Infrared radiation by a Dirac electron: First-order correction to the cross-section sum rule

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Bloch and Nordsieck [Phys. Rev. **52**, 54 (1937)] outlined a successive-approximation procedure for the calculation of finite infrared radiative corrections to the cross section for potential scattering of a Dirac electron, based on the use of coherent states for the description of the asymptotic motion of the electron in the presence of the radiation field and on the introduction of positive- and negative-energy projection operators. An elaboration of that procedure is described here in which the projection operators are used to develop an effective-Hamiltonian formulation of the scattering problem. This allows for a clear separation of the nearly singular contributions to the cross section from the remainder to which ordinary perturbation theory is applicable. With the aid of the optical theorem and the low-frequency approximation for single-photon spontaneous bremsstrahlung, a first-order correction to the Bloch-Nordsieck cross-section sum rule is obtained. The result is expressed in terms of measurable field-free scattering parameters. With the initial photon state chosen to be highly occupied, the formalism is directly applicable to the problem of scattering in an intense, slowly varying external field. Nearly singular terms arising from virtual Compton scattering in initial and final states, which appear when the field is not of the plane-wave type, are shown to cancel in the final form of the cross-section sum rule. This low-frequency approximation is then easily applied to problems involving realistically modeled fields in the form of wave packets, for example.

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I. INTRODUCTION

The Bloch-Nordsieck (BN) treatment of radiative corrections to scattering [1], based on the use of coherent states for the description of the asymptotic motion of the electron, provided the basis for the resolution of the infrared divergence problem in QED. At the same time, the BN calculation represents one of the earliest examples of a low-frequency approximation and this is of theoretical interest in its own right in view of the important role played subsequently by such approximations in the analysis of spontaneous bremsstrahlung [2,3] and scattering in a strong laser field [4]. Moreover, the BN formalism is directly applicable to external-field problems; one simply allows the initial photon state to be highly occupied. The utility of this approach is illustrated below with the derivation of a sum rule for stimulated bremsstrahlung in a low-frequency external field which need not be a monochromatic plane wave; this extends previous work based on standard treatments [4]. As seen in an earlier study [5], the low-frequency approximation for single-photon bremsstrahlung can be used to obtain an improved version of the BN sum rule for nonrelativistic scattering with the emission of an arbitrary number of soft (unobservable) photons—a correction to the sum rule, of first order in the soft-photon frequency, was derived [6]. Here we obtain an improved sum rule, of first-order accuracy, for the system originally considered by BN, namely, the potential scattering of a Dirac electron. Two features of this problem were not encountered in the earlier nonrelativistic calculations; negative-energy states must now be accounted for and electron-recoil effects must be treated exactly. (Recoil effects introduce corrections of order v/c. Such corrections were previously taken to be of first order [5] but here v/c is not treated as a small parameter.) These features require the use of a substantially different approach. Specifically, a projection-operator formalism is introduced to allow for a separation of positive- and negative-energy states. (It was in fact emphasized by BN that such an approach provides the basis for obtaining higher-order corrections to the approximation which they derived.) The BN coherent states are not exact asymptotic solutions in the positive-energy subspace but are sufficiently accurate for our present purposes, that is, for the calculation of the sum rule correct to first order. The formulation of the scattering problem in terms of an effective Hamiltonian operating in the positive-energy subspace is described in Sec. II. The calculation of the scattering cross section summed over final states of the radiation field is presented in Sec. III. The calculation is greatly simplified by the use of Low's theorem [2] on low-frequency single-photon bremsstrahlung—the version applicable to potential scattering of a Dirac electron is derived in the Appendix—and the application of the optical theorem relating the imaginary part of the forward-scattering amplitude to the total cross section. It is hoped that methods developed here, in the context of the simplified BN model, will be useful in the derivation of improved infrared radiative corrections in a more general formulation of the theory, as well as in providing a uniform treatment of spontaneous and stimulated bremsstrahlung.

II. FORMULATION

A. Effective-Hamiltonian method

As a preliminary to the analysis of the effect of infrared radiation we begin with a review of the application of the Casimir projection-operator formalism to the problem of scattering in the absence of radiative interactions [7]. The free-particle Dirac equation is written, in units with $\hbar=c=1$ and with spin indices suppressed, as

$$H_D|\mathbf{p}\rangle\equiv(\boldsymbol{\alpha}\cdot\mathbf{p}_e+\beta m)|\mathbf{p}\rangle=E(\mathbf{p})|\mathbf{p}\rangle\ , \tag{2.1}$$

with $E(\mathbf{p})\equiv\sqrt{|\mathbf{p}|^2+m^2}\equiv p_0$. Here $\mathbf{p}_e$ is the electron momentum operator and the Dirac matrices satisfy the standard anticommutation relations. The solution of Eq. (2.1) is represented as

$$\langle\mathbf{r}|\mathbf{p}\rangle=(2\pi)^{-3/2}\exp(i\mathbf{p}\cdot\mathbf{r})\gamma(\mathbf{p})\ , \tag{2.2}$$

where $\gamma(\mathbf{p})$ is a free-particle spinor. The operators which project onto positive- and negative-energy states are defined as

$$\Lambda_\pm=\frac{E(\mathbf{p}_e)\pm H_D}{2E(\mathbf{p}_e)}\ . \tag{2.3}$$

In addition to the relations $\Lambda_\pm^2=\Lambda_\pm$ and $\Lambda_+\Lambda_-=0$, the properties

$$\Lambda_+\boldsymbol{\alpha}\Lambda_+=\mathbf{v}_e\Lambda_+,\quad \Lambda_+\beta\Lambda_+=\sqrt{1-|\mathbf{v}_e|^2}\Lambda_+\ , \tag{2.4}$$

along with similar relations satisfied by Λ_-, are easily verified.

Consider now the problem of scattering in a short-range potential $V(\mathbf{r})$ [8]. This problem may be reformulated in the space of positive-energy states, with an effective scattering potential of the form

$$v_{\text{eff}}(E)=\Lambda_+[V+Vg_-(E)V]\Lambda_+\ . \tag{2.5}$$

The Green's function in the negative-energy subspace satisfies

$$\Lambda_-(E-H_D-V)\Lambda_-g_-(E)=\Lambda_-\ . \tag{2.6}$$

We define the positive-energy Green's function, associated with the effective Hamiltonian, as the solution of

$$\Lambda_+[E-H_D-v_{\text{eff}}(E)]\Lambda_+g_{\text{eff}}(E)=\Lambda_+\ . \tag{2.7}$$

In terms of these operators, the scattering amplitude (in the absence of radiative interactions) takes the form $t(\mathbf{p}',\mathbf{p};E)=\langle\mathbf{p}'|f(E)|\mathbf{p}\rangle$, where

$$f(E)=v_{\text{eff}}(E)+v_{\text{eff}}(E)g_{\text{eff}}(E)v_{\text{eff}}(E)\ . \tag{2.8}$$

The condition which places this amplitude on the energy shell is $E=p_0=p_0'$. A generalization of this effective-Hamiltonian formalism that includes infrared radiative interactions will now be described.

The Hamiltonian governing the problem of scattering in a radiation field is

$$H=H_D+V+H_F+H'\ , \tag{2.9}$$

where

$$H_F=\sum_{j=0}^{s}\omega_j a_j^\dagger a_j \tag{2.10}$$

is the free-field Hamiltonian and $H'=-e\boldsymbol{\alpha}\cdot\mathbf{A}$ is the interaction energy in the radiation gauge. The vector potential, quantized in a box of volume Ω, is represented as

$$\mathbf{A}=\sum_{j=0}^{s}\left[\frac{2\pi}{\omega_j\Omega}\right]^{1/2}\boldsymbol{\varepsilon}_j(a_je^{i\mathbf{k}_j\cdot\mathbf{r}}+a_j^\dagger e^{-i\mathbf{k}_j\cdot\mathbf{r}})\ . \tag{2.11}$$

As in the original BN model only soft-photon modes, with maximum frequency ω_s, are included in the representation of the field.

Asymptotic solutions of the Dirac equation are required in the formulation of the scattering problem. Exact solutions in the absence of the scattering potential are not available. However, in the positive-energy subspace the coherent states introduced by BN, and reviewed below in Sec. II B, represent approximate eigenfunctions of the Hamiltonian

$$K=\Lambda_+(H_D+H_F+H')\Lambda_+\ , \tag{2.12}$$

correct to lowest order. Here and in the following the "order" is defined in terms of the two small parameters in the problem, namely the maximum soft-photon frequency ω_s and the electron-field interaction energy. (With an external field present a limitation on its intensity must be imposed to maintain the validity of this approximation procedure; we return to this point briefly later on.) It will be convenient to proceed, at this point, with the assumption that the exact solutions of the asymptotic wave equation

$$(H_D+H_F+H'-E)\psi=0 \tag{2.13}$$

are available. Approximations for these solutions will be introduced after the scattering problem has been formulated.

As a first step we follow the method suggested by BN and decompose the solution of Eq. (2.13) into components $\psi_\pm=\Lambda_\pm\psi$. Then, with $H^0\equiv H_D+H_F+H'$, one finds that the equation satisfied by ψ_+ is

$$\Lambda_+[H^0+H'G_-^0(E)H'-E]\Lambda_+\psi_+=0\ , \tag{2.14}$$

where the negative-energy Green's function is defined as the solution of

$$\Lambda_-(E-H^0)\Lambda_-G_-^0(E)=\Lambda_-\ . \tag{2.15}$$

The decoupling procedure leading to Eq. (2.15) also provides us with the representation

$$\psi_-=G_-^0(E)\Lambda_-H'\Lambda_+\psi_+\ . \tag{2.16}$$

The amplitude for scattering by the local potential $V(\mathbf{r})$ is represented, in the standard manner, in terms of the resolvent $G(E)=(E-H)^{-1}$ as

$$T=\langle\psi'|[V+VG(E)V]|\psi\rangle\ , \tag{2.17}$$

where ψ and ψ' represent the initial and final states, respectively; a more explicit labeling of states will be adopted later on. The introduction of an effective positive-energy scattering operator is accomplished with the sub-

stitution of the form (2.16) for the negative-energy component of the asymptotic solution and with the use of the identity [9]

$$G = G_- + (\Lambda_+ + G_- H) G_{\text{eff}} (\Lambda_+ + H G_-) \ . \tag{2.18}$$

Here we have introduced the negative-energy Green's function as the solution of

$$\Lambda_- (E - H^0 - V) \Lambda_- G_-(E) = \Lambda_- \ . \tag{2.19}$$

The effective positive-energy Green's function appearing in Eq. (2.18) satisfies the resolvent equation

$$\Lambda_+ [E - H^0 - V_{\text{eff}}(E)] \Lambda_+ G_{\text{eff}}(E) = \Lambda_+ \ , \tag{2.20}$$

with the effective potential defined as

$$V_{\text{eff}} = \Lambda_+ \{ V + [V + H'] G_-(E) [V + H'] \} \Lambda_+ \ . \tag{2.21}$$

The essential feature of the BN analysis is the recognition that the electron-field interaction, even when intrinsically weak, must be treated nonperturbatively in the construction of the asymptotic wave function ψ_+ owing to the appearance of near degeneracies in the positive-energy spectrum of the electron-field system. In the negative-energy subspace, on the other hand, ordinary perturbation theory is adequate (according to our assumption concerning the strength of an external field that may be present) so that the equation

$$G_-(E) = g_-(E - H_F) + g_-(E - H_F) H' G_-(E) \tag{2.22}$$

may be solved by iteration. Working to first order in H' we find, after a brief calculation, that the expression for the transition amplitude may be reduced to

$$T = \langle \psi'_+ | [(V^{(0)} + V^{(1)}) + (V^{(0)} + V^{(1)}) G_{\text{eff}} (V^{(0)} + V^{(1)})] | \psi_+ \rangle \ , \tag{2.23}$$

where

$$V^{(0)}(E) = \Lambda_+ [V + V g_-(E - H_F) V] \Lambda_+ \equiv v_{\text{eff}}(E - H_F) \ , \tag{2.24}$$

[as introduced earlier in Eq. (2.5)] and where

$$V^{(1)}(E) = \Lambda_+ [H' g_-(E) V + V g_-(E) H' + V g_-(E) H' g_-(E) V] \Lambda_+ \ . \tag{2.25}$$

Further reduction is achieved with the introduction of the expansion

$$G_{\text{eff}}(E) \cong g_{\text{eff}}(E - H_F) + g_{\text{eff}}(E)(H' + V^{(1)}) g_{\text{eff}}(E) \ , \tag{2.26}$$

where both the interaction energy and the field energy are treated as quantities of first order and terms of second order are ignored. We then find that the scattering amplitude may be represented, to first order, as

$$T = \langle \psi'_+ | F^{(0)} + F^{(1)} | \psi_+ \rangle \ , \tag{2.27}$$

with

$$F^{(0)} = f(E - H_F) \ , \tag{2.28a}$$

$$F^{(1)} = V^{(1)} + V^{(1)} g_{\text{eff}} V^{(0)} + V^{(0)} g_{\text{eff}} V^{(1)} + V^{(0)} g_{\text{eff}} (H' + V^{(1)}) g_{\text{eff}} V^{(0)} \ . \tag{2.28b}$$

The field-free scattering operator $f(E)$ is defined in Eq. (2.8) and each of the operators in Eq. (2.28b) is evaluated at the energy E.

B. Asymptotic solutions

The coherent states introduced by BN provide approximations to the asymptotic solutions appropriate to a treatment of both infrared-radiative corrections to scattering and to scattering in a low-frequency external field. More precisely, these states, when projected onto the positive-energy subspace, are approximate solutions of Eq. (2.14). We shall now review those properties of the coherent states that are relevant to the present analysis. As seen from the original derivation [1], the coherent states satisfy Eq. (2.14) only to the extent that terms of second order, associated with the effect of electron recoil, are neglected. It does not immediately follow that the error in these asymptotic states is of second order since the near-degeneracy of the states of the electron-photon system may lead to the introduction of small energy denominators and a consequent lowering of the nominal order of the error. To resolve the question one should derive an explicit form for the correction to the BN approximation to the asymptotic states, but this turns out to be unnecessary for our present purposes, as will now be explained. The asymptotic solutions in the positive-energy subspace enter into the construction of the scattering matrix element shown in Eq. (2.27). The scattering operator is decomposed into terms of zeroth order and of first order, as indicated in Eqs. (2.28). In evaluating the matrix element of the first-order component it is sufficient to adopt the BN approximation, with first-order accuracy in the matrix element assured. On the other hand, in calculating the component of the operator of zeroth order an approximation for the asymptotic states correct to first order is required. Fortunately, the calculation can be done very easily using only a general property of the exact solution, namely, the unitarity property of the wave operator that transforms the noninteracting electron-field state into the fully interacting state, as will be described more explicitly below [10]. As a final comment on these matters it should be remarked that the energy-level shift (associated with asymptotic electron-photon interactions) enters into our final formula for the cross section. An expression for this shift is provided by the BN approximation. The variational principle for the energy eigenvalue guarantees that this expression is correct to first order since the shifted energy is given, to first order, by the expectation value of the particle-field Hamiltonian.

To begin our review of those properties of the coherent states that will be needed here, we note that to first order the Hamiltonian appearing in Eq. (2.14) may be replaced by $K = \Lambda_+ H^0 \Lambda_+$. Using Eq. (2.4), along with the relation $E(\mathbf{p}_e) = \mathbf{v}_e \cdot \mathbf{p}_e + m\sqrt{1 - |\mathbf{v}_e|^2}$, one sees that this positive-energy Hamiltonian can be expressed as

$$K = \Lambda_+ [H_F + E(\mathbf{p}_e) - e \mathbf{v}_e \cdot \mathbf{A}] \Lambda_+ \ . \tag{2.29}$$

Let us assume that before the electron-field interaction is switched on the electron has momentum $\mathbf{p}$ and the field is in a state characterized by the set of mode occupation numbers $\{n_1, n_2, \ldots, n_s\}$ denoted collectively as n. The states ψ_+ that appear in the preceding analysis of the scattering problem will be represented as $\Lambda_+\psi_{n\mathbf{p}}$. An approximate solution of the modified Dirac equation

$$K\Lambda_+|\psi_{n\mathbf{p}}\rangle = E_{n\mathbf{p}}\Lambda_+|\psi_{n\mathbf{p}}\rangle \tag{2.30}$$

may be constructed by choosing $\psi_{n\mathbf{p}}$ to be of the form

$$|\psi_{n\mathbf{p}}\rangle = W(\{\rho\})|n\rangle|\mathbf{p}\rangle \ , \tag{2.31}$$

with the approximate wave operator given by

$$W(\{\rho\}) = \exp\left[\sum_{j=0}^{s}\rho_j(a_j e^{i\mathbf{k}_j\cdot\mathbf{r}} - a_j^\dagger e^{-i\mathbf{k}_j\cdot\mathbf{r}})\right] , \tag{2.32}$$

and with

$$\rho_j = -e\left[\frac{2\pi}{\omega_j\Omega}\right]^{1/2}\frac{\boldsymbol{\varepsilon}_j\cdot\mathbf{v}}{\eta_j} \ . \tag{2.33}$$

Here we define $\eta_j = \omega_j - \mathbf{v}\cdot\mathbf{k}_j$, with $\mathbf{v} = \mathbf{p}/p_0$ representing the velocity of the electron. The energy eigenvalue is

$$E_{n\mathbf{p}} = E(\mathbf{p}) + \sum_{j=0}^{s} n_j\omega_j + \Delta_{\mathbf{p}} \ , \tag{2.34a}$$

where

$$\Delta_{\mathbf{p}} = -\sum_{j=0}^{s}\eta_j\rho_j^2 \ . \tag{2.34b}$$

The parameters ρ_j are treated as quantities of zeroth order so that the level shift $\Delta_{\mathbf{p}}$ is of first order. (This formal ordering of parameters is justified in Sec. III by an explicit evaluation and numerical estimate of the level shift.) We note the property

$$W^\dagger(\{\rho\})a_j W(\{\rho\}) = a_j - \rho_j e^{-i\mathbf{k}_j\cdot\mathbf{r}} \ , \tag{2.35}$$

which follows directly from the commutation relations of the field operators. Another useful property is the relation

$$W = \exp(-i\mathbf{p}_F\cdot\mathbf{r})W_0\exp(i\mathbf{p}_F\cdot\mathbf{r}) \ , \tag{2.36}$$

where $\mathbf{p}_F = \sum_j \mathbf{k}_j a_j^\dagger a_j$ is the field momentum operator and W_0 is obtained from the expression shown in Eq. (2.32) by setting $\mathbf{r}=0$. The representation (2.36) is an expression of momentum conservation and applies as well to the exact wave operator.

III. CROSS-SECTION SUM RULE

The characteristic feature of the low-frequency approximation is that it allows one to express the cross section for a radiative transition in terms of that for a simpler, measurable process. It is our objective to relate the forward-scattering amplitude (and hence the total cross section, with the aid of the optical theorem) to the on-shell field-free scattering amplitude. This will be accomplished in two steps. We first show that the first-order approximation for the radiative transition amplitude, derived in Sec. II, can be related directly to the amplitude for spontaneous single-photon bremsstrahlung. In the second step (described in the Appendix) the low-frequency approximation for this bremsstrahlung amplitude will be invoked, and this introduces the physical field-free amplitude. Recall that the lowest-order version of the BN sum rule is expressed in terms of measurable field-free scattering parameters [1]. It is a remarkable fact, not obvious at the outset, that this remains true even when a correction of first order is included.

In the first-order approximation developed in Sec. II the transition amplitude is represented as $T_{n'\mathbf{p}';n\mathbf{p}} \cong T^{(0)}_{n'\mathbf{p}';n\mathbf{p}} + T^{(1)}_{n'\mathbf{p}';n\mathbf{p}}$, with

$$T^{(i)}_{n'\mathbf{p}';n\mathbf{p}} = \langle\psi_{n'\mathbf{p}'}|F^{(i)}(E_{n\mathbf{p}})|\psi_{n\mathbf{p}}\rangle, \quad i=0,1 \ . \tag{3.1}$$

The operators $F^{(i)}$ are defined in Eqs. (2.28). Since we shall make use of the optical theorem we examine the forward-scattering amplitude, setting $\mathbf{p}'=\mathbf{p}$ and taking the initial and final photon states to be identical. Let us consider first the amplitude

$$T^{(0)}_{n\mathbf{p};n\mathbf{p}} = \langle\psi_{n\mathbf{p}}|f(E_{n\mathbf{p}} - H_F)|\psi_{n\mathbf{p}}\rangle \ , \tag{3.2a}$$

which, with the aid of Eqs. (2.31) and (2.36), may be expressed as

$$\begin{aligned} T^{(0)}_{n\mathbf{p};n\mathbf{p}} &= \langle n|W_0^\dagger\langle\mathbf{p}|\exp(i\mathbf{p}_{Fn}\cdot\mathbf{r})f(E_{n\mathbf{p}} - H_F) \\ &\quad\times\exp(-i\mathbf{p}_{Fn}\cdot\mathbf{r})|\mathbf{p}\rangle W_0|n\rangle \\ &= \gamma^\dagger(\mathbf{p})\langle n|W_0^\dagger f(\mathbf{p}-\mathbf{p}_{Fn},\mathbf{p}-\mathbf{p}_{Fn};E_{n\mathbf{p}} - H_F) \\ &\qquad\times W_0|n\rangle\gamma(\mathbf{p}) \ . \end{aligned} \tag{3.2b}$$

Here we introduced the abbreviation $\mathbf{p}_{Fn} \equiv \mathbf{p}_F - \mathbf{p}_n$, where $\mathbf{p}_n$ is the momentum of the field in state $|n\rangle$. Similarly, we write $H_{Fn} \equiv H_F - E_n$, with E_n representing the field energy. The scattering matrix is now expanded about its on-shell value. The result is then simplified considerably with the aid of the relations

$$\langle n|W_0^\dagger H_{Fn}W_0|n\rangle = \sum_{j=0}^{s}\omega_j\rho_j^2 \ , \tag{3.3a}$$

$$\langle n|W_0^\dagger\mathbf{p}_{Fn}W_0|n\rangle = \sum_{j=0}^{s}\mathbf{k}_j\rho_j^2 \ . \tag{3.3b}$$

These relations are verified using the properties, shown in Eq. (2.35), satisfied by the approximate wave operator. The expansion in Eq. (3.2b) is carried out by transforming to new scalar variables; we write $f(\mathbf{p}',\mathbf{p};E) \to f(\nu,\tau,\xi,\xi')$ with

$$\nu = p_0, \quad \tau = (\mathbf{p}'-\mathbf{p})^2, \quad \xi = p_0 - E, \quad \xi' = p_0' - E \ . \tag{3.4}$$

Using the relation $\sqrt{(\mathbf{p}-\mathbf{p}_{Fn})^2+m^2} \cong p_0 - \mathbf{v}\cdot\mathbf{p}_{Fn}$, we find that

$$\begin{aligned} &\xi = \xi' = -\mathbf{v}\cdot\mathbf{p}_{Fn} - \Delta_{\mathbf{p}} + H_{Fn} \ , \\ &\nu = p_0 - \mathbf{v}\cdot\mathbf{p}_{Fn} \ , \\ &\tau = 0 \ . \end{aligned} \tag{3.5}$$

With the neglect of second- and higher-order terms in the Taylor series expansion of the scattering matrix we ob-

tain the result

$$T^{(0)}_{n\mathbf{p};n\mathbf{p}}=\gamma^{\dagger}(\mathbf{p})\left[f-\left[\sum_j \mathbf{v}\cdot\mathbf{k}_j\rho_j^2\right]\frac{\partial f}{\partial\nu}-2\Delta_{\mathbf{p}}\left[\frac{\partial f}{\partial\xi}+\frac{\partial f}{\partial\xi'}\right]\right]\gamma(\mathbf{p}) , \tag{3.6}$$

where on the right-hand side f and its derivatives are evaluated at $\nu=p_0$, $\xi=\xi'=\tau=0$. Note that in evaluating the first term on the right-hand side of zeroth order, it was necessary to retain first-order accuracy in the wave operator. However, it was only the unitarity property of the wave operator that had to be invoked [10] and this enabled us to complete the calculation very easily while maintaining sufficient accuracy.

We now consider the first-order correction term

$$T^{(1)}_{n\mathbf{p};n\mathbf{p}}=\langle\psi_{n\mathbf{p}}|F^{(1)}|\psi_{n\mathbf{p}}\rangle , \tag{3.7}$$

where $F^{(1)}$ is given by Eq. (2.28b). Our objective is to relate this expression to the correction to the single-photon bremsstrahlung amplitude and then to use the low-frequency approximation to evaluate it. To illustrate the procedure we examine the contribution arising from the first term in Eq. (2.28b), which is

$$\langle\psi_{n\mathbf{p}}|V^{(1)}|\psi_{n\mathbf{p}}\rangle=\langle\mathbf{p}|\langle n|W^{\dagger}\Lambda_+(H'g_-V+Vg_-H'+Vg_-H'g_-V)\Lambda_+W|n\rangle|\mathbf{p}\rangle . \tag{3.8}$$

With recoil effects ignored in this first-order term we have, for example,

$$\langle n|W_0^{\dagger}\Lambda_+H'\Lambda_-W_0|n\rangle=\langle n|\Lambda_+W_0^{\dagger}(-e\boldsymbol{\alpha}\cdot\mathbf{A})W_0\Lambda_-|n\rangle=\Lambda_+\sum_j(-2\rho_j)\left[\frac{2\pi}{\omega_j\Omega}\right]^{1/2}(-e\boldsymbol{\alpha}\cdot\boldsymbol{\varepsilon}_j)\Lambda_- . \tag{3.9}$$

The same form is obtained for each term in Eq. (3.8); we then write

$$T^{(1)}_{n\mathbf{p};n\mathbf{p}}=\sum_j(-2\rho_j)\gamma^{\dagger}(\mathbf{p})B_j\gamma(\mathbf{p}) . \tag{3.10}$$

The matrix B_j may be related to the nonsingular contribution to the amplitude for emission of a photon into mode j during the course of the collision. The method used for making this identification will now be described.

The transition amplitude defined using coherent states may be interpreted as a generating function for N-photon bremsstrahlung amplitudes. That is, with level-shift corrections ignored, the expansion of the transition amplitude in powers of the charge generates expressions corresponding to the lowest nonvanishing order of perturbation theory for each of these bremsstrahlung amplitudes. To first order, for example, we have (omitting a mode index for the photon)

$$W\cong 1+\rho(ae^{i\mathbf{k}\cdot\mathbf{r}}-a^{\dagger}e^{-i\mathbf{k}\cdot\mathbf{r}}) ,$$
$$W^{\dagger}\cong 1-\rho'(ae^{i\mathbf{k}\cdot\mathbf{r}}-a^{\dagger}e^{-i\mathbf{k}\cdot\mathbf{r}}) .$$

We first construct the amplitude $T_{n\mathbf{p}';0\mathbf{p}}$, with $|n\rangle=a^{\dagger}|0\rangle$, for single-photon emission from the vacuum; $\mathbf{p}$ and $\mathbf{p}'$ will be taken to have the same direction at the end of the calculation. When evaluated to first order, and after being stripped of initial- and final-state spinors, the resultant amplitude takes the form

$$M=-\rho f(\mathbf{p}',\mathbf{p}-\mathbf{k};p_0-\omega)+\rho' f(\mathbf{p}'+\mathbf{k},\mathbf{p};p_0)+B , \tag{3.11}$$

where B is constructed by forming the matrix element of the operator $F^{(1)}$, according to Eq. (3.1). The expression for B obtained in this way is found to differ from the matrix B_j appearing in Eq. (3.10) only in the absence of a mode index. The first two terms in Eq. (3.11) are expanded about their on-shell values. This time, with $f(\mathbf{p}',\mathbf{p};E)\to f(\nu,\tau,\xi,\xi')$, we take as our scalar variables

$$\nu=p_0,\quad \tau=(\mathbf{p}'+\mathbf{k}-\mathbf{p})^2,\quad \xi=p_0-E,\quad \xi'=p_0'-E . \tag{3.12}$$

In the first term we have $\nu=p_0-\mathbf{v}\cdot\mathbf{k}$, $\xi=\eta$, $\xi'=0$, and in the second term we have $\nu=p_0$, $\xi=0$, $\xi'=-\eta'$. The first two terms, evaluated to first order in the frequency, then add to

$$M^{(0)}=(\rho'-\rho)f(p_0,\tau)-\eta\rho\left[\frac{\partial f}{\partial\xi}+\frac{\partial f}{\partial\xi'}\right]+\rho\mathbf{v}\cdot\mathbf{k}\frac{\partial f}{\partial\nu} . \tag{3.13}$$

To the appropriate order, then, we have $B=M_{\mathrm{LFA}}-M^{(0)}$, where the low-frequency approximation, derived in the Appendix, is

$$M_{\mathrm{LFA}}=\rho' f(p_0,\tau)-\rho f(p_0-\omega,\tau)+S . \tag{3.14}$$

The spin-dependent term is

$$S=\frac{-e}{2\eta p_0}\left[\frac{2\pi}{\omega\Omega}\right]^{1/2}[\boldsymbol{\alpha}\cdot\boldsymbol{\varepsilon}(\boldsymbol{\alpha}\cdot\mathbf{k}+\omega)f(p_0,0)+f(p_0,0)(\boldsymbol{\alpha}\cdot\mathbf{k}+\omega)\boldsymbol{\alpha}\cdot\boldsymbol{\varepsilon}] . \tag{3.15}$$

After supplying a subscript j to the expression for B just determined it may be combined with Eq. (3.10) and this yields the result

$$T^{(1)}_{n\mathbf{p};n\mathbf{p}}=\gamma^{\dagger}(\mathbf{p})\left[2\Delta_{\mathbf{p}}\frac{\partial f}{\partial\nu}+2\Delta_{\mathbf{p}}\left[\frac{\partial f}{\partial\xi}+\frac{\partial f}{\partial\xi'}\right]+\sum_j(-2\rho_j)S_j\right]\gamma(\mathbf{p}) . \tag{3.16}$$

Taking into account Eq. (3.6) we find for the sum $T^{(0)}_{n\mathbf{p};n\mathbf{p}}+T^{(1)}_{n\mathbf{p};n\mathbf{p}}$ the expression

$$T_{n\mathbf{p};n\mathbf{p}}=\gamma^{\dagger}(\mathbf{p})\left[f+\left[2\Delta_{\mathbf{p}}-\sum_j\mathbf{v}\cdot\mathbf{k}_j\rho_j^2\right]\frac{\partial f}{\partial\nu}+\sum_j(-2\rho_j)S_j\right]\gamma(\mathbf{p}) . \tag{3.17}$$

The off-shell contributions are seen to have cancelled. The on-shell matrix f and its energy derivative are evaluated at $\nu=p_0$ and $\tau=0$.

To simplify the expression (3.17) we define the modified spinor

$$\chi=\left[1+\sum_j \frac{e}{2(\mathbf{p}\cdot\mathbf{n}_j-p_0)}(\boldsymbol{\alpha}\cdot\mathbf{n}_j+1)\boldsymbol{\alpha}\cdot\langle\mathbf{A}_j\rangle\right]\gamma(\mathbf{p}) , \tag{3.18}$$

with $\mathbf{n}_j=\mathbf{k}_j/\omega_j$ and

$$\langle\mathbf{A}_j\rangle\equiv\langle n|W_0^\dagger\mathbf{A}_jW_0|n\rangle=(-2\rho_j)\left[\frac{2\pi}{\omega_j\Omega}\right]^{1/2}\boldsymbol{\varepsilon}_j . \tag{3.19}$$

In terms of these quantities we may put Eq. (3.17) in the form

$$T_{n\mathbf{p};n\mathbf{p}}=\chi^\dagger f(\mathbf{q},\mathbf{q};q_0)\chi , \tag{3.20}$$

where

$$q_0=p_0+2\Delta_{\mathbf{p}}-\sum_j \mathbf{v}\cdot\mathbf{k}_j\rho_j^2 . \tag{3.21a}$$

$\mathbf{q}$ has the direction of $\mathbf{p}$ and satisfies the on-shell condition $q^2+m^2=q_0^2$. In the limit of infinite quantization volume the sum in Eq. (3.21a) may be performed to give

$$q_0=p_0-\frac{e^2\omega_s}{2\pi}\left[-2+\ln\left[\frac{1+v}{1-v}\right]\left[\frac{1}{v}+v\right]\right] . \tag{3.21b}$$

In the same limit Eq. (3.18) becomes

$$\chi=\left[1-I(v)\frac{e^2\omega_s}{2\pi p_0}\boldsymbol{\alpha}\cdot\mathbf{v}\right]\gamma(\mathbf{p}) , \tag{3.22a}$$

with

$$I(v)=\frac{2}{v^2}\left[-2+\frac{1}{v}\ln\left[\frac{1+v}{1-v}\right]\right] . \tag{3.22b}$$

The total cross section, summed over final states of the field, may be obtained from the forward-scattering amplitude by application of the optical theorem. In a version appropriate to the problem at hand the theorem provides us with the relation

$$\mathrm{Im}T_{n\mathbf{p};n\mathbf{p}}=-\pi\sum_{n'}\int d^3p'\delta(E_{n'\mathbf{p}'}-E_{n\mathbf{p}})|T_{n'\mathbf{p}';n\mathbf{p}}|^2 . \tag{3.23}$$

The expression for the total cross section is

$$\sigma=\frac{p_0(2\pi)^4}{p}\sum_{n'}\int d^3p'\delta(E_{n'\mathbf{p}'}-E_{n\mathbf{p}})|T_{n'\mathbf{p}';n\mathbf{p}}|^2 , \tag{3.24}$$

from which we conclude that

$$\sigma=-\frac{p_0(2\pi)^2}{\pi p}\mathrm{Im}T_{n\mathbf{p};n\mathbf{p}} . \tag{3.25}$$

From Eq. (3.20) we have

$$\mathrm{Im}T_{n\mathbf{p};n\mathbf{p}}=(2i)^{-1}\chi^\dagger[f(\mathbf{q},\mathbf{q};q_0)-f^\dagger(\mathbf{q},\mathbf{q};q_0)]\chi . \tag{3.26}$$

The expression on the right-hand side may be evaluated through an application of the optical theorem for the field-free scattering problem, which we write as

$$f(\mathbf{q},\mathbf{q};q_0)-f^\dagger(\mathbf{q},\mathbf{q};q_0)=-(2\pi i)\int d^3q'\delta(q_0'-q_0)f^\dagger(\mathbf{q}',\mathbf{q};q_0)f(\mathbf{q}',\mathbf{q};q_0) . \tag{3.27}$$

The sum rule, correct to first order, then takes the form

$$\sigma=\frac{p_0(2\pi)^4}{p}\int d^3q'\delta(q_0'-q_0)\chi^\dagger f^\dagger(\mathbf{q}',\mathbf{q};q_0)\times f(\mathbf{q}',\mathbf{q};q_0)\chi . \tag{3.28}$$

The right-hand side may be determined from measurements of field-free scattering parameters. The approximate cross section, while accounting uniformly for spontaneous and stimulated radiation, shows no external-field effects to the order considered. This is justified, within the present approximation procedure, for external fields of moderate intensity. A discussion of the range of frequency and intensity parameters for which the external-field interaction may be consistently treated as a quantity of first order has been given previously [11]. The correction term arises from spontaneous emission and reabsorption and is extremely small since the relevant parameter, $e^2\omega_s/mc^3$, is of order 10^{-8} for $\hbar\omega_s$ of order 1 eV. The result of the calculation may therefore be viewed as a confirmation of the validity and accuracy of the sum rule in lowest order. It also demonstrates a systematic procedure for generating higher-order corrections that can be useful, particularly in calculations of laser-assisted scattering cross sections where such corrections become significant for sufficiently strong fields.

IV. DISCUSSION

In their Brief Report written many years ago [1], Bloch and Nordsieck presented the essential physical ideas that eventually led to the development of practical prescriptions for treating infrared radiative corrections in QED [12]. In addition to being important historically, that paper remains of methodological interest since it suggests theoretical techniques for dealing with the problem of scattering in a radiation field (external as well as spontaneously produced) that are still relevant to current research in a developing area of physics. In particular, the work of Bloch and Nordsieck provided one of the first examples of the use of a low-frequency approximation to relate the cross section for a radiative process to a measurable cross section for scattering in the absence of radiation. It also introduced a method for the nonperturbative treatment of electron-field interactions through the introduction of distorted plane waves, and suggested the applicability of a modified perturbation theory, based on the use of Casimir projection operators, to allow for systematic improvements in the accuracy of the calculation.

An attempt has been made in this paper to develop some of these ideas in the course of an evaluation of the cross section sum rule to the next order of accuracy

beyond that originally achieved by BN. The fact that the coherent states introduced by BN are only approximate solutions of the asymptotic wave equation represents one of the difficulties encountered in such a program. In particular, for the case of an external field, the approximate asymptotic states omit terms that are required to properly account for virtual Compton scattering in initial and final states. When the scattered photons have the same or very nearly equal frequency, such terms lead to the appearance of near singularities in the transition amplitude [11] and must therefore be treated with care. (There is a clear analogy between the present treatment of near singularities arising from virtual Compton scattering and the method used in the original work of BN for dealing with spontaneous bremsstrahlung of soft photons.) The difficulties associated with the inexact nature of the BN coherent states was surmounted here through the use of the unitarity property of the wave operator that generates the exact asymptotic solution in the positive-energy subspace, and through the introduction of an effective Hamiltonian formulation (analogous to one that is widely used in nuclear and atomic reaction theory [9]) to account for virtual transitions into states of negative energy. Such transitions introduce no near singularities and may be ignored in determining the leading term in the cross-section sum rule, but must be accounted for in higher order. This was accomplished by relating the correction term to a very similar term that appears in the amplitude for single-photon bremsstrahlung and that could be evaluated with the aid of the low-frequency approximation for that amplitude.

Owing to the nonsingular nature of the effective potential, higher-order corrections can be obtained through a straightforward expansion in powers of the electron-field interaction. An examination of the second-order term in this expansion shows that it can be related to the amplitude for two-photon bremsstrahlung, for which a low-frequency approximation is available [3]. Thus it seems possible that the successive approximation procedure can be extended to higher orders provided improved approximations for the positive-energy asymptotic solutions can be constructed. The development of such techniques could be useful in generating more effective methods for treating a variety of processes involving spontaneous or stimulated bremsstrahlung.

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APPENDIX: SINGLE-PHOTON BREMSSTRAHLUNG

Here we derive the low-frequency approximation for single-photon bremsstrahlung for scattering described by the single-particle Dirac equation. It will be convenient to follow not the original approach of Low [2], but rather that presented in Ref. [3], adapted to the potential scattering problem considered here. The on-shell field-free scattering amplitude is written as

$$t(\mathbf{p}',\mathbf{p};p_0)=\gamma^{\dagger}(\mathbf{p}')f(p_0,[\mathbf{p}'-\mathbf{p}]^2)\gamma(\mathbf{p}) . \tag{A1}$$

In treating the radiative process we include the scalar interaction eA_0; the derivation is based on gauge invariance so we do not here specialize to the radiation gauge. We express the amplitude for the emission of a photon of momentum $\mathbf{k}$, energy $\omega=|\mathbf{k}|$, and polarization $(\boldsymbol{\varepsilon},\varepsilon_0)$, as $-e(2\pi/\omega\Omega)^{1/2}\gamma^{\dagger}(\mathbf{p}')(L_1+L_2+R)\gamma(\mathbf{p})$. The sum of the matrices L_1 and L_2 will provide the low-frequency approximation with R, which will be shown to be of order ω, representing the remainder. L_1 accounts for photon emission either before or after the collision and is constructed with virtual transitions into negative-energy states ignored; it is the most nearly singular contribution, of order ω^{-1}. We have, with $p_0=p_0'+\omega$ and $\tau\equiv(\mathbf{p}'+\mathbf{k}-\mathbf{p})^2$, and with the matrix f understood to operate in the positive-energy subspace,

$$\begin{aligned}L_1=&(\boldsymbol{\alpha}\cdot\boldsymbol{\varepsilon}-\varepsilon_0)\frac{1}{p_0-[\boldsymbol{\alpha}\cdot(\mathbf{p}'+\mathbf{k})+\beta m]}f(p_0,\tau)\\&+f(p_0-\omega,\tau)\frac{1}{p_0-\omega-[\boldsymbol{\alpha}\cdot(\mathbf{p}-\mathbf{k})+\beta m]}\\&\times(\boldsymbol{\alpha}\cdot\boldsymbol{\varepsilon}-\varepsilon_0) .\end{aligned} \tag{A2}$$

After some algebra this can be put in the form

$$\begin{aligned}L_1=&[(\boldsymbol{\alpha}\cdot\boldsymbol{\varepsilon}-\varepsilon_0)(\boldsymbol{\alpha}\cdot\mathbf{k}+\omega)+2(\mathbf{p}'\cdot\boldsymbol{\varepsilon}-p_0\varepsilon_0)]\\&\times[2(p_0\omega-\mathbf{p}'\cdot\mathbf{k})]^{-1}f(p_0,\tau)\\&-f(p_0-\omega,\tau)[2(p_0\omega-\mathbf{p}\cdot\mathbf{k})]^{-1}\\&\times[2(\mathbf{p}\cdot\boldsymbol{\varepsilon}-p_0\varepsilon_0)-(\boldsymbol{\alpha}\cdot\mathbf{k}+\omega)(\boldsymbol{\alpha}\cdot\boldsymbol{\varepsilon}-\varepsilon_0)] .\end{aligned} \tag{A3}$$

Here we have omitted terms which vanish, by virtue of the Dirac equation satisfied by the spinors $\gamma(\mathbf{p})$ and $\gamma^{\dagger}(\mathbf{p})$, when the matrix element (A1) is formed. The expression (A3) is not gauge invariant since under the replacement $(\boldsymbol{\varepsilon},\varepsilon_0)\rightarrow(\mathbf{k},\omega)$ it reduces not to zero, but rather to $f(p_0-\omega,\tau)-f(p_0,\tau)$. With the choice

$$L_2=\frac{f(p_0,\tau)-f(p_0-\omega,\tau)}{\omega}\varepsilon_0 , \tag{A4}$$

we obtain a low-frequency approximation

$$M_{\mathrm{LFA}}=-e(2\pi/\omega\Omega)^{1/2}(L_1+L_2) , \tag{A5}$$

which is gauge invariant. It follows that the remainder R is gauge invariant and nonsingular in the zero-frequency limit; it is then easy to see that R must vanish in that limit. Note that in the radiation gauge L_2 vanishes.

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