Noise-dependent uncertainty relations for the harmonic oscillator

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The Gaussian noise model for a harmonic oscillator subjected to random linear fluctuations (e.g., thermal noise) is used to derive generalized Heisenberg and entropic uncertainty relations for the position-momentum and number-phase observables. The lower bounds are partitioned into pure quantum and pure noise terms, and the minimum-uncertainty states are determined. Formally similar results of Abe and Suzuki [Phys. Rev. A **41**, 4608 (1990)] for the position-momentum case, derived using the thermofield formalism, are shown to correspond to amplification noise rather than thermal noise.

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I. INTRODUCTION

An uncertainty relation for two observables A and B of a quantum system has the general form [1]

$$U(A,B|\rho)\geq\mathcal{B}(A,B)\ , \tag{1}$$

where U denotes some state-dependent measure of joint uncertainty and $\mathcal{B}$ denotes some state-independent lower bound. Those density operators ρ which saturate inequality (1) are the minimum-uncertainty states corresponding to U.

A well-known example is the Heisenberg uncertainty relation for a position coordinate X and its conjugate momentum P [2]:

$$U_H(X,P|\rho)=\Delta X\Delta P\geq\tfrac{1}{2}\ , \tag{2}$$

where ΔA denotes the root-mean-square deviation of observable A (units are chosen throughout such that $\hbar=1$). The minimum-uncertainty states corresponding to (2) have the position representation [2]

$$\begin{aligned}\psi(x)&=\langle x|\psi\rangle\\&=[2\pi(\Delta X)^2]^{-1/4}\\&\quad\times\exp[-(x-\langle X\rangle)^2/(2\Delta X)^2+i\langle P\rangle x]\ .\end{aligned} \tag{3}$$

A lesser-known example is the *entropic* uncertainty relation for position and momentum [3]:

$$U_E(X,P|\rho)=S(X|\rho)+S(P|\rho)\geq\ln\pi e\ , \tag{4}$$

where, if $\Pi(a|\rho)$ denotes the probability distribution of observable A for state ρ, then

$$S(A|\rho)=-\int da\ \Pi(a|\rho)\ln\Pi(a|\rho) \tag{5}$$

denotes the associated entropy. Inequality (2) in fact follows from inequality (4) [3], and hence the latter is a stronger uncertainty relation. It may be checked that the minimum-uncertainty states corresponding to (4) are again of the form (3). Entropic uncertainty relations for finite Hilbert spaces are discussed in [1,4].

While uncertainty relations (2) and (4) are universally valid, stronger relations may be obtained for quantum systems that are subject to noise from their environment. In particular, the joint uncertainty of two observables will tend to increase in the presence of noise, leading to a larger lower bound in (1). In Sec. II of this paper the case of a harmonic oscillator subject to random linear fluctuations (e.g., thermal noise) is considered, and generalizations of relations (2) and (4) are obtained for this case in Sec. III. A generalized entropic uncertainty relation for the number and phase observables of the oscillator is obtained in Sec. IV, and a noise-dependent Heisenberg uncertainty relation for number and phase is conjectured there.

The case of the harmonic oscillator is of interest not only because of its wide applicability as a model for quantum systems, but also because of its special connection with the minimum-uncertainty states (3). These states are in fact the well-known "squeezed states" of quantum optics [5,6] and, choosing units such that the oscillator mass and frequency are of unit magnitude, evolve under the Hamiltonian

$$H=\tfrac{1}{2}(P^2+X^2) \tag{6}$$

such that [6]

$$\begin{aligned}&(\Delta X)_t(\Delta P)_t\\&\quad=\tfrac{1}{2}\{1+\tfrac{1}{4}[(\Delta X)_0/(\Delta P)_0-(\Delta P)_0/(\Delta X)_0]^2\sin^2 2t\}^{1/2}\ .\end{aligned} \tag{7}$$

Thus the minimum-uncertainty property of squeezed states is periodically recovered for an oscillator system (at times $t=\pi/2,\pi,3\pi/2,\ldots$), and indeed is always present in the case

$$(\Delta X)_0=(\Delta P)_0=2^{-1/2}\ ,$$

i.e., when $|\psi\rangle$ is a *coherent* state [2,5,6].

It will be shown in Sec. III that *only* the coherent states remain minimum-uncertainty states of position-momentum when the oscillator is degraded by noise. Similarly, it is shown in Sec. IV that, while all energy eigenstates of the oscillator are minimum-uncertainty states of number-phase in the absence of noise, only the ground state remains a minimum-uncertainty state when noise is added.

In Sec. V it is demonstrated that formally similar results of Abe and Suzuki for the position-momentum case [7,8], derived using the thermofield formalism [9], correspond to amplification noise rather than to thermal noise. The interpretation of results is discussed in Sec. VI.

II. THE NOISY OSCILLATOR

The description of a harmonic oscillator subjected to random linear excitations was first considered by Glauber [10] for the case of the ground state and has since been extended to the cases of coherent states [11], squeezed states [12,13], and energy eigenstates [14]. The general description corresponds to a *Gaussian noise* model [15,16], and a systematic exposition is given in [16]. It should be noted that [10–16] work primarily with the annihilation operator $a=2^{-1/2}(X+iP)$, rather than with X and P separately.

The effect of adding random linear excitations to an oscillator described by state ρ is generically modeled (in the interaction picture) by replacing ρ with the density operator [15,16]

$$\Gamma(\rho)=\int\int dx\, dp\, p_\gamma(x,p;n_\gamma)D_{x,p}\rho D_{x,p}^\dagger \ . \tag{8}$$

Here, integration is over the x-p plane, $p_\gamma(x,p;n_\gamma)$ is a Gaussian distribution of the form

$$p_\gamma(x,p;n_\gamma)=(2\pi n_\gamma)^{-1}\exp[-(x^2+p^2)/(2n_\gamma)] \ , \tag{9}$$

n_γ is a dimensionless variance parameter, and $D_{x,p}$ is the unitary displacement operator [10]

$$D_{x,p}=\exp(ipX-ixP) \ . \tag{10}$$

Equation (8) implies that the oscillator is displaced by an amount (x,p) in phase space with probability $p_\gamma(x,p;n_\gamma)$. The form of this probability distribution essentially arises from the central limit theorem of classical probability theory, which states that the statistics of a large number of random fluctuations is typically Gaussian [16].

From (10) one has the relations [10]

$$D_{x,p}XD_{x,p}^\dagger=X-x \ , \quad D_{x,p}PD_{x,p}^\dagger=P-p \ , \tag{11}$$

and hence, using (8) and (9), one finds [15,16]

$$\mathrm{tr}[H\Gamma(\rho)]=\mathrm{tr}[H\rho]+n_\gamma \ . \tag{12}$$

Thus the variance parameter n_γ determines the *average noise energy* added to the oscillator. Adding noise to the ground state $|0\rangle\langle 0|$ yields the thermal state [10–16]

$$\Gamma(|0\rangle\langle 0|)=\sum_m |m\rangle\langle m|n_\gamma^m/(n_\gamma+1)^{m+1} \ , \tag{13}$$

where $|m\rangle$ denotes the mth energy eigenstate. Hence, an *effective noise temperature* T_γ may be defined via the Planck relation

$$n_\gamma=[\exp(\hbar\omega/kT_\gamma)-1]^{-1} \ , \tag{14}$$

where units have been restored in (14). For $T_\gamma\approx 300$ K and $\omega\approx 10^9-10^{15}$ Hz, one finds from (14) that $n_\gamma\approx 10^{-11}-10^4$.

Substitution of an arbitrary operator A for ρ in (8) defines the operator $\Gamma(A)$. Some useful properties of the mapping $A\rightarrow\Gamma(A)$, to be used later, are [15,16]

$$\Gamma(1)=1 \ , \tag{15}$$

$$\mathrm{tr}[A\Gamma(B)]=\mathrm{tr}[\Gamma(A)B] \ , \tag{16}$$

$$\Gamma(D_{x,p}AD_{x,p}^\dagger)=D_{x,p}\Gamma(A)D_{x,p}^\dagger \ . \tag{17}$$

These may easily be derived from (8)–(10). Note from (12), (15), and (16) that

$$\Gamma(H)=H+n_\gamma \ . \tag{18}$$

III. POSITION AND MOMENTUM

It will be demonstrated here that

$$U_H(X,P|\Gamma(\rho))\geq \tfrac{1}{2}+n_\gamma \ , \tag{19}$$

$$U_E(X,P|\Gamma(\rho))\geq \ln\pi e+\ln(2n_\gamma+1) \ , \tag{20}$$

thus generalizing uncertainty relations (2) and (4) to the case of the noisy oscillator.

First, it follows from (8), (9), (11), and (16) [cf. Eq. (21a) of [16]] that Gaussian noise increases the variances of X and P by an amount n_γ. Hence, from (2) one has

$$U_H(X,P|\Gamma(\rho))=[(\Delta X)^2+n_\gamma]^{1/2}[(\Delta P)^2+n_\gamma]^{1/2} \ , \tag{21}$$

where $\Delta X,\Delta P$ refer to state ρ. Minimizing this expression for a fixed value of $\Delta X\Delta P$ yields the condition $\Delta X=\Delta P$, which combined with the inequality in (2) implies relation (19) as desired. Noting that inequality (2) is saturated by the squeezed states (3), the condition $\Delta X=\Delta P$ implies that (19) is saturated only when ρ is a coherent state.

Second, writing ρ as a mixture of pure states

$$\rho=\sum_j p_j|\psi_j\rangle\langle\psi_j| \ , \quad \sum_j p_j=1=\langle\psi_j|\psi_j\rangle \ , \tag{22}$$

it follows from the concavity of the entropy functional (5) and the linearity of the mapping (8) that

$$S(A|\Gamma(\rho))=S\left[A\Big|\sum_j p_j\Gamma(|\psi_j\rangle\langle\psi_j|)\right]$$
$$\geq\sum_j p_jS(A|\Gamma(|\psi_j\rangle\langle\psi_j|)) \ .$$

Hence, from (4),

$$U_E(X,P|\Gamma(\rho))\geq\sum_j p_j U_E(X,P|\Gamma(|\psi_j\rangle\langle\psi_j|))$$
$$\geq\inf_j\{U_E(X,P|\Gamma(|\psi_j\rangle\langle\psi_j|))\} \ , \tag{23}$$

implying that only *pure* states need be considered for the minimization of $U_E(X,P|\Gamma(\rho))$.

Writing

$$J=U_E(X,P|\Gamma(|\psi\rangle\langle\psi|))-\lambda(\langle\psi|\psi\rangle-1) \ , \tag{24}$$

where λ is a variational parameter constraining the normalization of $|\psi\rangle$, and noting for example from (8) and (16) that

$$\left[\frac{\partial}{\partial\langle\psi|}\right]\Pi(x|\Gamma(|\psi\rangle\langle\psi|))=\left[\frac{\partial}{\partial\langle\psi|}\right]\langle x|\Gamma(|\psi\rangle\langle\psi|)|x\rangle$$
$$=\left[\frac{\partial}{\partial\langle\psi|}\right]\langle\psi|\Gamma(|x\rangle\langle x|)|\psi\rangle$$
$$=\Gamma(|x\rangle\langle x|)|\psi\rangle\ ,$$

one obtains the variational equation

$$\left[\frac{\partial}{\partial\langle\psi|}\right]J=0=[\Gamma(A_\psi)-\lambda-2]|\psi\rangle\ , \tag{25}$$

where A_ψ denotes the Hermitian operator

$$\begin{aligned}A_\psi&=-\int dx\,|x\rangle\langle x|\ln\Pi(x|\Gamma(|\psi\rangle\langle\psi|))\\&\quad-\int dp\,|p\rangle\langle p|\ln\Pi(p|\Gamma(|\psi\rangle\langle\psi|))\\&=-\ln\Pi(X|\Gamma(|\psi\rangle\langle\psi|))-\ln\Pi(P|\Gamma(|\psi\rangle\langle\psi|))\ .\end{aligned} \tag{26}$$

It follows from (25) that $U_E(X,P|\Gamma(|\psi\rangle\langle\psi|))$ attains its external values whenever $|\psi\rangle$ is an eigenstate of $\Gamma(A_\psi)$.

The minimum-uncertainty solutions to (25) may be found by trialing the squeezed states (3), which are already known to be minimum-uncertainty states for $n_\gamma=0$. For such states one has [cf. Eq. (64) of [16]]

$$\Pi(x|\Gamma(|\psi\rangle\langle\psi|))=\{2\pi[(\Delta X)^2+n_\gamma]\}^{-1/2}\exp\left[\frac{-(x-\langle X\rangle)^2}{2[(\Delta X)^2+n_\gamma]}\right]\ , \tag{27a}$$

$$\Pi(p[|\Gamma(|\psi\rangle\langle\psi|))=\{2\pi[(\Delta P)^2+n_\gamma]\}^{-1/2}\exp\left[\frac{-(p-\langle P\rangle)^2}{2[(\Delta P)^2+n_\gamma]}\right]\ , \tag{27b}$$

and hence from (11) and (26) it follows that

$$A_\psi=\ln 2\pi(n_\gamma+\tfrac{1}{2})+(2n_\gamma+1)^{-1}D_{\langle X\rangle,\langle P\rangle}(X^2+P^2)D^\dagger_{\langle X\rangle,\langle P\rangle} \tag{28}$$

in the coherent-state case $\Delta X=\Delta P=2^{-1/2}$. But for this case one has from (3) and (10) that $|\psi\rangle=D_{x,p}|0\rangle$, to within a phase factor, and so using (6), (10), (17), and (18), it follows that

$$\begin{aligned}\Gamma(A_\psi)|\psi\rangle&=\ln 2\pi(n_\gamma+\tfrac{1}{2})|\psi\rangle\\&\quad+2(2n_\gamma+1)^{-1}D_{\langle X\rangle,\langle P\rangle}\Gamma(H)|0\rangle\\&=[\ln 2\pi(n_\gamma+\tfrac{1}{2})+1]|\psi\rangle\ ,\end{aligned} \tag{29}$$

i.e., Eq. (25) is satisfied for coherent states $|\psi\rangle$.

Thus, the coherent states are minimum-uncertainty states for $U_E(X,P|\Gamma(\rho))$, and the uncertainty relation (20) follows via Eqs. (27) [it may also be checked via (27) that general squeezed states do not remain minimum-uncertainty states when noise is added]. Inequality (19) may be derived from inequality (20) in a manner exactly analogous to the zero-noise case [3], so that the latter is in general the stronger relation.

IV. NUMBER AND PHASE

The (dimensionless) number operator of the oscillator is defined by

$$N=\tfrac{1}{2}(X-iP)(X+iP)=H-\tfrac{1}{2} \tag{30}$$

and has eigenvalues $0,1,2,\ldots$ corresponding to the energy eigenstates $|0\rangle$, $|1\rangle$, $|2\rangle,\ldots$ of the oscillator. The conjugate observable to N is the *phase*, Φ. While phase cannot be represented by a Hermitian operator [17], the *canonical phase distribution* $\Pi(\phi|\rho)$ of state ρ may be calculated via the simple formula [18–20]

$$\Pi(\phi|\rho)=(2\pi)^{-1}\langle e^{i\phi}|\rho|e^{i\phi}\rangle\ , \tag{31}$$

where $|e^{i\phi}\rangle$ is the ket

$$|e^{i\phi}\rangle=\sum_n e^{in\phi}|n\rangle \tag{32}$$

(in [19,20] it is shown that this prescription gives results equivalent to the less compact Pegg-Barnett phase formalism [21]).

An entropic uncertainty relation for number and phase follows from results in [3] as

$$U_E(N,\Phi|\rho)=S(N|\rho)+S(\Phi|\rho)\geq\ln 2\pi\ , \tag{33}$$

as was demonstrated in [22]. The corresponding minimum-uncertainty states are just the energy eigenstates (two classes of *approximate* minimum-uncertainty states, with well-defined phase properties, are noted in Sec. IV of [22]).

It will be shown here that (33) may be generalized to

$$U_E(N,\Phi|\Gamma(\rho))\geq\ln 2\pi+\ln(n_\gamma+1)+n_\gamma\ln(1+n_\gamma^{-1}) \tag{34}$$

for the noisy oscillator. The only minimum-uncertainty state corresponding to this uncertainty relation for $n_\gamma>0$ is the ground state. Note from (20) and (34) that for $n_\gamma\gg 1$, one has

$$U_E(N,\Phi),U_E(X,P)\gtrsim\ln 2\pi e n_\gamma\ . \tag{35}$$

Hence, both pairs of conjugate observables, (X,P) and (N,Φ), share a common uncertainty bound in the high noise limit.

The proof of (34) is similar to that of (20) in Sec. III. First, as per (22) and (23), only pure states need be considered for the minimization problem, so that one writes

$$K=U_E(N,\Phi|\Gamma(|\psi\rangle\langle\psi|))-\lambda(\langle\psi|\psi\rangle-1) \tag{36}$$

in analogy to (24). As per Eqs. (25) and (26), the extremal states then satisfy

$$\left[\frac{\partial}{\partial\langle\psi|}\right]K=0=[\Gamma(B_\psi)-\lambda-2]|\psi\rangle\ , \tag{37}$$

where B_ψ denotes the Hermitian operator

$$B_\psi = -\sum_n |n\rangle\langle n| \ln\Pi(n|\Gamma(|\psi\rangle\langle\psi|)) - (2\pi)^{-1}\int d\phi |e^{i\phi}\rangle\langle e^{i\phi}| \ln\Pi(\phi|\Gamma(|\psi\rangle\langle\psi|)) \ . \quad (38)$$

The solutions to (37) are given by the energy eigenstates, as may be checked using (31), (32), and (38) and the property that $\Gamma(|n\rangle\langle n|)$ is diagonal with respect to the energy basis $\{|0\rangle,|1\rangle,|2\rangle,\ldots\}$ [14–16]. This last property also implies that

$$S(\Phi|\Gamma(|n\rangle\langle n|)) = \ln 2\pi \ , \quad (39a)$$

$$S(N|\Gamma(|n\rangle\langle n|)) = -\mathrm{tr}[\Gamma(|n\rangle\langle n|)\ln\Gamma(|n\rangle\langle n|)] \ . \quad (39b)$$

Combining these equations with the result [15]

$$-\mathrm{tr}[\Gamma(\rho)\ln\Gamma(\rho)] \geq -\mathrm{tr}[\Gamma(|0\rangle\langle 0|)\ln\Gamma(|0\rangle\langle 0|)] \ , \quad (40)$$

valid for all density operators ρ (with equality *only* for the coherent states), the uncertainty relation (34) follows as desired via (13), and is clearly saturated only by the ground state.

The difficulty of defining variance for periodic observables such as phase is discussed in [22]. However, one may consider the second moment of the canonical phase distribution (31) with respect to an arbitrary reference phase θ:

$$(\Delta_\theta\Phi)^2 = \int_{\theta-\pi}^{\theta+\pi} d\phi\,(\phi-\theta)^2\Pi(\phi|\rho) \ , \quad (41)$$

which satisfies the Heisenberg-type uncertainty relation [22]

$$U_H(N,\Phi|\rho;\theta) = \Delta N \Delta_\theta\Phi \geq \tfrac{1}{2}|1-2\pi\Pi(\theta+\pi|\rho)| \ . \quad (42)$$

The lower bound in (42) is state dependent, and reduces to the trivial value zero when ρ is an energy eigenstate. Based solely on the minimum-uncertainty property of the ground state for the entropic uncertainty relation (34), it is *conjectured* here that

$$U_H(N,\Phi|\Gamma(\rho);\theta) \geq U_H(N,\Phi|\Gamma(|0\rangle\langle 0|);\theta) = \pi[n_\gamma(n_\gamma+1)/3]^{1/2} \ , \quad (43)$$

where the last line follows via (13) and (31). Note that the conjectured lower bound scales linearly with noise energy in the limit $n_\gamma \gg 1$, in analogy with relation (19) for position and momentum.

V. THERMOFIELDS AND AMPLIFICATION

Inequalities formally similar to (19) and (20) have been previously stated by Abe and Suzuki [7,8] for states $|\psi,\tilde{\psi};\beta\rangle$ of the "doubled" Hilbert space $\mathcal{H}\otimes\tilde{\mathcal{H}}$. Here, $\tilde{\mathcal{H}}$ is isomorphic to the oscillator Hilbert space $\mathcal{H}$, and β is identified as an inverse temperature. Motivated by their use of the thermofield formalism [9] (more typically applied to the calculation of thermal averages in field theory), Abe and Suzuki interpret their inequalities as "thermal uncertainty relations," pertaining to an oscillator in equilibrium with a thermal reservoir. However, it will be shown here that, even under a suitable restriction of the states $|\psi,\tilde{\psi};\beta\rangle$ (to ensure the inequalities are valid), the results of [7,8] in fact pertain to an oscillator which has undergone linear amplification by a factor $(1-e^{-\beta})^{-1}$.

First, while the states $|\psi,\tilde{\psi};\beta\rangle$ are interpreted in [7,8] as corresponding to some inverse temperature β of the oscillator, no technical specification is actually made other than that they are states in $\mathcal{H}\otimes\tilde{\mathcal{H}}$. This invalidates the derivations in [7,8] as they stand [the crucial Eq. (41) of [7] does *not* hold for general states in $\mathcal{H}\otimes\tilde{\mathcal{H}}$, as is most easily seen by choosing $z=\tilde{z}=0$ in that equation]. Indeed, violations of inequalities (43) and (52) of [7] (see relations (47) below), for all values of β, can be obtained simply by choosing $|\psi,\tilde{\psi};\beta\rangle$ to be the doubled ground state $|0\rangle\otimes|\tilde{0}\rangle$.

It follows that some restriction of the states $|\psi,\tilde{\psi};\beta\rangle$ is necessary if the results of [7,8] are to be recovered. The definition of "thermal coherent states" in [7,8] does suggest a possible restriction, to states of the form

$$|\psi,\tilde{\psi};\beta\rangle = e^{-iG}|\psi\rangle\otimes|\tilde{\psi}\rangle \ . \quad (44)$$

Here, $|\psi\rangle,|\tilde{\psi}\rangle$ are arbitrary state vectors on $\mathcal{H}$ and $\tilde{\mathcal{H}}$, respectively, and

$$-iG = \theta(a^\dagger\tilde{a}^\dagger - a\tilde{a}) \ , \quad (45)$$

where

$$\theta = \arctan(e^{-\beta/2}) \ , \quad (46)$$

and $a,\tilde{a}$ denote the annihilation operators on $\mathcal{H},\tilde{\mathcal{H}}$, respectively.

It may be checked that the derivations in [7,8] do indeed go through under restriction (44) [noting that the second term in Eq. (38) of [7] should be added rather than subtracted, and that the normal-ordering symbols should be removed from Eq. (40) of [7]]. In fact, since only expectation values of operators on $\mathcal{H}$ are calculated, the results may be generalized to the density operator form

$$U_H(X,P|A(\rho)) \geq \tfrac{1}{2}\cosh(2\theta) \ , \quad (47a)$$

$$U_E(X,P|A(\rho)) \geq \ln\pi e + \ln[\cosh(2\theta)] \ . \quad (47b)$$

In these inequalities [corresponding to inequalities (43) and (52) of [7]], the mapping $\rho\to A(\rho)$ generalizes the mapping $|\psi\rangle\to|\psi,\tilde{\psi};\beta\rangle$ in (44) via

$$A(\rho) = \mathrm{tr}_\sim[e^{-iG}\rho\otimes\tilde{\rho}e^{iG}] \ , \quad (48)$$

where $\tilde{\rho}$ is some density operator on $\tilde{\mathcal{H}}$, and $\mathrm{tr}_\sim[\,]$ denotes the trace over $\tilde{\mathcal{H}}$.

However, while inequalities (47) are formally similar to uncertainty relations (19) and (20) [identifying $\cosh\theta$ with $(n_\gamma+1)^{1/2}$], the mapping $\rho\to A(\rho)$ has an "amplification" rather than a "thermal" interpretation. This is readily apparent from Eqs. (45) and (48), which correspond to the *parametric amplification* of state ρ [23,24], where $\tilde{\rho}$ denotes an initial "idler" state and the energy gain is $\cosh^2\theta$.

Moreover, for the case $\tilde{\rho}=|\tilde{0}\rangle\langle\tilde{0}|$ in (48), one may use the relation [7–9]

$$e^{iG}ae^{-iG} = a\cosh\theta + \tilde{a}^\dagger\sinh\theta \quad (49)$$

to calculate the normally ordered characteristic function of $A(\rho)$ as

$$\chi_A(\xi)=\mathrm{tr}[e^{i\xi^* a^\dagger}e^{i\xi a}A(\rho)]$$
$$=\mathrm{tr}[e^{iG}e^{i\xi^* a^\dagger}e^{i\xi a}e^{-iG})\rho\otimes|\tilde{0}\rangle\langle\tilde{0}|]$$
$$=\exp(-|\xi|^2\sinh\theta)\chi(\xi\cosh\theta)\ , \quad (50)$$

where χ denotes the characteristic function of ρ. Comparing with results in [25], it follows that $A(\rho)$ in this case also describes *ideal linear amplification* of the oscillator state ρ, via interaction with a large number of excited two-level systems, again with an energy gain of $\cosh^2\theta$. Berman [26] has studied this case from the thermofield viewpoint for the choice $\rho=|n\rangle\langle n|$, inappropriately referring to the states $A(|n\rangle\langle n|)$ as "thermal excitations."

Thus inequalities (47) do *not* correspond to an oscillator in equilibrium with a thermal reservoir of inverse temperature β, as suggested in [7,8], but rather to an oscillator which has undergone amplification with gain $\cosh^2\theta=(1-e^{-\beta})^{-1}$. The connection between the thermofield formalism and amplification has been noted briefly in Sec. VI of [13]. Heisenberg uncertainty relations of the form (47a) have been given previously by Caves for general linear amplification models [24].

VI. DISCUSSION

It has been demonstrated that the uncertainty relations (2) and (4) for position and momentum may be generalized to inequalities (19) and (20), respectively, for the case of the noisy oscillator. Similarly, the uncertainty relation (33) for number and phase has been generalized to inequality (34), and a generalized Heisenberg uncertainty relation for these observables has been conjectured in inequality (43). While the "thermofield" inequalities (47) have a formal connection with uncertainty relations (19) and (20), they have been demonstrated to correspond to "amplification" rather than "thermal" uncertainty relations.

The generalized Heisenberg uncertainty relation (19) indicates how an open quantum system can exhibit classical behavior. In particular, the effective minimal area of a phase-space cell will be much larger than the "quantum" area $\hbar/2$ when n_γ in (19) is sufficiently large, thus washing out quantum effects. Indeed, choosing n_γ to be $\frac{1}{2}$ is already sufficient to destroy any quadrature and amplitude squeezing properties [16], and the quantum upper bound for information transfer via noisy oscillator systems [15] approaches the classical upper bound in the limit $n_\gamma \gg 1$ [16].

The lower bounds in uncertainty relations (19), (20), and (34) each split into pure noise and pure quantum components, which are of equal magnitude when $n_\gamma=\frac{1}{2}$, $n_\gamma=\frac{1}{2}(\pi e-1)\approx 3.8$, and $n_\gamma\approx 1.8$ respectively. It has already been noted in Eq. (35) that the entropic lower bounds for the conjugate pairs (X,P) and (N,Φ) become equivalent in the high noise limit. Similarly, the Heisenberg bounds (19) and (43) for these conjugate pairs both scale linearly with n_γ in this limit. This again suggests an approach to classical behavior with increasing noise levels, in that the coordinatizations (x,p) and (n,ϕ) of phase space become equivalent in the limit $n_\gamma \gg 1$.

Coherent states have been interpreted by Zurek, Habib, and Paz in [27] as the closest quantum counterparts to classical phase-space points, based on a minimum entropy property of these states under Brownian motion [see also inequality (40) of this paper]. The coherent states are seen here to also have a minimal spreading in area and entropy with respect to position and momentum [i.e., they saturate (19) and (20)], but not with respect to number and phase [they do not saturate inequality (34)]. This distinction between position-momentum and number-phase could be connected to the fact that classical phase-space points play two distinct roles: first, as states with zero statistical dispersion (and therefore minimum dispersion in the presence of noise) and, second, as measurement results of ideal measurements on phase space.

The minimum entropy property of coherent states discussed in [27] corresponds to the first role mentioned above, i.e., to the sharpest possible state specification for a noisy oscillator, as measured by the statistical entropy. However, the minimum-uncertainty property of coherent states for uncertainty relations (19) and (20) may be more closely connected with the second role, i.e., to the best possible joint measurement of position and momentum for the oscillator (which need not correspond to the best possible joint measurement of number and phase). It is hoped that this will be investigated elsewhere, aided by the two remarkable completeness relations

$$(2\pi)^{-1}\int\int dx\, dp\, D_{x,p}\rho D^\dagger_{x,p}=1\ , \quad (51a)$$

$$(2\pi)^{-1}\sum_{n=-\infty}^{\infty}\int_{-\pi}^{\pi} d\phi\, D_{n,\phi}\rho D^\dagger_{n,\phi}=1\ . \quad (51b)$$

In relations (51) (noted here without proof), ρ is an arbitrary density operator; $D_{x,p}$ is the Glauber displacement operator (10); and $D_{n,\phi}$ is the "number-phase" displacement operator

$$D_{n,\phi}=\begin{cases}(U^\dagger)^n e^{iN\phi}\ , & n\ge 0\\ U^{-n}e^{iN\phi}\ , & n\le 0\ ,\end{cases} \quad (52)$$

where U denotes the Susskind-Glogower phase operator [17]

$$U=\sum_{n\,(\ge 0)}|n\rangle\langle n+1|\ . \quad (53)$$

Finally, the entropic uncertainty relation (34) may be applied to determine an upper bound for the information $I(\Phi|\Gamma(\rho))$ obtained by a measurement of phase, regarding the value of a random phase shift applied to the noisy state $\Gamma(\rho)$ [22]. In particular, using relations (33) and (34) above and Eq. (27b) of [22], one has the inequality

$$I(\Phi|\Gamma(\rho))=\ln 2\pi-S(\Phi|\Gamma(\rho))$$
$$\leq S(N|\Gamma(\rho))-\ln(n_\gamma+1)-n_\gamma\ln(1+n_\gamma^{-1}) \ . \quad (54)$$

It would be of interest to determine the sharpness of this inequality for the coherent phase states [20,22], which maximize the upper bound in (54) for a fixed value of the energy $\mathrm{tr}[H\rho]$.

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