing frequency, arbitrary derivable $M(t)$ functions can be approximated in the same way defining $M_{\rm appr}(t)$

$$M_{\rm appr}(t) = \begin{cases} M_1 = M(t_0) , & t_0 \le t < t_1 \\ M_2 = M(t_1) , & t_1 \le t < t_2 \\ \vdots & \\ M_i = M(t_{i-1}) , & t_{i-1} \le t < t_i \\ \vdots \, . & \end{cases} \tag{48}$$

The time evolution is given by Eq. (27), where $u_{\rm appr}(t)$ and $v_{\rm appr}(t)$ are determined by Eqs. (32) and (33). In the limiting case we have for $u_{\rm appr}(t)$ and $v_{\rm appr}(t)$

$$\frac{du_{\rm appr}(t)}{dt} = -i\omega u_{\rm appr}(t) + \frac{1}{2M(t)} \frac{dM(t)}{dt} v^*_{\rm appr}(t) , \tag{49a}$$

$$\frac{dv_{\rm appr}(t)}{dt} = -i\omega v_{\rm appr}(t) + \frac{1}{2M(t)} \frac{dM(t)}{dt} u^*_{\rm appr}(t) , \tag{49b}$$

which means that they satisfy the exact equations of motion. The conditions for the steps are also similar

$$\omega(t_i - t_{i-1}) \ll 1 , \tag{50}$$

$$\left.\frac{d^2M}{dt^2}\right|_{t_i} (t_i - t_{i-1}) \ll \left.\frac{dM}{dt}\right|_{t_i} . \tag{51}$$

V. EXAMPLES

The approximating method we defined in Sec. IV is a very useful means to solve the equations of motions for different, practically interesting cases. It offers an effective numerical procedure to determine the time evolution of the oscillator. With the help of it we can have numerical results in the case of any derivable frequency or mass functions. As an example we first study the analytically solvable case when the frequency changes modeling a linear sweep in the restoring force

$$\omega^2(t) = \begin{cases} \omega_0^2 , & t < 0 \\ \omega_0^2 + (\omega_1^2 - \omega_0^2)\dfrac{t}{T} , & 0 \le t \le T \\ \omega_1^2 , & t > T . \end{cases} \tag{52}$$

This problem was treated in a recent paper by Agarwal and Kumar [8], who solved the equations of motion exactly. The analytical solutions for the time evolution for the coordinate and momentum operators are expressed as a linear combination of their initial values, where the coefficients are Bessel functions of time. Using the approximation defined in Sec. IV we calculated the time evolution of the creation and annihilation operators numerically. We determined the variance of the coordinate and momentum operators assuming that the oscillator was originally in its ground state. Our results are demonstrated in Fig. 1, where Fig. 1(a) shows the $\omega(t)$ functions and Fig. 1(b) shows the corresponding variances of the coordinate operator. The solid line is the curve for the exact case while the dotted line shows a three-step and the dashed line a ten-step approximation. It can be seen that choosing more steps in the approximating $\omega_{\rm appr}(t)$ function [Fig. 1(a)], the approximate solution will tend to the exact solution very quickly [Fig. 1(b)].

As an illustration of the effectiveness of our approximation we discuss three practically interesting functions of the restoring force (being determined, e.g., by the external electromagnetic field in a Paul trap). We model a damped oscillation of the external influence, considering the three cases of damping.

First if the damping is weak we have for the frequency

$$\omega^2(t)=\begin{cases}\omega_0^2\,, & t<0\\ \omega_0^2[1+A\exp(-t/B)\sin(Ct)]\,, & t\geq 0\,.\end{cases} \tag{53}$$

The amplitude A is chosen to be $A=0.5$. The frequency of the oscillation of the restoring force is $C=2\omega_0$ in order to be resonant with the original oscillator frequency. The considered frequency functions and the resulting variances of the coordinate operator for different values of B are shown in Fig. 2. When $B \ll 1/\omega_0$ the oscillation appearing in the variance will not be significant and as the frequency tends to a constant value the variance will

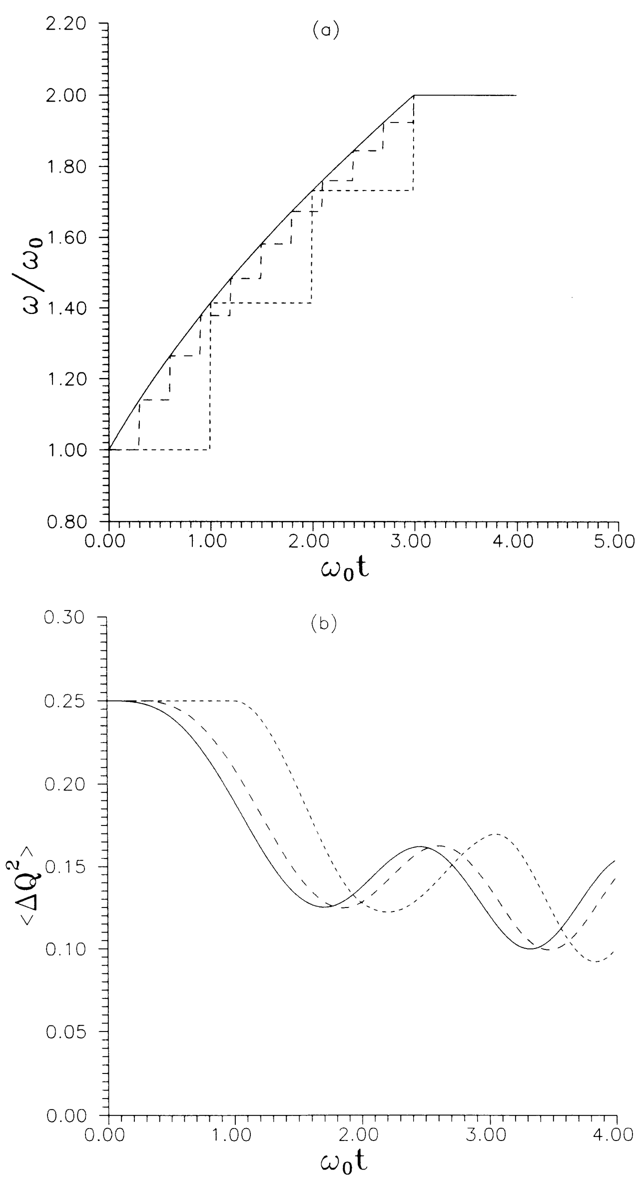

FIG. 1. Approximation of the $\omega(t)$ function given by Eq. (52) ($\omega_1/\omega_0=2$ and $\omega_0 T=3$). The number of steps is chosen to be (i) 3 steps (dotted line), (ii) 10 steps (dashed line), and (iii) the exact solution (solid line). (a) The $\omega(t)$ functions. (b) The time evolution of the variance of the dimensionless coordinate operator ($\hat{Q}=\sqrt{\omega_0 M/2}\hat{q}$).

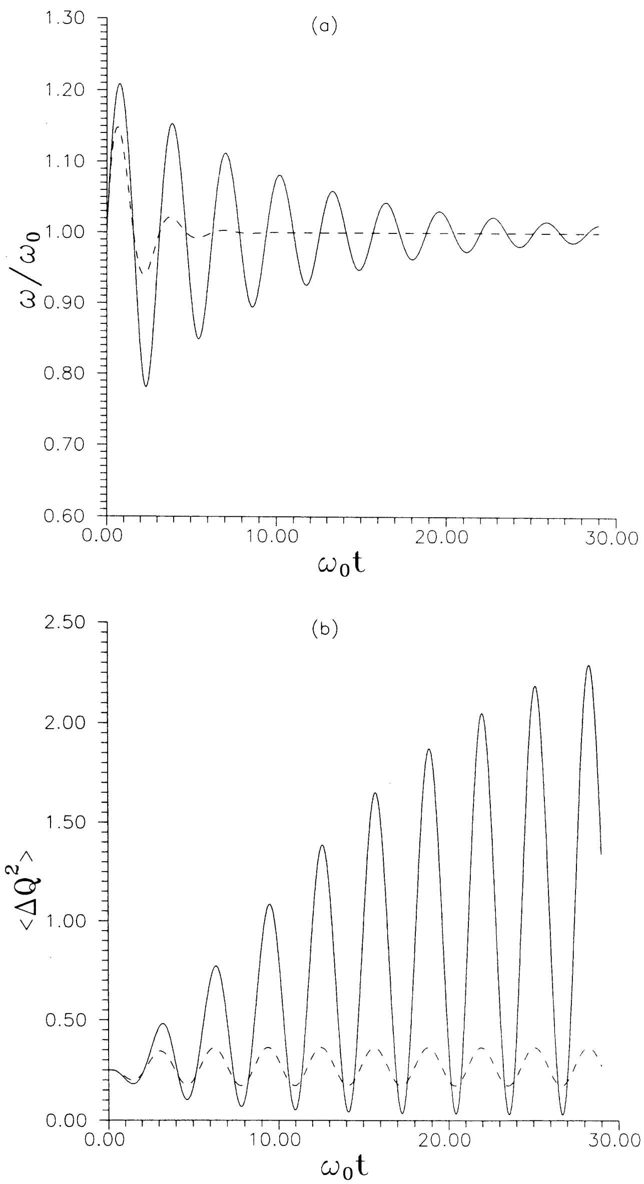

FIG. 2. Time evolution of a harmonic oscillator with a frequency function given by Eq. (53). The parameters are (i) $A=0.5$, $B=0.5\pi/\omega_0$, and $C=2\omega_0$ (dashed line), and (ii) $A=0.5$, $B=3\pi/\omega_0$, and $C=2\omega_0$ (solid line). (a) The $\omega(t)$ functions. (b) The time evolution of the variance of the dimensionless coordinate operator ($\hat{Q}=\sqrt{\omega_0 M/2}\hat{q}$).

show a simple harmonic oscillation of a squeezed state. If the damping is extremely weak ($B \gg 1/\omega_0$), resonance occurs and the minimal values of the pulsating variance of the coordinate tend to zero. We note that this behavior in the limiting case when there remains a simple oscillation of the restoring force is in agreement with the recently published paper of Abdalla and Colegrave [17], where absolute squeezing was found for resonant oscillation of the frequency.

The second example models the critical damping when there is no oscillation in the frequency

$$\omega^2(t)=\begin{cases}\omega_0^2\,, & t<0\\ \omega_0^2[1+At\exp(-t/B)]\,, & t\geq 0\,.\end{cases} \tag{54}$$

The frequency functions are shown in Fig. 3(a) for two different values of B and the corresponding variances can be seen in Fig. 3(b) ($A=0.5$). If B is large enough, then the oscillating variance of the coordinate operator will remain below its original value for a significant period of time.

The third case is overcritical damping with the frequency function

$$\omega^2(t)=\begin{cases}\omega_0^2\,, & t<0\\ \omega_0^2[1+At\exp(-t/B)\sinh(Ct)\,, & t\geq 0\,.\end{cases} \tag{55}$$

The shape of the $\omega(t)$ function [Fig. 4(a)] is quite similar to the previous example. The constants are $A=0.5$, $B=0.9/\omega_0$, and $C=\omega_0$; Fig. 4(b) shows the variance of

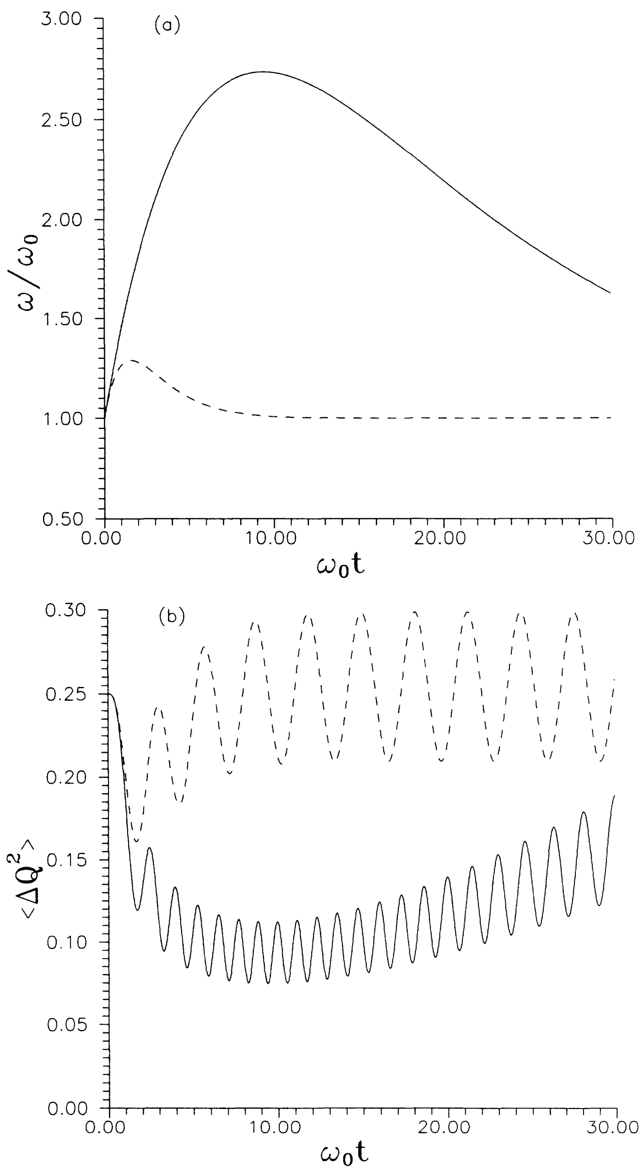

FIG. 3. Time evolution of a harmonic oscillator with a frequency function given by Eq. (54). The parameters are (i) $A=0.5$ and $B=0.5\pi/\omega_0$ (dashed line) and (ii) $A=0.5$ and $B=3\pi/\omega_0$ (solid line). (a) The $\omega(t)$ functions. (b) The time evolution of the variance of the dimensionless coordinate operator ($\hat{Q}=\sqrt{\omega_0 M/2}\hat{q}$).

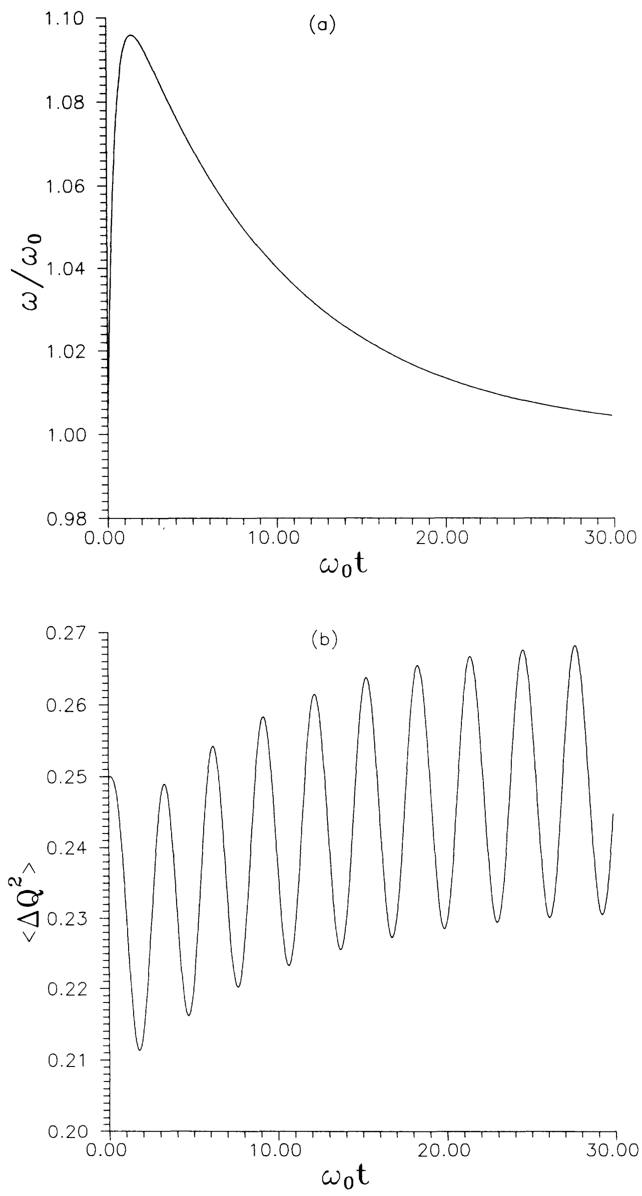

FIG. 4. Time evolution of a harmonic oscillator with a frequency function given by Eq. (55). The parameters are $A=0.5$, $B=0.9/\omega_0$, and $C=\omega_0$. (a) The $\omega(t)$ function. (b) The time evolution of the variance of the dimensionless coordinate operator ($\hat{Q}=\sqrt{\omega_0 M/2}\hat{q}$).

the coordinate.

The oscillator will be squeezed after a time in all the three cases, but the amount of squeezing strongly depends on the values of the parameters. Weaker damping in the oscillation of the restoring force will result in more enhanced squeezing.

VI. CONCLUSIONS

In the present paper we have determined the exact time evolution of the creation and annihilation operators of a harmonic oscillator with steplike time dependency of the frequency or the mass. Using these results we have developed an approximating method, which makes it possible to determine the time evolution of the oscillator with arbitrary derivable functions of the parameters. We have shown that the approximate solution in the limiting case tends to the exact solution of equations of motion.

We have applied the method to the case of the analytically solved linear time function of the restoring force and have illustrated how the approximation tends to the exact solution. As an other example we have studied a Hamiltonian describing a damped oscillation of the restoring force of a harmonic oscillator. We have discussed undercritical, and overcritical damping and have determined the time dependency of the variance of the coordinate operator in all the three cases.

The method of the present paper can be an easily applicable and effective means for determining the time evolution of time-dependent oscillators when the analytical solution is not known.

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