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Quantum-statistical properties of noise in a phase-sensitive linear amplifier

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We present a model for a two-photon phase-sensitive linear amplifier in which the phase sensitivity is introduced by injecting three-level atoms initially prepared in a coherent superposition of upper and lower levels. We have studied the quantum-statistical properties of noise and show that, under certain conditions, the additive noise is in a squeezed vacuum state whose squeezing parameter depends on the atomic variables.

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I. INTRODUCTION

The desire to retain nonclassical light properties through optical processing has encouraged research in the operation of quantum amplifiers [1–5]. The recent work on optical communication and high sensitivity quantum detectors has renewed interest in the quantum limits imposed on the amplifiers. Standard models of the linear amplifiers [6] indicate that the amplifier does indeed amplify an input signal but the output includes amplified noise, called added noise. The added noise arises from the coupling of the signal to the internal degrees of freedom of the amplifier. The nature of the added noise depends upon the state of these internal modes.

On the basis of classification by Caves [7], a phase-insensitive amplifier is the one which amplifies both the quadratures of the signal by the same factor and also adds equal noise to the two quadratures. It is, therefore, incapable of giving squeezed output for an unsqueezed input, whereas the phase-sensitive amplifier [8,9] is the one which responds differently to the two quadrature phases in the form of unequal gains or unequal noise or both. It has been suggested by many authors, that with the introduction and generation of squeezed states, it is possible to modify or "rig" the reservoir [10,11]. The reservoir state is produced in such a way that the processed boson mode is driven by the reservoir operators. The amplifier-added noise can be made widely different in two quadrature components of the field.

The idea of preparing atoms in a coherent superposition of atomic states has received attention because of its effectiveness in amplifying a squeezed-signal quadrature with reduced added noise as compared to the standard phase-insensitive amplifiers. It has been used for noise quenching by correlated-spontaneous-emission laser [12] and quantum-beat laser [13]. This idea has also been used in a phase-sensitive amplifier [14–16], in which atoms are prepared in a coherent superposition of atomic states. It is shown in these papers that under certain conditions the additive noise in one of the field quadratures goes to zero at the expense of enhanced noise in the conjugate quadrature.

In the present paper we consider the quantum-statistical properties of noise in a two-photon linear amplifier. We consider a system consisting of three-level atoms, initially prepared in a coherent superposition of atomic states. In particular, we show that in the Langevin picture, for an initial vacuum state, the expectation value of the normally ordered field operators is proportional to the expectation value of the corresponding antinormally ordered noise operators. This idea has motivated us to relate the P representation for the field operators to the corresponding Q representation for the noise operators. We also show that the additive noise is in a squeezed vacuum state and the effective squeezing parameter is a function of the initial atomic variables.

The organization of the paper is as follows. In Sec. II, using the Heisenberg-Langevin approach, it is shown for an initial vacuum state that the expectation value of normally ordered moments of the field operators is proportional to the expectation value of the corresponding antinormally ordered moments of the noise operators. A condition is obtained which relates the P representation for the field operators to the Q representation for the noise operators. In Sec. III, we define the model for the amplifier and the equation of motion for the reduced density matrix. In Sec. IV, an exact time-dependent solution of the Fokker-Planck equation for an initial vacuum state in the P representation has been obtained. A comparison between the P representation for the field operators and the Q representation for the noise operators shows that the additive noise is squeezed with the squeezing parameter being a function of the initial atomic variables. Section V contains the conclusion and a discussion of our results.

II. ADDITIVE NOISE IN THE AMPLIFIER

A linear amplifier by definition is the one whose output signal is linearly related to its input signal. It is now understood that the signal information is carried by the complex amplitudes of the relevant modes, rather than the number of quanta. The evolution equation for an operator representating a linear amplification process in the Langevin picture is

$$a_t = \sqrt{G}\, a_0 + \sqrt{(G-1)} N^\dagger \;, \tag{1}$$

where a_t and a_0 represent the annihilation operators for the field mode in the output and input, respectively, G is the gain of the amplifier, and $N^\dagger$ is the Langevin noise operator which is responsible for the amplifier's additive noise. The operator N itself is a boson operator. It is introduced in order that the operator a satisfies the boson commutation relation at the output.

The expectation value of the normally ordered product of creation and annihilation operator for the field mode are evaluated by using Eq. (1). For the field initially in a vacuum state $|0\rangle$,

$$\begin{aligned}\langle (a_t^\dagger)^n (a_t)^m\rangle &= \langle 0|[\sqrt{G}a_0^\dagger+\sqrt{(G-1)}N]^n \\ &\quad\times[\sqrt{G}a_0+\sqrt{(G-1)}N^\dagger]^m|0\rangle \\ &=(G-1)^{(n+m)/2}\langle N^n N^{\dagger m}\rangle \ . \end{aligned} \tag{2}$$

This result shows that for an initial vacuum state the expectation value of the normally ordered field operators in the output is proportional to the expectation value of the antinormally ordered noise operators. The proportionality established in Eq. (2) between the field operators and the noise operators is true only for the initial vacuum state. For any field other than vacuum, such a straightforward relationship does not hold. From Eq. (2) we have, for example,

$$\langle NN^\dagger\rangle = \frac{\langle a^\dagger a\rangle}{(G-1)} \ . \tag{3}$$

It may be of interest to note that the relationship established above between the normally ordered field operators and the corresponding antinormally ordered noise operators can also be represented in terms of a relationship between the P representation and Q representation for the output field and noise, respectively. It may be recalled that the P and Q representations can be used to evaluate the expectation values of the normally ordered and antinormally ordered operators, respectively. In view of Eq. (2), we can relate the P representation for the field variables to the Q representation for the noise variables via

$$P(\alpha,\alpha^*)\equiv Q_N(\mu,\mu^*) \ . \tag{4}$$

Here, the relation between μ and α is of the form

$$\mu=\frac{\alpha}{\sqrt{G-1}} \ . \tag{5}$$

It is thus apparent from Eq. (5) that the noise variable μ is dependent on the gain G of the amplifier. The expectation value of any antinormally ordered function $\langle F(N,N^\dagger)\rangle$ of N and $N^\dagger$ may be determined from $Q_N(\mu,\mu^*)$ via the relation [17]

$$\langle F(N,N^\dagger)\rangle = \int F(\mu,\mu^*)Q_N(\mu,\mu^*)d^2\mu \ . \tag{6a}$$

In particular we can write

$$\langle NN^\dagger\rangle = \int |\mu|^2 Q_N(\mu,\mu^*)d^2\mu \ . \tag{6b}$$

The function $Q_N(\mu,\mu^*)$ thus makes it possible to evaluate the expectation values of the antinormally ordered noise operators. We now turn to the specific model for the phase-sensitive linear amplifier.

III. THE AMPLIFIER MASTER EQUATION

Our system consists of three-level atoms in cascade configuration as shown in Fig. 1. The boson mode of frequency ω is assumed to be in resonance with two atomic transitions $|a\rangle\rightarrow|b\rangle$ and $|b\rangle\rightarrow|c\rangle$. The interaction Hamiltonian in the interaction picture and in the rotating-wave approximation is

$$H=\hbar g[a^\dagger(|b\rangle\langle a|+|c\rangle\langle b|)+a(|a\rangle\langle b|+|b\rangle\langle c|)] \ , \tag{7}$$

where $a^\dagger$ and a are the creation and the annihilation operators of the field and g is the atom-field coupling constant.

We consider a situation in which atoms are injected inside the amplifier at the rate r_a in a coherent superposition of states $|a\rangle$ and $|c\rangle$. The atomic wave function at time t is therefore,

$$|\psi(t)\rangle = C_a|a\rangle + C_c|c\rangle \ , \tag{8}$$

where C_a and C_c are the probability amplitudes for levels $|a\rangle$ and level $|c\rangle$, respectively. The atom-field density operator at time t is therefore

$$\begin{aligned}\rho_{\mathrm{AF}}(t)=[&\rho_{aa}|a\rangle\langle a|+\rho_{ac}|a\rangle\langle c| \\ &+\rho_{ca}|c\rangle\langle a|+\rho_{cc}|c\rangle\langle c|]\otimes\rho_F \ . \end{aligned}\tag{9}$$

In Eq. (9), ρ_{aa} and ρ_{cc} are the initial populations of levels $|a\rangle$ and $|c\rangle$, and ρ_{ac} and ρ_{ca} are responsible for the initial atomic coherence between levels $|a\rangle$ and $|c\rangle$.

The dynamics of the field mode equation of motion in the Born-Markov approximation governed by the master equation can be written as [14],

$$\begin{aligned}\frac{\partial\rho}{\partial t} &= -\tfrac{1}{2}\kappa\rho_{aa}(aa^\dagger\rho_F-2a^\dagger\rho_F a+\rho_F aa^\dagger) \\ &\quad-\tfrac{1}{2}\kappa\rho_{cc}(a^\dagger a\rho_F-2a\rho_F a^\dagger+\rho_F a^\dagger a) \\ &\quad-\tfrac{1}{2}\kappa\rho_{ca}(aa\rho_F-2a\rho_F a+\rho_F aa) \\ &\quad-\tfrac{1}{2}\kappa\rho_{ac}(a^\dagger a^\dagger\rho_F-2a^\dagger\rho_F a^\dagger+\rho_F a^\dagger a^\dagger) \ , \end{aligned}\tag{10}$$

where κ is the linear gain coefficient, the terms propor-

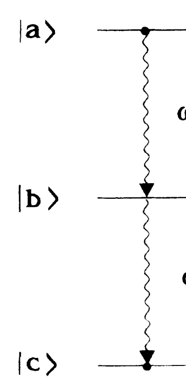

FIG. 1. Energy-level diagram for three-level atoms in cascade configuration.

tional to ρ_{aa} and ρ_{cc} correspond to the usual gain and absorption in the amplifier. The anomalous terms proportional to ρ_{ac} and ρ_{ca} are responsible for the phase-sensitive operation of the amplifier.

IV. FOKKER-PLANCK EQUATION AND ITS TIME-DEPENDENT SOLUTION

In this section we convert the density-matrix equation of motion for the field mode in a c-number Fokker-Planck equation using P representation for the density operator,

$$\rho=\int P(\alpha,\alpha^*)|\alpha\rangle\langle\alpha|d^2\alpha \ . \tag{11}$$

This representation provides a convenient way of evaluating the ensemble averages of normally ordered operators. The utility of normal ordering has long been recognized because of appropriateness of normally ordered products for the description of photon-absorption processes. Moreover, normal ordering has an application in the interpretation of photon counting and coherence experiments [18,19]. The correlation function $G^{(n)}$ for the quantized electromagnetic field are defined as the expectation values of normally ordered products of the annihilation and creation operators.

We transform the equation of motion (10) into the Fokker-Planck equation via the substitution [17],

$$a|\alpha\rangle\langle\alpha|=\alpha|\alpha\rangle\langle\alpha| \ , \tag{12a}$$

$$a^\dagger|\alpha\rangle\langle\alpha|=\left[\alpha^*+\frac{\partial}{\partial\alpha}\right]|\alpha\rangle\langle\alpha| \ . \tag{12b}$$

The resultant Fokker-Planck equation is of the form

$$\frac{\partial P}{\partial t}=\left[-\frac{\kappa}{2}(\rho_{aa}-\rho_{cc})\left[\frac{\partial}{\partial\alpha}\alpha+\frac{\partial}{\partial\alpha^*}\alpha^*\right]-\frac{\kappa}{2}\rho_{ac}\frac{\partial}{\partial\alpha^2}-\frac{\kappa}{2}\rho_{ca}\frac{\partial}{\partial\alpha^{*2}}+\kappa\rho_{aa}\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right]P \ . \tag{13}$$

For an initial vacuum state with complex amplitudes, the Fokker-Planck equation at time t has the form

$$P(\alpha,\alpha^*,t)=\frac{(N_1N_2)^{1/2}}{\pi}\exp\left\{-\frac{|\alpha|^2}{2}(N_1+N_2)-\tfrac{1}{4}[N_1(A-iC)^2+N_2(B-iD)^2]\alpha^2-\tfrac{1}{4}[N_1(A+iC)^2+N_2(B+iD)^2]\alpha^{*2}\right\} , \tag{14}$$

where

$$N_1=\frac{\rho_{aa}-\rho_{cc}}{(\rho_{aa}+|\rho_{ac}|)(G-1)} \ , \tag{15a}$$

$$N_2=\frac{\rho_{aa}-\rho_{cc}}{(\rho_{aa}-|\rho_{ac}|)(G-1)} \ , \tag{15b}$$

and

$$G=\exp[\kappa(\rho_{aa}-\rho_{cc})t] \ , \tag{16}$$

is the amplification factor. The amplification takes place when $\rho_{aa}>\rho_{cc}$. The values of the constants appearing in Eq. (14) are

$$A=\tfrac{1}{2}\left[(2|\rho_{ac}|-\rho_{ac}-\rho_{ca})/|\rho_{ac}|\right]^{1/2} , \tag{17a}$$

$$B=\tfrac{1}{2}\left[(2|\rho_{ac}|+\rho_{ac}+\rho_{ca})/|\rho_{ac}|\right]^{1/2} , \tag{17b}$$

$$C=\frac{i}{2}\frac{(\rho_{ac}-\rho_{ca})}{[(2|\rho_{ac}|-\rho_{ac}-\rho_{ca})|\rho_{ac}|]^{1/2}} \ , \tag{17c}$$

$$D=\frac{i}{2}\frac{(\rho_{ac}-\rho_{ca})}{[(2|\rho_{ac}|+\rho_{ac}+\rho_{ca})|\rho_{ac}|]^{1/2}} \ . \tag{17d}$$

In general, $|\rho_{ac}|^2\geq\rho_{aa}\rho_{cc}$. The above expression for $P(\alpha,\alpha^*,t)$ is valid for arbitrary value of coherence ρ_{ac}.

The solution (14) of the Fokker-Planck equation shows that the phase sensitivity which was introduced through the coherent atomic injection is still reflected in the constants N_1, N_2, A, B, C, and D. An interesting case arises when atoms are injected in perfect coherence, i.e.,

$$|\rho_{ac}|^2=\rho_{aa}\rho_{cc} \ . \tag{18}$$

We can then define a parameter η such that

$$\rho_{aa}=(1+\eta)/2 \ , \tag{19a}$$

$$\rho_{cc}=(1-\eta)/2 \ , \tag{19b}$$

$$\rho_{ac}=|\rho_{ac}|\exp(i\phi) \ , \tag{19c}$$

$$|\rho_{ac}|=(\tfrac{1}{2})(1-\eta^2)^{1/2} \ . \tag{19d}$$

On substituting for N_1, N_2, A, B, C, and D, solution (14) simplifies considerably and we obtain

$$P(\alpha,\alpha^*,t)=\frac{1}{\pi(G-1)}\left[\frac{2\eta}{1+\eta}\right]^{1/2}\times\exp\left\{-\frac{|\alpha|^2}{(G-1)}-\frac{1}{2(G-1)}(\alpha^2e^{-i\phi}+\alpha^{*2}e^{i\phi})\times\left[\frac{1-\eta}{1+\eta}\right]^{1/2}\right\} . \tag{20}$$

We next recall the relationship established between the P representation for the field operators and the corresponding Q representation for the noise operators in Eq. (4). It follows from Eqs. (4) and (20) that Q_N is given by the following:

$$Q_N(\mu,\mu^*,t)=\frac{1}{\pi(G-1)}\left[\frac{2\eta}{1+\eta}\right]^{1/2}\times\exp\left\{-|\mu|^2-\tfrac{1}{2}(\mu^2e^{-i\phi}+\mu^{*2}e^{i\phi})\times\left[\frac{1-\eta}{1+\eta}\right]^{1/2}\right\}. \tag{21}$$

Next we show that the Q representation for the noise operators corresponds to the Q representation for an ideal squeezed vacuum state. A squeezed vacuum state is given by

$$|\xi\rangle=S(\xi)|0\rangle\ , \tag{22}$$

where the unitary squeeze operator is defined by

$$S(\xi)=\exp(\tfrac{1}{2}\xi^*a^2+\tfrac{1}{2}\xi a^{\dagger 2})\ , \tag{23}$$

with $\xi=r\exp(i\theta)$ being an arbitrary complex number. Here, $r=|\xi|$ is called the squeeze parameter and θ is the reference phase of the squeezed field. Then for a state $|\xi\rangle$ [20]

$$Q(\alpha,\alpha^*,\xi)=\frac{1}{\pi}|\langle\alpha|\xi\rangle|^2=\frac{\text{sech}(r)}{\pi}\exp[-|\alpha|^2-\tfrac{1}{2}(\alpha^2e^{-i\theta}+\alpha^{*2}e^{i\theta})\times\tanh(r)]\ . \tag{24}$$

A comparison of Eqs. (21) and (24) shows that the Q representation for the added noise is identical to the Q representation for an ideal squeezed vacuum state if we identify

$$\theta=\phi \tag{25a}$$

and

$$\tanh(r)=\left[\frac{1-\eta}{1+\eta}\right]^{1/2}\ . \tag{25b}$$

The conclusion is that the additive noise is in a squeezed vacuum state with the effective squeezing parameter r.

On substituting the value of η in Eq. (25b) we find

$$\tanh^2(r)=\frac{\rho_{cc}}{\rho_{aa}}\ , \tag{26}$$

The additive noise is therefore squeezed with the squeezing parameter being a function of the initial atomic conditions. It is obvious that $r=0$ ($\rho_{cc}=0$) corresponds to incoherent excitation. The added noise, in this case, is due to vacuum fluctuations.

V. CONCLUSION

In conclusion, we have shown that a three-level phase-sensitive linear amplifier prepares the internal degrees of freedom in such a way that the additive noise is in a squeezed vacuum state. The associated squeezing parameter is a function of initial atomic variables.

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