Quantum inverse problem for the derivative nonlinear Schrödinger equation

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The quantum inverse problem for the nonultralocal nonlinear Schrödinger problem is formulated by using the idea of operator product expansion. It is demonstrated that the quantum R matrix so generated has the usual relation with the classical r matrix constructed through the approach of Tsyplyaev [Theor. Math. Phys. **48**, 580 (1981)]. The algebraic Bethe ansatz is then set up and the excitation spectrum of the problem is determined.

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INTRODUCTION

Quantization of an integrable nonlinear system is one of the most important problems of nonlinear theory [1]. Well-established methods have been set up by Fadeev *et al.* for classical integrable systems that are ultralocal, that is, whose Poisson structure involves only a δ function. On the other hand, no such prescription is known for nonultralocal systems, which abound in nature.

The pioneering attempt to treat the nonultralocal system was done by Tsyplyaev [2]. After that an independent formulation for the case of an extended derivative nonlinear Schrödinger equation was done by Roy Chowdhury and Sen [3], which was accidentally ultralocal. A similar situation was also observed for the case of Alfvén wave propagation in a plasma, which is governed by a set of equations very similar to a derivative nonlinear Schrödinger equation but again ultralocal [4]. On the other hand, significant progress for the case of a nonultralocal integrable system was done by deVega, Eiechenherr, and Maillet [5] and Maillet [6] who succeeded in developing a theory for the r matrix (classical case) for the nonultralocal case. Though it can be mentioned that their approach does not include every nonultralocal situation. The situation in the quantum case (for the nonultralocal system) is still not clear. So here we have tried to formulate the quantum inverse problem for the derivative nonlinear Schrödinger equation with the help of the concept of operator product expansion [7], a methodology immensely successful in the domains of high-energy physics and solid-state physics. We then show that our quantum R matrix possesses the same natural relation with the classical r matrix deduced *à la* Tsyplyaev [2]. In the next section we show how the algebraic Bethe ansatz [8] can be formulated and the excitation spectrum determined. Lastly we note that the quantum R matrix so deduced satisfies the Yang-Baxter equation [9].

FORMULATION

The derivative nonlinear Schrödinger equation is written as

$$iq_t+q_{xx}+\epsilon(|q|^2q)_x=0,\quad \epsilon=\pm 1 \tag{1}$$

The isospectral problem associated with it is

$$\Psi_x=iL(x,\lambda)\Psi\ , \tag{2}$$

$$L(x,\lambda)=\begin{bmatrix}\lambda^2 & -i\lambda q(x)\\ -i\lambda q^*(x) & -\lambda^2\end{bmatrix}. \tag{3}$$

The classical Poisson bracket due to the Hamiltonian structure of (1) is

$$\{q^*(x),q(y)\}=\frac{\partial}{\partial x}\delta(x-y)\ . \tag{3$'$}$$

The presence of the derivative of the δ function in the Poisson bracket is the source of nonultralocality.

We start with the derivation of the classical r matrix following Tsyplyaev. We can rewrite L as

$$L=\lambda^2\sigma_3-i\lambda q(x)\sigma_+-i\lambda q^*(x)\sigma_-\ , \tag{4}$$

where $\sigma_\pm=(\sigma_1\pm i\sigma_2)/2$, σ_i being Pauli matrices. Using Eq. (3′) we at once obtain

$$\left\{L(x,\lambda)\underset{,}{\otimes} L(y,\mu)\right\}=\omega\delta'(x-y)\ ,$$

where

$$\omega=-\lambda\mu(\sigma_+\otimes\sigma_-+\sigma_-\otimes\sigma_+)\ . \tag{5}$$

The transition matrix $\tau(\lambda)$ associated with the linear problem is defined by

$$\tau(\lambda)=\lim_{\substack{x\to+\infty\\ y\to-\infty}}\tau_+^{-1}(x|\lambda)\tau(x,y|\lambda)\tau_-(y|\lambda)\ , \tag{6}$$

where $\tau_\pm(x|\lambda)$ denotes the solutions of $[\partial_x-iL(x,\lambda)]\tau(x,y|\lambda)=0$ for the asymptotic matrix $L_\pm(\lambda)=\lim_{x\to\pm\infty}L(x,\lambda)$. The Jost functions are related to these solutions through

$$\Phi_\pm(x,\lambda)=\lim_{y\to\pm\infty}\tau(x,y|\lambda)\tau_\pm(y|\lambda)\ . \tag{7}$$

It is then easy to demonstrate that the Poisson bracket of $\tau(\lambda)$ and $\tau(\mu)$ is given as

$$\left\{\tau(\lambda) \underset{,}{\otimes} \tau(\mu)\right\} = \int \int_{-\infty}^{\infty} dx\, dy\, \Phi_+^{-1}(x,\lambda) \otimes \Phi_+^{-1}(y,\mu) K \Phi_-(x,\lambda) \Phi_-(y,\mu) \tag{8}$$

where

$$K = \left\{L(x,\lambda) \underset{,}{\otimes} L(y,\mu)\right\} .$$

Using

$$\begin{aligned} \partial_x \Phi_+^{-1}(x,\lambda) &= -\Phi_+^{-1}(x,\lambda) L(x\lambda) , \\ \partial_x \Phi_-(x,\lambda) &= L(x,\lambda) \Phi_-(x,\lambda) , \end{aligned} \tag{9}$$

the equation can be transformed to the following form:

$$\left\{\tau(\lambda) \underset{,}{\otimes} \tau(\mu)\right\} = \int_{-\infty}^{\infty} dx \{\Phi_+^{-1}(x,\lambda) \otimes \Phi_+^{-1}(x,\mu)\} \Omega(x|\lambda,\mu) \{\Phi_-(x,\lambda) \otimes \Phi_-(x,\mu)\} , \tag{10}$$

where

$$\Omega(x|\lambda,\mu) = \frac{\lambda\mu}{2} [\sigma_+ \otimes \sigma_- + \sigma_- \otimes \sigma_+, L(x,\lambda) \otimes 1 - 1 \otimes L(x,\mu)] . \tag{11}$$

We now demand, following Ref. [6], that the integrand of Eq. (10) be the total derivative of

$$\Phi_+^{-1}(x,\lambda) \otimes \Phi_+^{-1}(x,\mu) r(x|\lambda,\mu) \Phi_-(x,\lambda) \otimes \Phi_-(x,\mu) \tag{12}$$

which in turn leads to a differential equation for the classical r matrix $r(x|\lambda,\mu)$, viz.,

$$\partial_x r(x|\lambda\mu) + [r(x|\lambda,\mu), L(x,\lambda) \otimes 1 + 1 \otimes L(x,\mu)] = \Omega(x|\lambda,\mu) . \tag{13}$$

With Ω, given by (11), we try to solve Eq. (13) by setting

$$r(x|\lambda,\mu) = A\, 1 \otimes 1 + B \sigma_3 \otimes \sigma_3 + C \sigma + \otimes \sigma_- + D \sigma_- \otimes \sigma_+ . \tag{14}$$

Assuming the functions A, B, C, D to be independent of x, we get

$$\begin{aligned} &2B\lambda - C\mu = \frac{\lambda\mu^2}{2} , \\ &2B\mu - C\lambda = -\frac{\lambda^2\mu}{2} , \\ &B = \frac{\lambda^2\mu^2}{2(\lambda^2-\mu^2)}, \quad C = \frac{\lambda\mu(\lambda^2+\mu^2)}{2(\lambda^2-\mu^2)} = D \end{aligned} \tag{15}$$

and A remains arbitrary, whence we get

$$\begin{aligned} r(x|\lambda,\mu) &\equiv r(\lambda,\mu) \\ &= A\, 1 \otimes 1 + \frac{\lambda^2\mu^2}{2(\lambda^2-\mu^2)} \sigma_3 \otimes \sigma_3 \\ &\quad + \frac{\lambda\mu(\lambda^2+\mu^2)}{2(\lambda^2-\mu^2)}, \\ &\quad \times Q(\sigma_+ \otimes \sigma_- + \sigma_- \otimes \sigma_+). \end{aligned} \tag{16}$$

THE QUANTUM R MATRIX

Though the classical r matrix can be deduced as above, in the corresponding case, R cannot be constructed. So we here follow a different route. We adopt the fundamental idea that the field operators $q(x)$, $q^*(y)$, do not have a well-defined product at the same space-time point, and $q(x)q^*(x)$ is singular at x.

Consider the monodromy matrix $T(x,y,\lambda)$, which is a solution of the equation

$$[\partial_x - iL(x,\lambda)] T(x,y,\lambda) = 0 \tag{17}$$

with boundary condition $T(x,x;\lambda) = I$, where I represents the unit matrix.

The formal solution of (17) can be written as an ordered exponential and we get

$$T(x,y,\lambda) = P \exp\left\{ i \int_y^x L(\xi,\lambda) d\xi \right\} . \tag{18}$$

P denotes the ordering. Using the group property of the exponential we can write at once

$$T(x,y,\lambda) \otimes T(x,y,\mu) = S T(x-\Delta,y;\lambda) \otimes T(x-\Delta,y;\mu) , \tag{19}$$

where S is

$$\left\{ P \exp\left[i \int_{x-\Delta}^x L(\xi,\lambda) d\xi \right] \otimes P \exp\left[i \int_{x-\Delta}^x L(\xi,\mu) d\xi \right] \right\} \tag{20}$$

for arbitrary Δ. Differentiating Eq. (19) with respect to x we get

$$\partial_x\{T(x,y,\lambda)\otimes T(x,y,\mu)\}=\{iL(x,\lambda)P\exp[i\bar{L}(\lambda)]\otimes P\exp[i\bar{L}(\mu)]+P\exp[i\bar{L}(\lambda)]\otimes iL(x,\mu)$$
$$\times P\exp[i\bar{L}(\mu)]\}T(x-\Delta,y,\lambda)\otimes T(x-\Delta,y,\mu)\ . \tag{21}$$

The commutation relation of q^*, and q given from (3′) via the correspondence principle shows that $[q^*(x),q(y)] = -i\hbar(\partial/\partial x)\delta(x-y)\sim\Delta^{-2}$. So we expand the exponentials in (21) and retain terms up to second order in Δ. This leads to

$$\partial_x\{T(x,y,\lambda)\otimes T(x,y,\mu)\}=\Gamma(x,\lambda,\mu)T(x,y,\lambda)\otimes T(x,y,\mu) \tag{22}$$

as $\Delta\to 0$. In Eq. (21) above we have used the notation

$$\bar{L}(\lambda)=\int_{x-\Delta}^{x}L(\xi,\lambda)d\lambda,\quad \bar{L}(\mu)=\int_{x-\Delta}^{x}L(\zeta,\mu)d\zeta\ .$$

The expression for Γ can be obtained without much difficulty and we get

$$\Gamma(x,\lambda,\mu)=\Bigg[iL(x,\lambda)\otimes 1+1\otimes iL(x,\mu)-iL(x,\lambda)\otimes\int_{x-\Delta}^{x}\int_{x-\Delta}^{\xi}L(\zeta,\mu)L(\zeta',\mu)d\zeta'd\zeta$$
$$-\int_{x-\Delta}^{x}\int_{x-\Delta}^{\xi}L(\xi,\lambda)L(\xi',\lambda)d\xi'd\xi\otimes iL(x,\mu)\Bigg]\ . \tag{23}$$

In the present case we get

$$\Gamma(x,\lambda,\mu)=q(x)(\lambda\sigma_+\otimes I+\mu I\otimes\sigma_+)+q^*(x)(\lambda\sigma_-\otimes 1+\mu I\otimes\sigma_-)+i(\lambda^2\sigma_3\otimes 1+\mu^2 I\otimes\sigma_3)$$
$$+2\hbar\lambda\mu(\lambda^2+\mu^2)(\sigma_+\otimes\sigma-\sigma_-\otimes\sigma_+)+i\hbar\lambda\mu q(x)(\mu\sigma_+\otimes\sigma_3-\lambda\sigma_3\otimes\sigma_+)$$
$$+i\hbar\lambda\mu q^*(x)(\lambda\sigma_3\otimes\sigma_- -\mu\sigma_-\otimes\sigma_3)\ , \tag{24}$$

where I is the unit matrix. Now the quantum R matrix is defined to be an operator that is defined on the direct product space $V_1\otimes V_2$ and intertwines between $T(x,y,\lambda)\otimes T(x,y,\mu)$ and $T(x,y,\mu)\otimes T(x,y,\lambda)$, whence we demand

$$R(\lambda,\mu)\Gamma(x,\lambda,\mu)=\Gamma(x,\mu,\lambda)R(\lambda,\mu)\ . \tag{25}$$

Writing out R in the basis of the product space $V_1\otimes V_2$ we can easily solve (25) and get

$$R(\lambda,\mu)=\tfrac{1}{2}\left[1-2i\hbar\lambda\mu\left[\frac{\lambda+\mu}{\lambda-\mu}\right]\right]1\otimes 1$$
$$+\tfrac{1}{2}\left[1+2i\hbar\lambda\mu\left[\frac{\lambda+\mu}{\lambda+\mu}\right]\right]\sigma_3\otimes\sigma_3$$
$$+(\sigma_+\otimes\sigma_-+\sigma_-\otimes\sigma_+)\ , \tag{26}$$

which is exact up to the order of $\hbar$. Comparing with Eq. (16) we immediately observe that

$$R(\lambda,\mu)=P_r[I-4ihr(\lambda,\mu)]\ , \tag{27}$$

where P_r stands for the permutation operator

$$P_r=\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}.$$

So our quantum R matrix can be connected to the classical one as in the case of an ultralocal system.

ASYMPTOTIC LIMIT AND SCATTERING DATA

Since $R(\lambda,\mu)$ is independent of x and

$$R(\lambda,\mu)T(x,y,\lambda)\otimes T(x,y,\mu)=T(x,y,\mu)\otimes T(x,y,\lambda)R(\lambda,\mu)\ , \tag{28}$$

we can take the limit $x\to\pm\infty$ in this equation to deduce the commutation rules of the scattering data. For the solitonic fields $|q(x)|\to 0$ as $x\to\pm\infty$ and the solution of (2) behaves as $e^{i\lambda^2\sigma_3 x}$. So we set

$$\lim_{\substack{x\to\infty\\ y\to-\infty}} T(x,y,\lambda)=e^{i\lambda^2\sigma_3 x}T(\lambda)e^{-i\lambda^2\sigma_3 y}\ , \tag{29}$$

$T(\lambda)$ being the scattering data. So from Eq. (28) we get

$$R_1(\lambda,\mu)T(\lambda)\otimes T(\mu)=T(\mu)\otimes T(\lambda)R_2(\lambda,\mu) \tag{30}$$

where

$$R_1(\lambda,\mu)=\lim_{x\to\infty}(e^{-i\mu^2\sigma_3 x}\otimes e^{-i\lambda^2\sigma_3 x})R(\lambda,\mu)(e^{i\lambda^2\sigma_3 x}\otimes e^{i\mu^2\sigma_3 x}) ,$$
$$R_2(\lambda,\mu)=\lim_{y\to-\infty}(e^{-i\mu^2\sigma_3 y}\otimes e^{-i\lambda^2\sigma_3 y})R(\lambda,\mu)(e^{i\lambda^2\sigma_3 y}\otimes e^{i\mu^2\sigma_3 y}) . \tag{31}$$

Let

$$\Gamma_0(x,\lambda,\mu)=\lim_{x\to\infty}\Gamma(x,\lambda,\mu)=i(\lambda^2\sigma_3\otimes I+\mu^2 I\otimes\sigma_3)+2\hbar\lambda\mu(\lambda^2+\mu^2)(\sigma_+\otimes\sigma_- -\sigma_-\otimes\sigma_+) ,$$

whence

$$R\Gamma_0(x,\lambda,\mu)=\Gamma_0(x,\lambda,\mu)R ,$$
$$e^{-i\Gamma_0(x\lambda\mu)}R=Re^{-i\Gamma_0(x\lambda\mu)} . \tag{32}$$

Using Eqs. (31) and (32) we arrive at

$$R_+T(\lambda)\otimes T(\mu)=T(\mu)\otimes T(\lambda)R_- , \tag{33}$$

$$R_+=\lim_{x\to\infty}(e^{-i\mu^2\sigma_3 x}\otimes e^{-i\lambda^2\sigma_3 x})(e^{i\Gamma_0}Re^{-i\Gamma_0})\times(e^{i\lambda^2\sigma_3 x}\otimes e^{i\mu^2\sigma_3 x}) , \tag{34a}$$

$$R_-=\lim_{y\to-\infty}(e^{-i\mu^2\sigma_3 y}\otimes e^{-i\lambda^2\sigma_3 y})(e^{i\Gamma_0}Re^{-i\Gamma_0})\times(e^{i\lambda^2\sigma_3 y}\otimes e^{i\mu^2\sigma_3 y}) . \tag{34b}$$

If we now rewrite $R(\lambda,\mu)$ given in Eq. (26) as

$$R(\lambda\mu)=\begin{bmatrix} a(\lambda,\mu) & 0 & 0 & 0\\ 0 & b(\lambda\mu) & 1 & 0\\ 0 & 1 & b(\lambda\mu) & 0\\ 0 & 0 & 0 & a(\lambda\mu)\end{bmatrix} \tag{35}$$

then Eqs. (34a) and (34b) lead to

$$R_\pm(\lambda,\mu)=\lim_{x\to\pm\infty}\begin{bmatrix} a(\lambda,\mu) & 0 & 0 & 0\\ 0 & be^{2i(\lambda^2-\mu^2)x} & 1 & 0\\ 0 & 1 & be^{-2i(\lambda^2-\mu^2)x} & 0\\ 0 & 0 & 0 & a(\lambda,\mu)\end{bmatrix} . \tag{36}$$

Now the symmetry of the L operator dictates that

$$T(\lambda)=\begin{bmatrix} A(\lambda) & B(\lambda)\\ B^*(\lambda) & A^*(\lambda)\end{bmatrix} . \tag{37}$$

The commutation rules of the scattering data can be easily read off from Eq. (33) and yield

$$A(\lambda)B^*(\mu)=\tilde b B^*(\lambda)A(\mu)+\tilde a B^*(\mu)A(\lambda) ,$$
$$A^*(\lambda)B^*(\mu)=\tilde a^{-1}B^*(\mu)A^*(\lambda)+\frac{\tilde b}{\tilde a}A^*(\mu)B^*(\lambda) ,$$
$$A^*(\mu)B^*(\lambda)=\tilde a B^*(\lambda)A^*(\mu)+\tilde b B^*(\mu)A^*(\lambda) , \tag{38}$$

where

$$\tilde a=\mathrm{P}\left[1-\frac{4i\hbar\lambda\mu}{\lambda^2-\mu^2}\right] , \tag{39}$$

P standing for the principal value and

$$\tilde b=2\pi\hbar\lambda\mu(\lambda^2+\mu^2)\delta(\lambda^2-\mu^2) . \tag{40}$$

ALGEBRAIC BETHE ANSATZ

From the structure of the L operator and the definition of $T(\lambda)$ in terms of L, it is apparent that if we interpret $q(x)$ as a destruction operator and q^* as a creation operator, then due to the triangular structure of $L(x,\lambda)|0\rangle$ (where $|0\rangle$ represents a pseudovacuum), $B(\lambda)$ acts as a destruction operator and B^* as the creation operator for the Bethe ansatz. On the other hand, the Hamiltonian is given by

$$\mathrm{Tr}T(\lambda)=A(\lambda)+A^*(\lambda) . \tag{41}$$

So, let us assume

$$A(\lambda)|0\rangle=\varepsilon(\lambda)|0\rangle ,$$
$$A^*(\lambda)|0\rangle=\varepsilon^*(\lambda)|0\rangle . \tag{42}$$

The n particle state is

$$\Omega_n(\mu_1\cdots\mu_n)=B^*(\mu_1)B^*(\mu_2)\cdots B^*(\mu_n)|0\rangle . \tag{43}$$

Now, operating with $A(\lambda)+A^*(\lambda)$ on Ω_n and using the commutation rules (38), we can separate the wanted and unwanted terms and obtain the equation determining the equation for the eigenmomenta μ_i.

Since the calculation is straightforward we just state the final result. The Bethe ansatz equations determining μ_i are

$$\frac{\varepsilon^*(\mu_i)}{\varepsilon(\mu_i)}=-\prod_{\substack{j=1\\ j\neq i}}^{n}\left\{\frac{\tilde a^{2}(\mu_i,\mu_j)}{1+\tilde b^{2}(\mu_i,\mu_j)}\right\} , \tag{44}$$

whereas the energy eigenvalue of the state Ω_n is given as

$$E_n=\prod_{i=1}^{n}\tilde{a}(\lambda,\mu_i)\varepsilon(\lambda)+\prod_{i=1}^{n}\frac{1+\tilde{b}^{2}(\lambda,\mu_i)}{\tilde{a}(\lambda,\mu_i)}\varepsilon^*(\lambda)\ . \tag{45}$$

The expressions for $\varepsilon(\lambda)$ and $\varepsilon^*(\lambda)$ are most easily determined by a discretization procedure. If we denote the length of the interval on the x axis as L' then

$$\varepsilon^*=e^{-i\lambda^2L'},\quad \varepsilon=e^{i\lambda^2L'}\ .$$

Since the coupled set of Eqs. (44) are very difficult to solve we can take recourse to the usual approach of converting it into an integral equation. If we denote

$$\rho(\mu_i)=\frac{1}{L'(\mu_{i+1}-\mu_i)} \tag{46}$$

then following the standard procedure we at once obtain in the limit $L'\to\infty$ and in the continuous limit of the eigenvalues μ_i,

$$\begin{gathered}\mu=\frac{i\pi}{2}\rho(\mu)-2\hbar\int_{-\mu_F}^{\mu_F}\rho(\mu')F'(\mu')d\mu'\mu'\ ,\\ F'=\frac{\partial F}{\partial\mu'^{2}},\quad F=\ln\left[1+\frac{4ih}{\mu^{-2}-\mu'^{-2}}\right]\ ,\end{gathered} \tag{47}$$

subject to the condition

$$\int_{-\mu_F}^{\mu_F}\rho(\mu')d\mu'=\frac{N}{L'}\ .$$

DISCUSSIONS

In our above analysis we have shown that the idea of operator product expansion can lead to a reasonable derivation of the quantum R matrix for the nonultralocal case of the derivative nonlinear Schrödinger problem. It has the required limit when $\hbar\to 0$ and matches with the classical r matrix derived via the approach of Tsyplyaev. The algebraic Bethe ansatz can be set up and the usual analysis of the quantum inverse scattering method can be performed. Lastly it can be mentioned that the R matrix deduced in Eq. (35) does satisfy

$$R_{12}(\lambda,\mu)R_{13}(\lambda,\nu)R_{23}(\mu,\nu)=R_{23}(\mu,\nu)R_{13}(\lambda,\nu)R_{12}(\lambda,\mu)\ ,$$

which is nothing but the usual Yang-Baxter equation.

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