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Triorthogonal uniqueness theorem and its relevance to the interpretation of quantum mechanics

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Many-world, decoherence, and modal interpretations of quantum mechanics suffer from a "basis degeneracy problem" arising from the nonuniqueness of some biorthogonal decompositions. We prove that when a quantum state can be written in the triorthogonal form $\Psi=\sum_i c_i|A_i\rangle\otimes|B_i\rangle\otimes|C_i\rangle$, then, even if some of the c_i's are equal, no alternative bases exist such that Ψ can be rewritten $\sum_i d_i|A_i'\rangle\otimes|B_i'\rangle\otimes|C_i'\rangle$. Therefore the triorthogonal decomposition picks out a "special" basis. We can use this preferred basis to address the basis degeneracy problem.

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I. INTRODUCTION

We will prove a technical result that helps several interpretations of quantum mechanics (QM). Specifically, we show that when a quantum state vector describing three systems can be written in the triorthogonal form $\Psi=\sum_i c_i|A_i\rangle\otimes|B_i\rangle\otimes|C_i\rangle$, then there exists no *other* triorthogonal basis in terms of which Ψ can be expanded, even if some of the c_i's are equal. The triorthogonal decomposition picks out a "special" basis.

Several interpretations of QM can make use of this special basis. For instance, many-world adherents can claim that a branching of worlds occurs in the preferred basis picked out by the unique triorthogonal decomposition. Modal interpreters can postulate that the triorthogonal basis helps to pick out which observables possess definite values at a given time. And decoherence theorists can cite the uniqueness of the triorthogonal decomposition as a principled reason for asserting that pointer readings become "classical" upon interacting with the environment.

To motivate our technical results, we must show why triorthogonal decompositions, as opposed to biorthogonal decompositions, are sometimes needed to pick out a basis. But first we set the context by briefly reviewing the measurement problem.

II. THE MEASUREMENT PROBLEM

Consider a spin-$\frac{1}{2}$ particle initially described by a superposition of eigenstates of S_z, the z component of spin: $|\Phi\rangle=c_1|S_z=+\rangle+c_2|S_z=-\rangle$. Let $|R=+\rangle$ and $|R=-\rangle$ denote the "up" and "down" pointer-reading eigenstates of an S_z-measuring apparatus. According to QM (with no wave-function collapse), if the apparatus ideally measures the particle, the combined system evolves into an entangled superposition,

$$|\varphi\rangle=c_1|S_z=+\rangle\otimes|R=+\rangle+c_2|S_z=-\rangle\otimes|R=-\rangle . \tag{1}$$

Common sense insists that after the measurement, the pointer reading is definite. According to the "orthodox" value-assignment rule, however, the pointer reading is definite only if the quantum state is an eigenstate of $\hat{R}$, the pointer-reading operator. Since $|\varphi\rangle$ is not an eigenstate of $\hat{R}$, the pointer reading is indefinite.

The interpretations of QM mentioned above attempt to deal with this aspect of the measurement problem. But their solutions run into a technical difficulty we will call the "basis degeneracy problem."

III. BASIS DEGENERACY PROBLEM

To introduce the basis degeneracy problem, we will show how it arises in the context of many-world interpretations. Many-world interpretations [1] address the measurement problem by hypothesizing that when the combined system occupies state $|\varphi\rangle$, the two branches of the superposition split into separate worlds, in some sense. The pointer reading becomes definite relative to its branch. For instance, in the "up" world, the particle has spin up and the apparatus possesses the corresponding pointer reading. In this way, many-world interpreters explain why we always "see" definite pointer readings, instead of superpositions.

This approach suffers from a well-known technical problem, the basis degeneracy problem, which arises from the nonuniqueness of some biorthogonal decompositions. According to the biorthogonal decomposition theorem, any quantum state vector describing two systems can, for a certain choice of bases, be expanded in the simple form $\sum_i c_i|A_i\rangle\otimes|B_i\rangle$, where the $\{|A_i\rangle\}$ and $\{|B_i\rangle\}$ vectors are orthonormal, and are therefore eigenstates of Hermitian operators (observables) $\hat{A}$ and $\hat{B}$ associated with systems 1 and 2, respectively. This "biorthogonal" expansion picks out the "Schmidt" basis. The basis degeneracy problem arises because the biorthogonal decomposition is unique just in case all of the nonzero $|c_i|$'s are different.

When $|c_1|=|c_2|$, we can biorthogonally expand φ in an infinite number of bases. For instance, we can construct $\hat{S}_x$ eigenstates out of linear combinations of $\hat{S}_z$ eigenstates. And, similarly, we can introduce a new apparatus observable $\hat{R}'$, whose eigenstates are superpositions of pointer-reading eigenstates:

$$|S_x=\pm\rangle=2^{-1/2}[|S_z=+\rangle\pm|S_z=-\rangle] ,$$

$$|R'=\pm\rangle=2^{-1/2}[|R=+\rangle\pm|R=-\rangle] .$$

When $c_1=c_2=2^{-1/2}$, we can rewrite Eq. (1) as

$$|\varphi\rangle=2^{-1/2}[|S_x=+\rangle\otimes|R'=+\rangle+|S_x=-\rangle\otimes|R'=-\rangle . \quad (2)$$

These two ways of writing $|\varphi\rangle$ correspond to two ways of writing the reduced density operator ρ_a that describes the apparatus. Trace over the particle's states, using Eq. (1) and then Eq. (2), to get

$$\rho_a=2^{-1}[|R=+\rangle\langle R=+|+|R=-\rangle\langle R=-|]$$

$$=2^{-1}[|R'=+\rangle\langle R'=+|+|R'=-\rangle\langle R'=-|] .$$

From these considerations, we can see that nothing is special about the pointer-reading basis. The formalism gives us no more reason to assert that the universe splits into pointer-reading eigenstates than it gives us to assert that the universe splits into $\hat{R}'$ eigenstates. For this reason, the basis degeneracy problem leaves many-world interpreters without a purely formal algorithm for deciding how splitting occurs.

Our technical results solve the basis degeneracy problem for the many-world interpretation. Here is why: As the decoherence theorists [2,3] show, when the environment interacts with the combined particle-apparatus system, the following state results:

$$|\Psi\rangle=c_1|S_z=+\rangle\otimes|R=+\rangle\otimes|E_+\rangle$$

$$+c_2|S_z=-\rangle\otimes|R=-\rangle\otimes|E_-\rangle , \quad (3)$$

where $|E_\pm\rangle$ is the state of the rest of the universe after the environment interacts with the apparatus. As time passes, these environmental states quickly approach orthogonality: $\langle E_+|E_-\rangle\rightarrow 0$. In this limit, Eq. (3) is a triorthogonal decomposition of $|\Psi\rangle$. We will prove in Sec. V that *even if* $c_1=c_2$, the triorthogonal decomposition is unique. In other words, no transformed bases exist such that $|\Psi\rangle$ can be expanded as $d_1|S'=+\rangle\otimes|R'=+\rangle\otimes|E'_+\rangle + d_2|S'=-\rangle\otimes|R'=-\rangle\otimes|E'_-\rangle$. Therefore, Eq. (3) picks out a "preferred" basis. Many-world interpreters can postulate that this basis determines the branches into which the universe splits.

We did not come up with the idea that the environment picks out the pointer-reading basis. Decoherence theorists have been stressing this point for years. Indeed, Zurek's [2] "existential interpretation," a sophisticated variant of the many-world view, relies on the environment to select the "correct" basis.

Unfortunately, this interpretation suffers from a version of the basis degeneracy problem. Zurek [2] emphasizes that if Eq. (3) describes the universe, and if $\langle E_+|E_-\rangle=0$, then the reduced density operator $\rho_{p\&a}$ describing the particle and apparatus (found by "tracing over" the environmental degrees of freedom) is

$$\rho_{p\&a}=|c_1|^2|S_z=+\rangle\langle S_z=+||R=+\rangle\langle R=+|$$

$$+|c_2|^2|S_z=-\rangle\langle S_z=-||R=-\rangle\langle R=-| ,$$

the same mixture as would be obtained upon wave-function collapse. If $c_1=c_2$, however, then we can decompose this mixture into another basis, in which case the pointer reading loses its "special" status. For example, define $|q_\pm\rangle\equiv 2^{-1/2}[|S_z=+\rangle\otimes|R=+\rangle\pm|S_z=-\rangle\otimes|R=-\rangle]$. If $c_1=c_2=2^{-1/2}$, then we can rewrite $\rho_{p\&a}$ as

$$\rho_{p\&a}=2^{-1}[|q_+\rangle\langle q_+|+|q_-\rangle\langle q_-|] .$$

Although decoherence-based interpretations can deal with their basis degeneracy problem in many ways, a particularly "clean," formal solution is to invoke the uniqueness of the triorthogonal decomposition in Eq. (3). As noted above, uniqueness holds even when $c_1=c_2$.

A third kind of interpretation aided by our technical result is "modal" interpretations [4–8] that rely on the biorthogonal decomposition theorem. According to most modal interpretations, if $\sum_i c_i|A_i\rangle\otimes|B_i\rangle$ is the unique biorthogonal decomposition of the quantum state, then system 1 *has* a definite value for observable $\hat{A}$, and system 2 *has* a definite value for $\hat{B}$. For instance, consider Eq. (1), the state of the particle-apparatus system after an ideal spin measurement. According to modal interpretations, if $|c_1|\neq|c_2|$, then the particle has a definite z component of spin, and the apparatus has a definite pointer reading. These possessed values result not from a world splitting; the entangled wave function still exists entirely in our world, and continues to determine the dynamical evolution of the system. Rather, these modally possessed values are a kind of hidden variable. (Since few observables possess values at any given time, and since some of these observables are nonlocal, modal interpretations do not fall prey to "no-go" theorems such as Bell's or Kochen and Specker's.)

In short, according to modal interpretations, an observable can possess a definite value even when the quantum state is not an eigenstate of that observable. The unique biorthogonal decomposition determines which observables take on definite values.

For this reason, modal interpretations suffer from the basis degeneracy problem, just as many-world interpretations do, when $|c_1|=|c_2|$. But our technical results can help. When the particle-apparatus system interacts with its environment, it evolves into $|\Psi\rangle$, which is (uniquely) triorthogonally decomposed in Eq. (3). By allowing unique triorthogonal decompositions—as well as unique biorthogonal decompositions—to pick out which observables receive definite values, modal interpreters can explain why all ideal measurements have definite results (i.e., after the measurement, the pointer reading is definite). Importantly, the basis selected by a triorthogonal decomposition never conflicts with the basis picked out by the unique biorthogonal decomposition, when one exists. In summary, by proving the uniqueness of the triorthogonal decomposition, we can help many-world, modal, and decoherence interpretations deal with the basis degeneracy problem.

IV. FACTORIZABLE VERSUS ENTANGLED STATES

To discuss our technical results, we need to review some terminology. Assume throughout that all vectors are normalized.

Two vectors are collinear if the modulus of their inner product equals 1. For instance, $|\zeta\rangle$ and $e^{i\phi}|\zeta\rangle$ are collinear. A nontrivial set of vectors, denoted $\{|a_i\rangle\}$, must contain two or more vectors. $\{|a_i\rangle\}$ is "noncollinear" if any two vectors in the set are not collinear. A set of vectors $\{|a_i'\rangle\}$ "differs nontrivially" from $\{|a_i\rangle\}$ only if some of the $\{|a_i'\rangle\}$ vectors are not collinear with any of the $\{|a_i\rangle\}$ vectors.

$\{|a_i\rangle\}$ is linearly independent only if no vector in the set can be written as a linear combination of other vectors in the set. $\{|a_i\rangle\}$ is orthogonal if $\langle a_i|a_j\rangle=\delta_{ij}$. A linearly independent set of vectors can serve as a basis for the Hilbert space (or the subspace thereof) spanned by those vectors.

First, we will prove a lemma about factorizable states. In some cases, a two-system quantum state is *factorizable*: $|\Psi\rangle=|\phi\rangle\otimes|\zeta\rangle$, where $|\phi\rangle$ is the state of system 1, and $|\zeta\rangle$ is the state of system 2. If so, then we cannot rewrite $|\Psi\rangle$ in the "entangled" form $\sum_i d_i|A_i\rangle\otimes|B_i\rangle$. In other words, a factorizable state cannot be rewritten as an entangled state, or vice versa. We will now formalize this result.

Lemma 1: Suppose $|\Psi\rangle$, the combined state of system 1 and system 2, is factorizable. Then there exists no orthogonal set of vectors $\{|A_i\rangle\}$, and no noncollinear set of vectors $\{|B_i\rangle\}$, such that $|\Psi\rangle=\sum_i d_i|A_i\rangle\otimes|B_i\rangle$ (for two or more nonzero d_i's).

Proof: By assumption, $|\Psi\rangle=|\phi\rangle\otimes|\zeta\rangle$. Let $\{|A_i'\rangle\}$ denote an orthogonal set of vectors such that $|\phi\rangle=\sum_j c_j|A_j'\rangle$, where each c_j is nonzero. (An infinite number of such sets exist.) So, $|\Psi\rangle=\sum_j c_j|A_j'\rangle\otimes|\zeta\rangle$.

Now let $|\Psi'\rangle=\sum_i d_i|A_i\rangle\otimes|B_i\rangle$, where $\{|A_i\rangle\}$ is orthogonal and contains the same number of vectors as $\{|A_i'\rangle\}$ does. Clearly, $|\Psi\rangle=|\Psi'\rangle$ only if $\{|A_i\rangle\}$ and $\{|A_i'\rangle\}$ span the same subspace of H_1. Therefore, since $\{|A_i\rangle\}$ and $\{|A_i'\rangle\}$ are both linearly independent sets of vectors, the primed and unprimed vectors can be expanded as linear combinations of each other: $|A_j'\rangle=\sum_i e_{ji}|A_i\rangle$. So, from $|\Psi\rangle=\sum_j c_j|A_j'\rangle\otimes|\zeta\rangle$, we get $|\Psi\rangle=\sum_j\sum_i c_j e_{ji}|A_i\rangle\otimes|\zeta\rangle=\sum_i h_i|A_i\rangle\otimes|\zeta\rangle$, where $h_i\equiv\sum_j c_j e_{ji}$. Since $\{|A_i\rangle\}$ is a linearly independent set, $|\Psi\rangle=\sum_i h_i|A_i\rangle\otimes|\zeta\rangle$ and $|\Psi'\rangle=\sum_i d_i|A_i\rangle\otimes|B_i\rangle$ are the same vector only if $d_i|B_i\rangle=h_i|\zeta\rangle$ for each i. Therefore, $|\Psi\rangle=|\Psi'\rangle$ only if all the $\{|B_i\rangle\}$ vectors are collinear. This proves the lemma.

V. THEOREMS

In this section, we prove the triorthogonal uniqueness theorem and related results. We will begin with a trivial but crucial lemma.

Lemma 2. Let $\{|\alpha_i\rangle\}$ and $\{|C_i\rangle\}$ be linearly independent sets of vectors in H_1 and H_2. Let $\{|C_i'\rangle\}$ be a linearly independent set of vectors that differs nontrivially from $\{|C_i\rangle\}$. If $|\psi\rangle=\sum_i c_i|\alpha_i\rangle\otimes|C_i\rangle$, then $|\psi\rangle=\sum_i d_i|\alpha_i'\rangle\otimes|C_i'\rangle$ *only if* at least one of the $\{|\alpha_i'\rangle\}$ vectors is a linear combination of (at least two) $\{|\alpha_i\rangle\}$ vectors. In symbols, for some k, $|\alpha_k'\rangle=\sum_i g_{ki}|\alpha_i\rangle$, where at least two of the g_{ki}'s are nonzero.

Proof. Let $|\psi'\rangle=\sum_i d_i|\alpha_i\rangle\otimes|C_i'\rangle$. We will first prove that $|\psi'\rangle\neq|\psi\rangle$.

$|\psi'\rangle$ equals $|\psi\rangle$ only if $\{|C_i'\rangle\}$ and $\{|C_i\rangle\}$ span the same subspace of H_2. Therefore, since both of those sets of vectors are linearly independent, the $\{|C_i'\rangle\}$ vectors are linear combinations of the $\{|C_i\rangle\}$ vectors: $|C_i'\rangle=\sum_j e_{ij}|C_j\rangle$. By assumption, at least one of the $|C_i'\rangle$ vectors is not collinear with any of the $|C_j\rangle$ vectors. Therefore, for some i, $e_{ij}\neq 0$ for at least two values of j.

Since $|C_i'\rangle=\sum_j e_{ij}|C_j\rangle$, we have $|\psi'\rangle=\sum_i d_i|\alpha_i\rangle\otimes(\sum_j e_{ij}|C_j\rangle)$. Because the $\{|\alpha_i\rangle\}$ vectors are linearly independent, $|\psi'\rangle=\sum_i d_i|\alpha_i\rangle\otimes(\sum_j e_{ij}|C_j\rangle)$ is the same vector as $|\psi\rangle=\sum_i c_i|\alpha_i\rangle\otimes|C_i\rangle$ only if $|C_i\rangle=(d_i/c_i)\sum_j e_{ij}|C_j\rangle$ for each i. Since the $\{|C_i\rangle\}$ vectors are linearly independent, $|C_i\rangle=(d_i/c_i)\sum_j e_{ij}|C_j\rangle$ only if $e_{ij}=0$ for all $j\neq i$. Therefore, for each i, $e_{ij}\neq 0$ for exactly one value of j, namely $j=i$. This contradicts the conclusion of the previous paragraph.

By similar reasoning, you can prove that $|\psi'\rangle=\sum_i d_i|\alpha_i'\rangle\otimes|C_i'\rangle$ cannot equal $|\psi\rangle$ if all the $|\alpha_i'\rangle$'s are collinear with the $|\alpha_i\rangle$'s. Therefore, $|\psi\rangle=\sum_i d_i|\alpha_i'\rangle\otimes|C_i'\rangle$ only if $\{|\alpha_i'\rangle\}$ is nontrivially distinct from $\{|\alpha_i\rangle\}$. Since $\{|\alpha_i'\rangle\}$ and $\{|\alpha_i\rangle\}$ must span the same subspace of H_1 (or else $\sum_i d_i|\alpha_i'\rangle\otimes|C_i'\rangle$ could not equal $\sum_i c_i|\alpha_i\rangle\otimes|C_i\rangle$), it follows that at least one $|\alpha_i'\rangle$ vector is a linear combination of $\{|\alpha_i\rangle\}$ vectors. In symbols, for some k, $|\alpha_k'\rangle=\sum_i g_{ki}|\alpha_i\rangle$, where at least two of the g_{ki}'s are nonzero.

Given this lemma, we can now prove triorthogonal uniqueness. Although a three-system quantum state cannot in general be triorthogonally decomposed, our proof shows that if such a decomposition exists, it is unique.

Triorthogonal uniqueness theorem. Suppose $|\Psi\rangle=\sum_i c_i|A_i\rangle\otimes|B_i\rangle\otimes|C_i\rangle$, where $\{|A_i\rangle\}$, $\{|B_i\rangle\}$, and $\{|C_i\rangle\}$ are orthogonal (and therefore linearly independent) sets of vectors in H_1, H_2, and H_3. Then, even if some of the $|c_i|$'s are equal, no alternative orthogonal sets of vectors $\{|A_i'\rangle\}$, $\{|B_i'\rangle\}$, and $\{|C_i'\rangle\}$ exist such that $|\Psi\rangle=\sum_i d_i|A_i'\rangle\otimes|B_i'\rangle\otimes|C_i'\rangle$, unless each alternative set of vectors differs only trivially from the set it replaces.

Proof. Assume, without loss of generality, that $\{|C_i\rangle\}$ differs nontrivially from $\{|C_i'\rangle\}$. It is given that $|\Psi\rangle=\sum_i c_i|\alpha_i\rangle\otimes|C_i\rangle$, where $|\alpha_i\rangle\equiv|A_i\rangle\otimes|B_i\rangle$.

Here is a proof by contradiction. Suppose $|\Psi\rangle=\sum_i d_i|A_i'\rangle\otimes|B_i'\rangle\otimes|C_i'\rangle$. Then $|\Psi\rangle=\sum_i d_i|\alpha_i'\rangle\otimes|C_i'\rangle$, where $|\alpha_i'\rangle\equiv|A_i'\rangle\otimes|B_i'\rangle$. By lemma 1, we cannot rewrite the factorizable state $|\alpha_k'\rangle=|A_k'\rangle\otimes|B_k'\rangle$ as an entangled state.

But according to lemma 2, since $|\Psi\rangle=\sum_i c_i|\alpha_i\rangle\otimes|C_i\rangle$ and since $\{|C_i\rangle\}$ differs nontrivially from $\{|C_i'\rangle\}$, it follows that $|\Psi\rangle=\sum_i d_i|\alpha_i'\rangle\otimes|C_i'\rangle$ only if $|\alpha_k'\rangle=\sum_i g_{ki}|\alpha_i\rangle$, where at least two of the g_{ki}'s are nonzero. Since $|\alpha_i\rangle\equiv|A_i\rangle\otimes|B_i\rangle$, it follows that $|\alpha_k'\rangle$ is an entangled state, $|\alpha_k'\rangle=\sum_i g_{ki}|A_i\rangle\otimes|B_i\rangle$. This contradiction establishes that $|\Psi\rangle\neq\sum_i d_i|A_i'\rangle\otimes|B_i'\rangle\otimes|C_i'\rangle$.

We can generalize this result, because nowhere did we need to assume that $\{|B_i\rangle\}$ is orthogonal, or even linearly independent. In lemma 1, we assumed only that $\{|B_i\rangle\}$ is noncollinear. In lemma 2, we assumed that $\{|C_i\rangle\}$ and $\{|\alpha_i\rangle\}$ are both linearly independent sets of vectors. But if $|\alpha_i\rangle\equiv|A_i\rangle\otimes|B_i\rangle$, then the linear independence of $\{|\alpha_i\rangle\}$ follows entirely from the linear independence of $\{|A_i\rangle\}$.

Finally, as Allen Stairs first pointed out, lemma 1 holds even if $\{|A_i\rangle\}$ is a linearly independent, instead of orthogonal, set of vectors. Putting all this together, we get the tridecompositional uniqueness theorem.

Tridecompositional uniqueness theorem.

Suppose $|\Psi\rangle=\sum_i c_i|A_i\rangle\otimes|B_i\rangle\otimes|C_i\rangle$, where $\{|A_i\rangle\}$ and $\{|C_i\rangle\}$ are linearly independent sets of vectors, while $\{|B_i\rangle\}$ is merely noncollinear. Then there exist no alternative linearly independent sets of vectors $\{|A_i'\rangle\}$ and $\{|C_i'\rangle\}$, and no alternative noncollinear set $\{|B_i'\rangle\}$, such that $|\Psi\rangle=\sum_i d_i|A_i'\rangle\otimes|B_i'\rangle\otimes|C_i'\rangle$. (Unless each alternative set of vectors differs only trivially from the set it replaces.)

At this point, for any n, we can prove the *n-decompositional uniqueness theorem.*

Suppose $\Psi=\sum_i c_i|A_i\rangle\otimes|U_i\rangle\otimes|V_i\rangle\otimes\cdots\otimes|C_i\rangle$, where $\{|A_i\rangle\}$ and $\{|C_i\rangle\}$ are linearly independent sets of vectors, $\{|U_i\rangle\}$ is a noncollinear set, and $\{|V_i\rangle\}$, etc., are any sets of vectors. No nontrivially-different alternative sets of vectors exist such that $\Psi=\sum_i d_i|A_i'\rangle\otimes|U_i'\rangle\otimes|V_i'\rangle\otimes\cdots\otimes|D_i'\rangle$.

Proof. Define $|B_i\rangle\equiv|U_i\rangle\otimes|V_i\rangle\otimes\cdots$, and $|B_i'\rangle\equiv|U_i'\rangle\otimes|V_i'\rangle\otimes\cdots$. The noncollinearity of $\{|U_i\rangle\}$ and $\{|U_i'\rangle\}$ implies the noncollinearity of $\{|B_i\rangle\}$ and $\{|B_i'\rangle\}$. Now plug these $\{|B_i\rangle\}$'s and $\{|B_i'\rangle\}$'s into lemma 1 and the triorthogonal uniqueness theorem. (Lemma 2 does not invoke $|B_i\rangle$ vectors.) The proofs go through unchanged. Therefore, the n-orthogonal uniqueness theorem holds.

VI. CONCLUSION

The tridecompositional uniqueness theorem provides many-world interpretations, decoherence interpretations, and modal interpretations with a rigorous solution to the basis degeneracy problem. Recall from above that when a device ideally measures a spin-$\frac{1}{2}$ particle's z component of spin, and then the combined system interacts with its environment; the final state is

$$|\Psi\rangle=c_1|S_z=+\rangle\otimes|R=+\rangle\otimes|E_+\rangle + c_2|S_z=-\rangle\otimes|R=-\rangle\otimes|E_-\rangle \ .$$

If $c_1=c_2$, then the biorthogonal decomposition of the apparatus with the particle-environment system is not unique, and therefore gives us no principled reasoning for singling out the pointer-reading basis. This is the basis degeneracy problem. We have just proven, however, that even if $c_1=c_2$, and even if the *environmental states are not strictly orthogonal,* no alternative bases exist such that Ψ can be expanded as $d_1|S'=+\rangle\otimes|R'=+\rangle\otimes|E_+'\rangle+d_2|S'=-\rangle\otimes|R'=-\rangle\otimes|E_-'\rangle$. Therefore, we have pinpointed a rigorous formal reason for calling the pointer-reading basis "special."

Many-world interpreters can claim that worlds "split" along that preferred basis. Decoherence theorists can invoke tridecompositional uniqueness to support their claim that environmental interactions make the pointer reading become "classical" in some sense. And modal interpreters can claim that when a quantum state vector can be a tridecomposed (with two of the three bases orthogonal), the observables picked out by the tridecomposition have definite values.

In [9], one of us argues that ideal measurements of most observables are impossible. If this is correct, then the unique biorthogonal decomposition of the measuring device with the rest of the universe, if it exists, usually picks out some apparatus basis other than the pointer-reading basis. Furthermore, a nonideal measurement usually results in a particle-apparatus-environment state that cannot be tridecomposed. Our technical results cannot help to solve these deep-seated technical problems associated with nonideal measurements.

In conclusion, at least for idealized cases, the tridecompositional uniqueness theorem can help several interpretations of QM deal with the basis degeneracy problem that arises from nonunique biorthogonal decompositions.

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[1] *The Many-Worlds Interpretation of Quantum Mechanics,* edited by B. DeWitt and R. N. Graham (Princeton University Press, Princeton, 1973).
[2] W. H. Zurek, Prog. Theor. Phys. **89**, 281 (1993).
[3] E. Joos and H. D. Zeh, Z. Phys. B **59**, 223 (1985).
[4] R. Healey, *The Philosophy of Quantum Mechanics: An Interactive Interpretation* (Cambridge University Press, Cambridge, England, 1989).
[5] S. Kochen, in *Symposium on the Foundations of Modern Physics,* edited by P. Lahti and P. Mittelstaedt (World Scientific, Singapore, 1985), p. 151.
[6] D. Dieks, Phys. Lett. A **142**, 439 (1989).
[7] B. van Fraassen, *Quantum Mechanics: An Empiricist View* (Clarendon, Oxford, England, 1991).
[8] J. Bub, Found. Phys. **22**, 737 (1992).
[9] A. Elby, Found. Phys. Lett. **6**, 5 (1993).