Theory of electron-counting processes

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A theory of electron-counting processes is formulated in terms of the Liouville space method. The developed model is based on the quantum Markov processes developed by Davies and Srinivas. The time evolution of an electron system interacting with an electron counter and the counting probability distribution are investigated. The average value, fluctuation, and correlation function of the electron number registered by the counter are calculated, and the sub-Poissonian distribution and antibunching correlation of the electrons are obtained for an arbitrary noncorrelated initial state of electrons. For a correlated initial state, depending on the initial correlation among the electrons and on the electron counters used in the measurement, the statistics for the electron number registered by the counter is characterized as a sub-Poissonian, Poissonian, or super-Poissonian distribution, and the intensity correlation function exhibits antibunching or bunching correlation. As an example, the state with correlation between the up-spin and down-spin electrons is considered. The electron counting processes under the influence of a chaotic electron source are also considered, and the effect of the chaotic electron source on the counting statistics is investigated.

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I. INTRODUCTION

The quantum counting probability and the intensity correlation function are useful for investigating the nonclassical properties of light beams. This is also true for electron beams. For a light beam, a sub-Poissonian counting probability and antibunching correlation are characteristic of the nonclassical states of lights. Although photon-counting processes have been extensively studied by several authors [1–9], only a few works have addressed electron-counting processes [10–12]. We calculated the electron-counting probability for an arbitrary noncorrelated electron state [13], using a model based on the quantum Markov processes developed by Davies [14–17] and Srinivas and Davies [1,2,18]. Using an axiomatic treatment for the quantum Markov process enables us to investigate the electron-counting measurement process systematically rather than having to use the conventional method [19–21]. It was found that the statistics for the electron number registered by the electron counter is characterized by a sub-Poissonian distribution [12,13]. Furthermore, it is shown that the coincidence counting probability in the two-counter measurement proposed by Hanbury-Brawn and Twiss [22,23] exhibits antibunching correlation of electrons [10–13]. It seems that these characteristics of electrons are due to the Pauli exclusion principle [24].

This paper further develops the theory of the electron-counting processes. We first present a general theory of the electron-counting process in the framework of the Liouville space formulation and the quantum Markov process. Using this result, we investigate the time evolution of an electron system interacting with the electron counter and the statistics for the electron numbers registered by the counters. We take into account a chaotic electron source and consider its influence on the counting statistics. Although we address only an electron system in this paper, all the results obtained here are also valid for an arbitrary fermion system.

This paper is organized as follows. In Sec. II, we briefly summarize the Liouville space formulation [25–32] for a fermion system in a way suitable for investigating electron counting processes. We also review the model of the quantum counting process proposed by Davies and Srinivas [1,2,14–18] within the framework of the Liouville space formulation. In Sec. III, for an arbitrary initial electron state, we investigate the time-evolution property of a system in contact with a counter and the counting statistics of the electron number registered by the counter. As an example, we show that any noncorrelated initial electron state leads to the sub-Poissonian counting probability and antibunching correlation among electrons. In Sec. IV, we calculate the electron-counting probabilities and the intensity correlation functions of correlated electrons. We show that the counting statistics for the electron number registered by the electron counter obeys a sub-Poissonian, Poissonian, or super-Poissonian distribution, depending on the initial correlation of the electrons and on the electron counters used in the measurement. As an example, we consider an initial state in which there exists a correlation between up-spin and down-spin electrons. In Sec. V, we consider the electron-counting processes under the influence of a chaotic electron source which is a model for the thermal or field emission of electrons. The effect of the chaotic electron source on the counting statistics is investigated. A summary is given in Sec. VI.

II. QUANTUM COUNTING PROCESSES IN THE LIOUVILLE SPACE

A. Liouville space formulation

The Liouville space formulation is a powerful method for investigating various kinds of physical phenomena

[25–36]. Although there are several ways to construct the Liouville space formulation, in this paper we use the one based on the tilde conjugation to investigate the electron counting processes. The Liouville space $\mathcal{L}$ can be expressed as a direct product of the two Hilbert spaces $\mathcal{H} \otimes \tilde{\mathcal{H}}$, where $\mathcal{H}$ is the usual Hilbert space in quantum mechanics and $\tilde{\mathcal{H}}$ is the space derived from $\mathcal{H}$ by the tilde conjugation. When A is an arbitrary operator acting on state vectors in $\mathcal{H}$, the tilde conjugation of A gives an operator $\tilde{A}$ acting on state vectors belonging to $\tilde{\mathcal{H}}$. The tilde conjugation is defined by

$$(AB)^{\sim} = \tilde{A}\tilde{B}, \quad (A^\dagger)^{\sim} = (\tilde{A})^\dagger,$$
$$(aA + bB)^{\sim} = a^*\tilde{A} + b^*\tilde{B}, \quad (\tilde{A})^{\sim} = \sigma_F A, \tag{2.1}$$

where A and B are arbitrary operators and a and b are c numbers. In (2.1), $\sigma_F = 1$ for a bosonic operator A and $\sigma_F = -1$ for a fermionic operator A. It is assumed that any bosonic (fermionic) tilde operator commutes (anticommutes) with all bosonic (fermionic) operators without the tilde.

Now we consider a fermionic system in the Liouville space $\mathcal{L}$. We ignore momentum and spin indices, for simplicity, since the generalization to a many-body fermionic system is straightforward. Such a fermionic system is described in the two-dimensional Hilbert space $\mathcal{H}$. The complete orthonormal basis in $\mathcal{H}$ is given by $\{|0\rangle, |1\rangle\}$, where $|1\rangle = c^\dagger|0\rangle$ and $c|0\rangle = 0$. Here c and $c^\dagger$ are fermionic annihilation and creation operators satisfying $[c, c^\dagger]_+ = 1$ and $[c, c]_+ = [c^\dagger, c^\dagger]_+ = 0$. The tilde conjugation then gives a complete orthonormal basis in $\tilde{\mathcal{H}}$: $\{|\tilde{0}\rangle, |\tilde{1}\rangle\}$, where $|\tilde{1}\rangle = \tilde{c}^\dagger|\tilde{0}\rangle$ and $\tilde{c}|\tilde{0}\rangle = 0$. Here $\tilde{c}$ and $\tilde{c}^\dagger$ are the tilde conjugates of c and $c^\dagger$, respectively, which satisfy $[\tilde{c}, \tilde{c}^\dagger]_+ = 1$ and $[\tilde{c}, \tilde{c}]_+ = [\tilde{c}^\dagger, \tilde{c}^\dagger]_+ = 0$. It should be noted from (2.1) that $(\tilde{c})^{\sim} = -c$ and $(\tilde{c}^\dagger)^{\sim} = -c^\dagger$ are satisfied. Thus the Liouville space of a fermionic system is spanned by a complete orthonormal set $\{|m, n\rangle = |m\rangle \otimes |\tilde{n}\rangle \mid m, n = 0, 1\}$. In this case, the state $|i, j\rangle$ is constructed by operating $c^\dagger$ and $\tilde{c}^\dagger$ on the vacuum $|0, 0\rangle$; $|1, 0\rangle = c^\dagger|0, 0\rangle$, $|0, 1\rangle = \tilde{c}^\dagger|0, 0\rangle$ and $|1, 1\rangle = c^\dagger\tilde{c}^\dagger|0, 0\rangle$, where $c|0, 0\rangle = \tilde{c}|0, 0\rangle = 0$.

One of the most important state vectors in the Liouville space $\mathcal{L}$ is

$$|1\rangle\!\rangle = |0, 0\rangle + |1, 1\rangle, \tag{2.2}$$

which satisfies the relations $c|1\rangle\!\rangle = \tilde{c}^\dagger|1\rangle\!\rangle$ and $\tilde{c}|1\rangle\!\rangle = -c^\dagger|1\rangle\!\rangle$. State $|1\rangle\!\rangle$ gives the relations between the tilde and nontilde operators, which are called the thermal state conditions in thermofield dynamics [28–30].

In fermionic Liouville space, an arbitrary state $|\Psi\rangle\!\rangle$ of the system can be expanded as

$$|\Psi\rangle\!\rangle = \sum_{m=0}^{1}\sum_{n=0}^{1} g_{mn}|m, n\rangle, \tag{2.3}$$

where the normalization condition of a state vector is satisfied if $g_{00}+g_{11} = 1$. For an operator that is expressed only in terms of c and $c^\dagger$, we obtain from (2.2) and (2.3)

$$\langle\!\langle 1|A(c, c^\dagger)|\Psi\rangle\!\rangle = \sum_{m=0}^{1}\sum_{n=0}^{1} \langle n|A(c, c^\dagger)|m\rangle g_{mn}. \tag{2.4}$$

It is clear that if g_{mn} is the matrix element of the statistical operator ρ in the Hilbert space $\mathcal{H}$, such that $g_{mn} = \langle m|\rho|n\rangle$, the right hand side of (2.4) is identical to $\mathrm{Tr}(A\rho)$, where Tr is trace operation in $\mathcal{H}$. Thus we find that in the Liouville space $\mathcal{L}$, the average value of $A(c, c^\dagger)$ can be expressed by the matrix element $\langle A(c, c^\dagger)\rangle = \langle\!\langle 1|A(c, c^\dagger)|\Psi\rangle\!\rangle$. For example, when we set $g_{00} = 1 - \bar{n}$, $g_{11} = \bar{n}$, and $g_{01} = g_{10} = 0$, $|\Psi\rangle\!\rangle$ becomes

$$|\Psi\rangle\!\rangle = (1 - \bar{n})|0, 0\rangle + \bar{n}|1, 1\rangle. \tag{2.5}$$

This is the chaotic state (or thermal state) of an electron, in which the entropy of the system is maximized. It is well known [26,29] that the state vector $|m, n\rangle$ in $\mathcal{L}$ corresponds to the operator $|m\rangle\langle n|$ in $\mathcal{H}$. In the density matrix form in $\mathcal{H}$, (2.5) is expressed as

$$\rho = |0\rangle(1 - \bar{n})\langle 0| + |1\rangle\bar{n}\langle 1|. \tag{2.6}$$

In the Liouville space $\mathcal{L}$, the time evolution of a state vector $|\Psi(t)\rangle\!\rangle$ is determined by

$$\frac{\partial}{\partial t}|\Psi(t)\rangle\!\rangle = -i\hat{H}|\Psi(t)\rangle\!\rangle, \quad \hat{H} = H - \tilde{H} + i\hat{\Pi}, \tag{2.7}$$

where H is the Hamiltonian of the system, $\tilde{H}$ is its tilde conjugate, and $\hat{\Pi}$ is a damping operator. This equation corresponds to the Liouville–von Neumann equation (or the quantum master equation) in the usual Hilbert space $\mathcal{H}$. The details of the Liouville space formulation described here are given in Refs. [25–32], and its application to quantum optics is discussed in Refs. [37–40].

B. Quantum counting process

In this section, we describe the quantum counting processes in terms of the Liouville space formulation. The model for the quantum counting processes used here was first developed by Srinivas and Davies [1,2] and is based on the theory of the quantum Markov process [14–18]. According to the theory by Srinivas and Davies, the quantum counting processes in the Liouville space can be formulated as follows.

It is first assumed that there exists an operator $\hat{\mathcal{N}}_m(t)$ acting on a state vector in the Liouville space $\mathcal{L}$. This operator describes a process in which m particles are registered by a counter during measurement time t. For any normalized state vector $|\Psi\rangle\!\rangle$ in $\mathcal{L}$ such that $\langle\!\langle 1|\Psi\rangle\!\rangle = 1$, the operator $\hat{\mathcal{N}}_m(t)$ satisfies the following relations:

$$0 \leq \langle\!\langle 1|\hat{\mathcal{N}}_m(t)|\Psi\rangle\!\rangle \leq 1, \quad \lim_{t\to 0}\hat{\mathcal{N}}_0(t)|\Psi\rangle\!\rangle = |\Psi\rangle\!\rangle, \tag{2.8a}$$

$$\hat{\mathcal{N}}_m(t_1 + t_2) = \sum_{n=0}^{m}\hat{\mathcal{N}}_n(t_1)\hat{\mathcal{N}}_{m-n}(t_2),$$
$$\sum_{m=1}^{\infty}\langle\!\langle 1|\hat{\mathcal{N}}_m(t)|\Psi\rangle\!\rangle < Ct, \tag{2.8b}$$

where C is a finite constant. When an operator $\hat{\mathcal{T}}(t)$ is defined by

$$\hat{\mathcal{T}}(t) = \sum_{m=0}^{\infty} \hat{\mathcal{N}}_m(t), \tag{2.9}$$

the relation $\langle\langle 1|\hat{\mathcal{T}}(t)|\Psi\rangle\rangle = 1$ is established for any normalized state vector $|\Psi\rangle\rangle$ in $\mathcal{L}$.

By using $\hat{\mathcal{N}}_m(t)$, the counting probability distribution $P_m(t)$ that m particles are registered by a counter during measurement time t is given by

$$P_m(t) = \langle\langle 1|\hat{\mathcal{N}}_m(t)|\Psi\rangle\rangle, \tag{2.10}$$

which satisfies $\sum_{m=0}^{\infty} P_m(t) = 1$ and $\lim_{t\to 0} P_0(t) = 1$.

When we define an operator $\hat{S}(t)$ as

$$\hat{S}(t) = \hat{\mathcal{N}}_0(t), \tag{2.11}$$

the state vector given by

$$|\Psi(t)\rangle\rangle = \frac{\hat{S}(t)|\Psi\rangle\rangle}{\langle\langle 1|\hat{S}(t)|\Psi\rangle\rangle} \tag{2.12}$$

represents the state of the system at time t if the counter does not register any particle in the interval $[0, t)$. The state $|\Psi_m(t)\rangle\rangle$ of the system after m electrons have been registered by the counter is expressed in terms of $\hat{\mathcal{N}}_m(t)$ and $P_m(t)$,

$$|\Psi_m(t)\rangle\rangle = \frac{1}{P_m(t)} \hat{\mathcal{N}}_m(t)|\Psi\rangle\rangle. \tag{2.13}$$

When we do not refer to the result indicated by the counter, though counting is performed, the state $|\tilde{\Psi}(t)\rangle\rangle$ of the system is given by

$$|\tilde{\Psi}(t)\rangle\rangle = \sum_{m=0}^{\infty} \hat{\mathcal{N}}_m(t)|\Psi\rangle\rangle = \hat{\mathcal{T}}(t)|\Psi\rangle\rangle. \tag{2.14}$$

Under the above assumptions, we have the following results from the theory of the quantum Markov process [17]. The two sets of operators $\{\hat{\mathcal{T}}(t)\,|\,t \geq 0\}$ and $\{\hat{S}(t)\,|\,t \geq 0\}$ become one-parameter semigroups which satisfy

$$\hat{\mathcal{T}}(t_1)\hat{\mathcal{T}}(t_2) = \hat{\mathcal{T}}(t_1 + t_2), \quad \hat{S}(t_1)\hat{S}(t_2) = \hat{S}(t_1 + t_2), \tag{2.15}$$

and $\hat{\mathcal{T}}(t)$ and $\hat{S}(t)$ are strongly continuous. There exists a bounded positive operator $\hat{J}$ acting on a state vector in the Liouville space $\mathcal{L}$ such that

$$\hat{J}|\Psi\rangle\rangle = \lim_{t\to 0} \frac{1}{t} \hat{\mathcal{N}}_1(t)|\Psi\rangle\rangle. \tag{2.16}$$

Operator $\hat{J}$ determines the change in the state when the counter registers one particle, so $\hat{J}$ characterizes the one-count process. Then the one-count process at time t transforms any state vector $|\Psi(t)\rangle\rangle$ into $|\Psi(t_+)\rangle\rangle$, defined by

$$|\Psi(t_+)\rangle\rangle = \frac{\hat{J}|\Psi(t)\rangle\rangle}{\langle\langle 1|\hat{J}|\Psi(t)\rangle\rangle}. \tag{2.17}$$

Furthermore, operators $\hat{S}(t)$, $\hat{\mathcal{T}}(t)$, and $\hat{J}$ satisfy the following relation:

$$\hat{\mathcal{T}}(t) = \hat{S}(t) + \int_0^t d\tau\, \hat{\mathcal{T}}(t-\tau)\hat{J}\hat{S}(\tau). \tag{2.18}$$

When we denote the generator of $\hat{S}(t)$ as $\hat{Y}$, so that

$$\hat{S}(t) = \exp[t\hat{Y}], \tag{2.19}$$

the following relation between $\hat{J}$ and $\hat{Y}$ is established for any state vector $|\Psi\rangle\rangle$:

$$\langle\langle 1|\hat{Y}|\Psi\rangle\rangle = -\langle\langle 1|\hat{J}|\Psi\rangle\rangle. \tag{2.20}$$

This relation leads the fact that the probability that the counter registers more than one particle during an infinitesimal time can be neglected. Furthermore, it should be noted that $\hat{Y}$ and $\hat{J}$ are invariant under the tilde conjugation $(\hat{Y})^{\sim} = \hat{Y}$ and $(\hat{J})^{\sim} = \hat{J}$.

It is found that operators $\hat{Y}$ and $\hat{J}$ completely determine the quantum counting processes. Operator $\hat{S}(t)$ is given by (2.19), and $\hat{\mathcal{N}}_m$ and $\hat{\mathcal{T}}(t)$ are expressed as

$$\hat{\mathcal{N}}_m(t) = \int_0^t dt_m \int_0^{t_m} dt_{m-1} \cdots \times \int_0^{t_2} dt_1\, \hat{S}(t - t_m)\hat{J} \times \hat{S}(t_m - t_{m-1})\hat{J} \cdots \hat{J}\hat{S}(t_1), \tag{2.21a}$$

$$\hat{\mathcal{T}}(t) = \hat{S}(t) T \exp\left[\int_0^t d\tau\, \hat{S}(-\tau)\hat{J}\hat{S}(\tau)\right] = \exp[t(\hat{Y} + \hat{J})], \tag{2.21b}$$

where T is the time-ordered product and (2.19) is used in the second equality of (2.21b).

It can be seen from (2.10) and (2.21a) that the counting probability distribution $P_m(t)$ that m particles are registered by the counter during measurement time t is given by

$$P_m(t) = \int_0^t dt_m \int_0^{t_m} dt_{m-1} \cdots \int_0^{t_2} dt_1 \langle\langle 1|\hat{S}(t - t_m)\hat{J}\hat{S}(t_m - t_{m-1})\hat{J} \cdots \hat{J}\hat{S}(t_1)|\Psi\rangle\rangle. \tag{2.22}$$

Furthermore, the elementary probability distribution $P_e(t; t_m, t_{m-1}, \ldots, t_1)$ [1] that one particle is registered by the counter at each of times $t_m > t_{m-1} > \cdots > t_1$ ($t > t_m$, $t_1 > 0$) and any particle is not registered in the rest of the interval $[0, t)$ is given by

$$P_e(t; t_m, t_{m-1}, \ldots, t_1)$$
$$= \langle\!\langle 1|\hat{S}(t-t_m)\hat{J}\hat{S}(t_m - t_{m-1})\hat{J}\cdots\hat{J}\hat{S}(t_1)|\Psi\rangle\!\rangle, \tag{2.23}$$

and the coincidence probability distribution $P_c(t; t_m, t_{m-1}, \ldots, t_1)$ [1] that one particle is registered at each of times $t_m > t_{m-1} > \cdots > t_1$, together with the other possible counts in the rest of the interval $[0, t)$, is given by

$$P_c(t; t_m, t_{m-1}, \ldots, t_1)$$
$$= \langle\!\langle 1|\hat{\mathcal{T}}(t-t_m)\hat{J}\hat{\mathcal{T}}(t_m - t_{m-1})\hat{J}\cdots\hat{J}\hat{\mathcal{T}}(t_1)|\Psi\rangle\!\rangle. \tag{2.24}$$

When we define a generating functional by

$$G(t; [\mu(\tau)]) = \langle\!\langle 1|T\exp\left\{\int_0^t d\tau\,[\hat{Y} + \mu(\tau)\hat{J}]\right\}|\Psi\rangle\!\rangle, \tag{2.25}$$

the elementary and coincidence probability distributions are calculated as follows:

$$P_e(t; t_m, t_{m-1}, \ldots, t_1)$$
$$= \left.\frac{\delta^m}{\delta\mu(t_m)\delta\mu(t_{m-1})\cdots\delta\mu(t_1)}G(t; [\mu(\tau)])\right|_{\mu(\tau)=0}, \tag{2.26a}$$

$$P_c(t; t_m, t_{m-1}, \ldots, t_1)$$
$$= \left.\frac{\delta^m}{\delta\mu(t_m)\delta\mu(t_{m-1})\cdots\delta\mu(t_1)}G(t; [\mu(\tau)+1])\right|_{\mu(\tau)=0}. \tag{2.26b}$$

It is easily seen from (2.19) and (2.21a) that the counting probability distribution $P_m(t)$ can also be expressed as

$$P_m(t) = \left.\frac{1}{m!}\frac{\partial^m}{\partial\mu^m}P(t;\mu)\right|_{\mu=0}, \tag{2.27}$$

where we have defined

$$P(t;\mu) = \langle\!\langle 1|\hat{\mathcal{N}}(t;\mu)|\Psi\rangle\!\rangle, \quad \hat{\mathcal{N}}(t;\mu) = \exp[t(\hat{Y} + \mu\hat{J})]. \tag{2.28}$$

It should be noted that $\hat{\mathcal{T}}(t)$ is expressed as $\hat{\mathcal{T}}(t) = \hat{\mathcal{N}}(t;1)$.

The moment of particle number recorded by the counter during time t is calculated by

$$\overline{n^k} = \sum_{n=0}^{\infty} n^k P_n(t). \tag{2.29}$$

Using relation (2.27), we can obtain the mth factorial moment of the particle number registered by the counter during time t as follows:

$$\overline{n(n-1)\cdots(n-m+1)} = \left.\frac{\partial^m}{\partial\mu^m}P(t;\mu+1)\right|_{\mu=0}. \tag{2.30}$$

Of special interest, $\bar{n}$ and $\overline{n^2}$ are given by

$$\bar{n} = \left.\frac{\partial}{\partial\mu}P(t;\mu+1)\right|_{\mu=0}, \quad \overline{n^2} = \bar{n} + \left.\frac{\partial^2}{\partial\mu^2}P(t;\mu+1)\right|_{\mu=0}. \tag{2.31}$$

Thus the quantity $P(t;\mu)$ completely determines the quantum counting statistics.

III. ELECTRON-COUNTING PROCESS

A. General theory of electron counting

Using the general theory of quantum counting processes in the Liouville space $\mathcal{L}$ developed in the preceding section, we will formulate the electron-counting processes. In this case, we can assume that the one-count process $\hat{J}$ for electrons is described by

$$\hat{J} = -\sum_k \lambda_k c_k \tilde{c}_k, \tag{3.1}$$

where $\{\lambda_k\}$ characterizes the measurement performed by the electron counter, k indicates the momentum and spin of an electron, and c_k and $c_k^\dagger$ are the annihilation and creation operators of the electron; $[c_k, c_l^\dagger]_+ = \delta_{kl}$, $[c_k, c_l]_+ = [c_k^\dagger, c_l^\dagger]_+ = 0$, and $\tilde{c}_k$ and $\tilde{c}_k^\dagger$ are the tilde conjugates of c_k and $c_k^\dagger$. It is easily seen that $\hat{J}$ defined by (3.1) transforms a state with n electrons into one with $n-1$ electrons. This is reasonable since an electron is removed from the system when it is registered by the counter. From relation (2.20) and the tilde invariance of $\hat{Y}$, the generator $\hat{Y}$ of $\hat{S}(t)$ should be

$$\hat{Y} = -i(H - \tilde{H}) - \tfrac{1}{2}\sum_k \lambda_k(c_k^\dagger c_k + \tilde{c}_k^\dagger\tilde{c}_k), \tag{3.2}$$

where H is a certain Hermitian operator. If we assume that the time evolution of the system is determined by the Hamiltonian in the absence of the electron-counting measurement ($\lambda_k = 0$), H becomes the Hamiltonian of the electron system and $\tilde{H}$ is its tilde conjugate. Since we consider the free propagation of electrons, we have $H = \sum_k \omega_k c_k^\dagger c_k$. It is easily seen that the operator $H - \tilde{H}$ commutes with $\hat{J}$ and $\hat{Y}$. Thus $H - \tilde{H}$ produces only unimportant phase factors and we can neglect $H - \tilde{H}$ in (3.2). Consequently, the time-evolution generator $\hat{S}(t)$ without counting electrons during time t is given by

$$\hat{S}(t) = \exp\left[-\tfrac{1}{2}t\sum_k \lambda_k(c_k^\dagger c_k + \tilde{c}_k^\dagger \tilde{c}_k)\right]. \tag{3.3}$$

It should be noted that $\hat{S}(t)$ is a nonunitary operator. This nonunitarity is caused by the backaction of the continuous measurement with the electron counter [41]. Furthermore, the operator $\hat{\mathcal{N}}(t;\mu)$ becomes

$$\hat{\mathcal{N}}(t;\mu) = \exp\left[-\tfrac{1}{2}t\sum_k \lambda_k(c_k^\dagger c_k + \tilde{c}_k^\dagger \tilde{c}_k) - \mu t\sum_k \lambda_k c_k \tilde{c}_k\right]. \tag{3.4}$$

Let us define generators of su(2) Lie algebra, $J_+(k)$, $J_-(k)$, and $J_0(k)$, by

$$J_+(k) = c_k\tilde{c}_k, \quad J_-(k) = \tilde{c}_k^\dagger c_k^\dagger,$$
$$J_0(k) = \tfrac{1}{2}(c_k^\dagger c_k + \tilde{c}_k^\dagger \tilde{c}_k - 1). \tag{3.5}$$

The operators $\hat{J}$, $\hat{S}(t)$, and $\hat{\mathcal{N}}(t;\mu)$ can then be expressed as follows:

$$\hat{J} = -\sum_k \lambda_k J_-(k), \tag{3.6a}$$
$$\hat{S}(t) = \exp\left[-t\sum_k \lambda_k[J_0(k) + \tfrac{1}{2}]\right],$$

$$\hat{\mathcal{N}}(t;\mu) = \exp\left[-t\sum_k \lambda_k[J_0(k) + \tfrac{1}{2}] - \mu t\sum_k \lambda_k J_-(k)\right]. \tag{3.6b}$$

By using the Baker-Campbell-Hausdorff formula [42,43], $\hat{\mathcal{N}}(t;\mu)$ can be expressed as

$$\hat{\mathcal{N}}(t;\mu) = \exp\left[-t\sum_k \lambda_k[J_0(k) + \tfrac{1}{2}]\right] \times \exp\left[-\mu\sum_k \xi_k(t)J_-(k)\right], \tag{3.7}$$

with $\xi_k(t) = 1 - e^{-\lambda_k t}$. Therefore, the electron-counting process based on the quantum Markov process in the Liouville space $\mathcal{L}$ can be well described by su(2) Lie algebra. It is of interest to remember that the photon-counting process in the Liouville space can be described by su(1,1) Lie algebra [13,40]

When one electron is registered by the counter, it is found from (2.17) and (3.1) that state $|\Psi\rangle\rangle$ changes into the following form:

$$|\Psi_+\rangle\rangle = -\frac{\sum_k \lambda_k c_k\tilde{c}_k|\Psi\rangle\rangle}{\sum_k \lambda_k\langle N_k\rangle}, \tag{3.8}$$

where $N_k = c_k^\dagger c_k$ and $\langle\,\rangle$ means the average calculated by $\langle\langle 1|\,|\Psi\rangle\rangle$. The change in the average number of electrons in the system caused by the one-count process can then be calculated as

$$\langle N\rangle_+ - \langle N\rangle = \frac{\sum_k\sum_l \lambda_l[\langle N_k N_l\rangle - \langle N_k\rangle\langle N_l\rangle]}{\sum_k \lambda_k\langle N_k\rangle} - 1, \tag{3.9}$$

where $N = \sum_k N_k$ and $\langle\,\rangle_+ = \langle\langle 1|\,|\Psi_+\rangle\rangle$. If parameter λ_k is independent of k, (3.9) reduces to

$$\langle N\rangle_+ - \langle N\rangle = \frac{(\Delta N)^2}{\langle N\rangle} - 1, \tag{3.10}$$

with $(\Delta N)^2 = \langle N^2\rangle - \langle N\rangle^2$. We can then obtain $\langle N\rangle - 1 < \langle N\rangle_+ < \langle N\rangle$ if the statistics of the electron number of the system obeys a sub-Poissonian distribution, $\langle N\rangle = \langle N\rangle_+$ for a Poissonian distribution, and $\langle N\rangle < \langle N\rangle_+$ for a super-Poissonian distribution. For photon counting process, these results were given in Ref. [5].

When the counter registers no electrons during time t, from (2.12) and (3.3) the state $|\Psi(t)\rangle\rangle$ becomes

$$|\Psi(t)\rangle\rangle = \frac{\exp\left[-\tfrac{1}{2}t\sum_k \lambda_k(N_k + \tilde{N}_k)\right]|\Psi\rangle\rangle}{\left\langle \exp\left[-t\sum_k \lambda_k N_k\right]\right\rangle}. \tag{3.11}$$

By using (3.11), the time evolution of the average number of electrons is calculated to be

$$\langle N\rangle_t = \frac{\left\langle N\exp\left[-t\sum_k \lambda_k N_k\right]\right\rangle}{\left\langle \exp\left[-t\sum_k \lambda_k N_k\right]\right\rangle}, \tag{3.12}$$

where we set $\langle\,\rangle_t = \langle\langle 1|\,|\Psi(t)\rangle\rangle$. By differentiating (3.12) with respect to time t, we obtain the following equation of motion:

$$\frac{d}{dt}\langle N\rangle_t = -\sum_k\sum_l \lambda_k[\langle\langle N_k N_l\rangle\rangle_t - \langle\langle N_k\rangle\rangle_t\langle\langle N_l\rangle\rangle_t], \tag{3.13}$$

where we have defined $\langle\langle\,\rangle\rangle_t$ by

$$\langle\langle A\rangle\rangle_t = \frac{\left\langle A\exp\left[-t\sum_k \lambda_k N_k\right]\right\rangle}{\left\langle \exp\left[-t\sum_k \lambda_k N_k\right]\right\rangle}. \tag{3.14}$$

When parameter λ_k is independent of k and we consider a very short time region ($\lambda t \ll 1$), (3.13) can be simplified to

$$\frac{d}{dt}\langle N\rangle = -\lambda\langle N\rangle - \lambda[(\Delta N)^2 - \langle N\rangle]. \qquad (3.15)$$

From this result, we draw the following conclusions. When the fluctuation of the electron number in the system obeys a sub-Poissonian distribution, the decay of the average number of electrons is slower than the exponential decay ($\sim e^{-\lambda t}$). On the other hand, when the fluctuation of the electron number in the system is subject to a super-Poissonian distribution, the decay of the average number of electrons is faster than the exponential decay. If the statistics of the electron number in the system is a Poissonian, the exponential decay is matched.

Next we consider the elementary probability distribution $P_e(t; t_m, t_{m-1}, \ldots, t_1)$ and the coincidence probability distribution $P_c(t; t_m, t_{m-1}, \ldots, t_1)$ defined by (2.23) and (2.24), respectively. Using the semigroup property of $\hat{S}(t)$ and $\hat{\mathcal{T}}(t)$, we can obtain the following expressions:

$$P_e(t; t_m, t_{m-1}, \ldots, t_1) = \langle\langle 1|\hat{S}(t)\hat{J}(t_m)\hat{J}(t_{m-1})\cdots\hat{J}(t_1)|\Psi\rangle\rangle, \qquad (3.16a)$$

$$P_c(t; t_m, t_{m-1}, \ldots, t_1) = \langle\langle 1|\hat{J}(t_m)\hat{J}(t_{m-1})\cdots\hat{J}(t_1)|\Psi\rangle\rangle, \qquad (3.16b)$$

where $t > t_m > t_{m-1} > \cdots > t_1 > 0$ and $\hat{J}(t)$ is defined by

$$\hat{J}(t) = \hat{S}(-t)\hat{J}\hat{S}(t) = \hat{\mathcal{T}}(-t)\hat{J}\hat{\mathcal{T}}(t) = -\sum_k \lambda_k e^{-\lambda_k t} c_k \tilde{c}_k. \qquad (3.17)$$

In deriving (3.16b), we used the relation $\langle\langle 1|\hat{\mathcal{T}}(t) = \langle\langle 1|$, which is proven by using $c_k|1\rangle\rangle = \tilde{c}_k^\dagger|1\rangle\rangle$ and $\tilde{c}_k|1\rangle\rangle = -c_k^\dagger|1\rangle\rangle$. It should be noted that the coincidence probability distribution is independent of t. When we set $K_k = -c_k\tilde{c}_k$ and $N_k = c_k^\dagger c_k$, where $\langle\langle 1|K_k = \langle\langle 1|N_k$ is satisfied, (3.16a) and (3.16b) can be expressed as

$$P_e(t; t_m, t_{m-1}, \ldots, t_1) = \sum_{k_m}\cdots\sum_{k_1} f_m(\mathbf{k}, \mathbf{t})\langle\langle 1|\exp\left[-t\sum_k \lambda_k N_k\right] \times K_{k_m}\cdots K_{k_1}|\Psi\rangle\rangle, \qquad (3.18a)$$

$$P_c(t; t_m, t_{m-1}, \ldots, t_1) = \sum_{k_m}\cdots\sum_{k_1} f_m(\mathbf{k}, \mathbf{t})\langle\langle 1|K_{k_m}\cdots K_{k_1}|\Psi\rangle\rangle, \qquad (3.18b)$$

with $f_m(\mathbf{k}, \mathbf{t}) = \lambda_{k_m}\cdots\lambda_{k_1} e^{-(\lambda_{k_m}t_m + \cdots + \lambda_{k_1}t_1)}$. The generating functional (2.25) for the elementary and coincidence probability distributions is given by

$$G(t, [\mu(\tau)]) = \langle\langle 1|T\exp\left\{-\int_0^t d\tau \sum_k \lambda_k[\tfrac{1}{2}(c_k^\dagger c_k + \tilde{c}_k^\dagger \tilde{c}_k) + \mu(\tau)c_k\tilde{c}_k]\right\} \times|\Psi\rangle\rangle. \qquad (3.19)$$

If parameter λ_k is independent of k, which means that the counter is insensitive to the momentum of electron, (3.16a) and (3.16b) reduce to

$$P_e(t; t_m, t_{m-1}, \ldots, t_1) = \lambda^m e^{-\lambda(t_m + t_{m-1} + \cdots + t_1)}\langle\langle 1|e^{-\lambda t N}K^m|\Psi\rangle\rangle, \qquad (3.20a)$$

$$P_c(t; t_m, t_{m-1}, \ldots, t_1) = \lambda^m e^{-\lambda(t_m + t_{m-1} + \cdots + t_1)}\langle\langle 1|K^m|\Psi\rangle\rangle, \qquad (3.20b)$$

where we have set $K = \sum_k K_k$ and $N = \sum_k N_k$.

From (2.27), (2.28), and (3.4), the electron-counting probability $P_m(t)$ that m electrons are registered by the counter during time t is given by

$$P_m(t) = \frac{1}{m!}\langle\langle 1|\exp\left[-t\sum_k \lambda_k N_k\right]\left[\sum_k \xi_k(t)K_k\right]^m|\Psi\rangle\rangle, \qquad (3.21)$$

with $\xi_k(t) = 1 - e^{-\lambda_k t}$. It should be noted that state $|1\rangle\rangle$ of the system now considered is given by

$$|1\rangle\rangle = \prod_k[|0_k, 0_k\rangle + |1_k, 1_k\rangle], \qquad (3.22)$$

where $|0_k, 0_k\rangle$ is the vacuum state of electrons defined by $c_k|0_k, 0_k\rangle = \tilde{c}_k|0_k, 0_k\rangle = 0$, and $|1_k, 0_k\rangle$, $|0_k, 1_k\rangle$, and $|1_k, 1_k\rangle$ are given by

$$\begin{aligned} |1_k, 0_k\rangle &= c_k^\dagger|0_k, 0_k\rangle, \\ |0_k, 1_k\rangle &= \tilde{c}_k^\dagger|0_k, 0_k\rangle, \\ |1_k, 1_k\rangle &= c_k^\dagger\tilde{c}_k^\dagger|0_k, 0_k\rangle. \end{aligned} \qquad (3.23)$$

Using (3.4) and (3.22), $P(t;\mu)$ can be expressed as

$$P(t;\mu) = \langle\langle 1|\prod_k[1 + (\mu - 1)\xi_k(t)N_k]|\Psi\rangle\rangle \qquad (3.24)$$

$$= \langle\langle 1|\exp\left[(\mu - 1)\sum_k \xi_k(t)K_k\right]|\Psi\rangle\rangle, \qquad (3.25)$$

where we have used the relations $K_k^2 = 0$ and $\langle\langle 1|K_k = \langle\langle 1|N_k$. If λ_k is independent of k, (3.21) simplifies as follows:

$$\begin{aligned} P_m(t) &= \frac{1}{m!}\xi(t)^m\langle\langle 1|e^{-\lambda t N}K^m|\Psi\rangle\rangle \\ &= \frac{1}{m!}\xi(t)^m\langle\langle 1|e^{-\xi(t)K}K^m|\Psi\rangle\rangle. \end{aligned} \qquad (3.26)$$

In the second equality, we have used the relation $\langle\langle 1|e^{-\lambda t N} = \langle\langle 1|e^{-\xi(t)K}$.

By using (2.29), (2.30), and (3.21), the mth factorial moment of the electron number registered by the counter becomes

$$\overline{n(n-1)(n-2)\cdots(n-m+1)}$$
$$= \langle\!\langle 1| \left[\sum_k \xi_k(t) K_k \right]^m |\Psi\rangle\!\rangle. \quad (3.27)$$

Furthermore, using the relation $\langle\!\langle 1|\tilde{c}_k = \langle\!\langle 1|c_k^\dagger$, (3.27) can be expressed as a normal-ordered product

$$\overline{n(n-1)\cdots(n-m+1)}$$
$$= \sum_{k_m} \cdots \sum_{k_1} \xi_{k_m}(t)\cdots\xi_{k_1}(t)\langle c_{k_m}^\dagger \cdots c_{k_1}^\dagger c_{k_1}\cdots c_{k_m}\rangle, \quad (3.28)$$

with $\langle\,\rangle = \langle\!\langle 1|\,|\Psi\rangle\!\rangle$. It is found from these results that information about the state of the electrons in the system, such as $\langle c_{k_m}^\dagger \cdots c_{k_1}^\dagger c_{k_1}\cdots c_{k_m}\rangle$ can be obtained from the statistics of the electron number registered by the counter, such as $\overline{n(n-1)\cdots(n-m+1)}$. Of particular interest, we have

$$\bar{n} = \sum_k \xi_k(t)\langle N_k\rangle, \quad \Delta n^2 = \bar{n} + \bar{n}^2(\Delta - \kappa), \quad (3.29)$$

where $\Delta n^2 = \overline{n^2} - \bar{n}^2$, and Δ and κ are given by

$$\Delta = \frac{1}{\bar{n}^2}\sum_k \sum_{\substack{l \\ (l\neq k)}} \xi_k(t)\xi_l(t)[\langle N_k N_l\rangle - \langle N_k\rangle\langle N_l\rangle],$$
$$\kappa = \frac{1}{\bar{n}^2}\sum_k [\xi_k(t)\langle N_k\rangle]^2. \quad (3.30)$$

According to $\Delta < \kappa$, $\Delta = \kappa$ or $\Delta > \kappa$, we have a sub-Poissonian, Poissonian, or super-Poissonian statistics, respectively, in the electron-counting measurement.

When $\xi_k(t) \ll 1$, (3.25) can be approximated by

$$P(t,\mu) \approx \{1 + \tfrac{1}{2}(\mu-1)^2(\Delta-\kappa)\bar{n}^2\}e^{(\mu-1)\bar{n}}. \quad (3.31)$$

Thus the counting probability distribution $P_m(t)$ becomes

$$P_m(t) \approx \frac{1}{m!}\bar{n}^m e^{-\bar{n}}\{1 + \tfrac{1}{2}(\Delta-\kappa)[m(m-1) - 2m\bar{n} + \bar{n}^2]\}. \quad (3.32)$$

The factor in braces in (3.32) represents the deviation from the Poissonian distribution.

Next we consider electron-counting measurement using two counters, where we can obtain the intensity correlation function which is somewhat similar to the second-order coherence measured in the Hanbury-Brown and Twiss setup [22,23]. In this case, we have to take account of the one-count processes by the two counters, so we assume that the one-count processes are specified by two operations:

$$\hat{J}_1 = -\sum_k \lambda_1(k)c_k\tilde{c}_k, \quad \hat{J}_2 = -\sum_k \lambda_2(k)c_k\tilde{c}_k, \quad (3.33)$$

where $\{\lambda_1(k)\}$ and $\{\lambda_2(k)\}$ characterize the measurement performed by the two electron counters. Since the one-count process is described by $\hat{J} = \hat{J}_1 + \hat{J}_2$, the nonunitary time evolution with no-count during time t is obtained from (2.20) as

$$\hat{S}(t) = \exp\left\{-\tfrac{1}{2}t\sum_k [\lambda_1(k)+\lambda_2(k)](c_k^\dagger c_k + \tilde{c}_k^\dagger \tilde{c}_k)\right\}. \quad (3.34)$$

Using the same method as that used for the one-counter measurement, we obtain the counting probability distribution $P_{m_1m_2}(t)$ that m_1 electrons are registered by one-counter and m_2 electrons by the other counter during measurement time t:

$$P_{m_1m_2} = \frac{1}{m_1!m_2!}\frac{\partial^{m_1+m_2}}{\partial\mu_1^{m_1}\partial\mu_2^{m_2}}P(t;\mu_1,\mu_2)\bigg|_{\mu_1=\mu_2=0}, \quad (3.35)$$

where $P(t;\mu_1,\mu_2)$ is given by

$$P(t;\mu_1,\mu_2) = \langle\!\langle 1|\hat{\mathcal{N}}(t;\mu_1,\mu_2)|\Psi\rangle\!\rangle, \quad (3.36a)$$

$$\hat{\mathcal{N}}(t;\mu_1,\mu_2) = \exp\left[-\tfrac{1}{2}t\sum_k \bar{\lambda}(k)(c_k^\dagger c_k + \tilde{c}_k^\dagger\tilde{c}_k)\right] \times \exp\left[-\sum_k \bar{\mu}(k)\bar{\xi}_k(t)c_k\tilde{c}_k\right]. \quad (3.36b)$$

Here $\bar{\lambda}(k)$, $\bar{\xi}(k)$, and $\bar{\mu}(k)$ are defined by

$$\bar{\lambda}(k) = \lambda_1(k) + \lambda_2(k),$$
$$\bar{\xi}_k(t) = 1 - \exp[-\bar{\lambda}(k)t], \quad (3.37)$$
$$\bar{\mu}(k) = \frac{\mu_1\lambda_1(k) + \mu_2\lambda_2(k)}{\lambda_1(k)+\lambda_2(k)}.$$

It is thus seen from (3.35) and (3.36) that the counting probability distribution $P_{m_1m_2}(t)$ becomes

$$P_{m_1m_2}(t) = \frac{1}{m_1!m_2!}\langle\!\langle 1|\exp\left[-t\sum_k \bar{\lambda}(k)N_k\right] \times \left(\sum_k \bar{\lambda}_1(k)\bar{\xi}_k(t)K_k\right)^{m_1} \times \left(\sum_k \bar{\lambda}_2(k)\bar{\xi}_k(t)K_k\right)^{m_2} |\Psi\rangle\!\rangle, \quad (3.38)$$

where we set $\bar{\lambda}_j(k) = \lambda_j(k)/\bar{\lambda}(k)$.

Since the moment $\overline{n_1^{m_1}n_2^{m_2}}$ of the electron numbers registered by the two counters is calculated by

$$\overline{n_1^{m_1}n_2^{m_2}} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} n_1^{m_1}n_2^{m_2}P_{n_1n_2}(t), \quad (3.39)$$

and the counting probability distribution $P_{m_1m_2}(t)$ is given by (3.35), we obtain the factorial moment as follows:

$$\overline{n_1(n_1-1)\cdots(n_1-m_1+1)n_2(n_2-1)\cdots(n_2-m_2+1)}$$
$$= \frac{\partial^{m_1+m_2}}{\partial\mu_1^{m_1}\partial\mu_2^{m_2}}P(t;\mu_1+1,\mu_2+1)\bigg|_{\mu_1=\mu_2=0}. \quad (3.40)$$

Substituting (3.36) into (3.40), we can obtain the following expression:

$$\overline{n_1(n_1-1)\cdots(n_1-m_1+1)n_2(n_2-1)\cdots(n_2-m_2+1)}$$

$$\begin{aligned} &= \langle\!\langle 1|\left(\sum_k \bar\lambda_1(k)\bar\xi_k(t)K_k\right)^{m_1}\left(\sum_k \bar\lambda_2(k)\bar\xi_k(t)K_k\right)^{m_2}|\Psi\rangle\!\rangle \\ &= \sum_{k_{m_1}}\cdots\sum_{k_1}\sum_{l_{m_2}}\cdots\sum_{l_1}\bar\lambda_1(k_{m_1})\cdots\bar\lambda_1(k_1)\bar\lambda_2(l_{m_2})\cdots\bar\lambda_2(l_1) \\ &\quad\times\bar\xi_{k_{m_1}}(t)\cdots\bar\xi_{k_1}(t)\bar\xi_{l_{m_2}}(t)\cdots\bar\xi_{l_1}(t)\langle c^\dagger_{l_{m_2}}\cdots c^\dagger_{l_1}c^\dagger_{k_{m_1}}\cdots c^\dagger_{k_1}c_{k_1}\cdots c_{k_{m_1}}c_{l_1}\cdots c_{l_{m_2}}\rangle. \end{aligned} \tag{3.41}$$

The average number, second-order moment, and cross correlation function of the electron numbers registered by the two counters are given by

$$\bar n_j = \sum_k \bar\lambda_j(k)\bar\xi_k(t)\langle c^\dagger_k c_k\rangle, \tag{3.42a}$$

$$\overline{n_j^2} = \bar n_j + \sum_k\sum_l \bar\lambda_j(k)\bar\lambda_j(l)\bar\xi_k(t)\bar\xi_l(t)\langle c^\dagger_k c^\dagger_l c_l c_k\rangle,$$

$$\overline{n_1n_2} = \sum_k\sum_l \bar\lambda_1(k)\bar\lambda_2(l)\bar\xi_k(t)\bar\xi_l(t)\langle c^\dagger_k c^\dagger_l c_l c_k\rangle, \tag{3.42b}$$

with $j=1,2$. When we define quantities κ_{ij} and Δ_{ij} by

$$\kappa_{ij} = \frac{1}{\bar n_i\bar n_j}\sum_k \bar\lambda_i(k)\bar\lambda_j(k)[\bar\xi_k(t)\langle N_k\rangle]^2, \tag{3.43a}$$

$$\Delta_{ij} = \frac{1}{\bar n_i\bar n_j}\sum_k \sum_{\substack{l\\(l\neq k)}} \bar\lambda_i(k)\bar\lambda_j(l)\bar\xi_k(t)\bar\xi_l(t) \times[\langle N_kN_l\rangle - \langle N_k\rangle\langle N_l\rangle], \tag{3.43b}$$

we obtain the following results:

$$\begin{aligned} \Delta n_j^2 - \bar n_j &= \bar n_j^2[\Delta_{jj}-\kappa_{jj}], \\ \overline{n_1n_2} - \bar n_1\bar n_2 &= \bar n_1\bar n_2[\Delta_{12}-\kappa_{12}]. \end{aligned} \tag{3.44}$$

Thus we find that according to $\Delta_{jj} < \kappa_{jj}$, $\Delta_{jj} = \kappa_{jj}$, or $\Delta_{jj} > \kappa_{jj}$, the statistics of the electron number registered by each counter is subject to a sub-Poissonian, Poissonian, or super-Poissonian distribution. We also see that the correlation becomes antibunching for $\Delta_{12} < \kappa_{12}$ and bunching for $\Delta_{12} > \kappa_{12}$.

When $\bar\xi_k(t) \ll 1$, $P(t;\mu_1,\mu_2)$ can be approximated by

$$\begin{aligned} P(t;\mu_1,\mu_2) \approx \{&1+\tfrac{1}{2}(\mu_1-1)^2(\Delta_{11}-\kappa_{11})\bar n_1^2 \\ &+(\mu_1-1)(\mu_2-1)(\Delta_{12}-\kappa_{12})\bar n_1\bar n_2 \\ &+\tfrac{1}{2}(\mu_2-1)^2(\Delta_{22}-\kappa_{22})\bar n_2^2\} \\ &\times e^{(\mu_1-1)\bar n_1+(\mu_2-1)\bar n_2}. \end{aligned} \tag{3.45}$$

Thus the counting probability distribution $P_{m_1m_2}(t)$ becomes

$$\begin{aligned} P_{m_1m_2}(t) = \frac{\bar n_1^{m_1}\bar n_2^{m_2}}{m_1!m_2!}e^{-\bar n_1-\bar n_2}\{&1+\tfrac{1}{2}(\Delta_{11}-\kappa_{11})[m_1(m_1-1)-2m_1\bar n_1+\bar n_1^2]+(\Delta_{12}-\kappa_{12})(m_1-\bar n_1)(m_2-\bar n_2) \\ &+\tfrac{1}{2}(\Delta_{22}-\kappa_{12})[m_2(m_2-1)-2m_2\bar n_2+\bar n_2^2]\}, \end{aligned} \tag{3.46}$$

where Δ_{ij} and κ_{ij} are given by (3.43a) and (3.43b). This shows that the counting probability distribution deviates slightly from a Poissonian distribution. The results obtained in this section are summarized in Table I.

B. Electron counting for a noncorrelated initial state

Thus far, we have not assumed the initial state of electrons $|\Psi\rangle\!\rangle$. Consequently, the results obtained in Sec. III A are valid for an arbitrary initial state of electrons. Let us now assume the initial state of electrons, so we can perform more explicit calculations. In this section, we consider an arbitrary noncorrelated initial state of electrons. In the Liouville space $\mathcal{L}$, such a state is expressed as

$$\begin{aligned} |\Psi\rangle\!\rangle = \prod_k[&g_{00}(k)|0_k,0_k\rangle + g_{10}(k)|1_k,0_k\rangle \\ &+g_{01}(k)|0_k,1_k\rangle + g_{11}(k)|1_k,1_k\rangle], \end{aligned} \tag{3.47}$$

where the normalization condition is given by $\langle\!\langle 1|\Psi\rangle\!\rangle = g_{00}(k)+g_{11}(k) = 1$. For example, when we set $g_{00}(k) = 1-\bar n_k$ and $g_{11}(k) = \bar n_k$, (3.47) becomes the chaotic state (or thermal state) of electrons, where $\bar n_k$ is the fermion

TABLE I. List of the results of the electron-counting measurements.

	One-counter measurement	Two-counter measurement
Counting probability	Eqs. (3.21) and (3.32)	Eqs. (3.38) and (3.46)
Fluctuation and correlation	Eqs. (3.29) and (3.30)	Eqs. (3.43) and (3.44)

distribution function. When we assume that $g_{00}(k) = 0$ and $g_{11}(k) = 1$ for $k \in (k_{\min}, k_{\max})$ and that $g_{00}(k) = 1$ and $g_{11}(k) = 0$ for $k \notin (k_{\min}, k_{\max})$, where $k_{\min}$ and $k_{\max}$ are certain constants, then (3.47) is an eigenstate of the electron number.

When the counter does not register any electron in the interval $[0, t)$, the state of the system at time t is obtained from (3.11),

$$|\Psi(t)\rangle\rangle = \prod_k \frac{g_{00}(k)|0_k, 0_k\rangle + e^{-\frac{1}{2}\lambda_k t}[g_{10}(k)|1_k, 0_k\rangle + g_{01}(k)|0_k, 1_k\rangle] + e^{-\lambda_k t} g_{11}(k)|1_k, 1_k\rangle}{1 - \xi_k(t) g_{11}(k)}, \tag{3.48}$$

where we have used the relation $g_{00}(k) + g_{11}(k) = 1$. In the limit as $t \to \infty$, (3.48) reduces to the vacuum state of electrons $|\Psi(\infty)\rangle\rangle = \prod_k |0_k, 0_k\rangle$. This means that there are no electrons in the system if the counter does not register any electrons during an arbitrarily long measurement time. By using (3.48), the average value of the electron number in the system is given by

$$\langle N \rangle_t = \sum_k \frac{e^{-\lambda_k t} g_{11}(k)}{1 - \xi_k(t) g_{11}(k)}. \tag{3.49}$$

If the initial state $|\Psi\rangle\rangle$ is the electron-number eigenstate, we have $\langle N \rangle_t = \langle N \rangle_{t=0}$. This means that the electron number remains constant if the counter does not register any electron. In general, however, the average number of electron numbers decreases in time, even if no electron is registered by the counter. This decrease is caused by the change in our knowledge of the electron system, which is the result of continuous measurement by means of the electron counter.

If one electron is registered by the counter at time t, we obtain the change in the state from (2.17), (3.1), and (3.48),

$$\begin{aligned} |\Psi(t+0)\rangle\rangle = {} & \left[\sum_p \lambda_p \frac{e^{-\lambda_p t} g_{11}(p)}{1 - \xi_p(t) g_{11}(p)}\right]^{-1} \sum_k \lambda_k \frac{e^{-\lambda_k t} g_{11}(k)}{1 - \xi_k(t) g_{11}(k)} |0_k, 0_k\rangle \\ & \times \prod_{\substack{l \\ (l \neq k)}} \frac{g_{00}(l)|0_l, 0_l\rangle + e^{-\frac{1}{2}\lambda_l t}[g_{10}(l)|1_l, 0_l\rangle + g_{01}(l)|0_l, 1_l\rangle] + e^{-\lambda_l t} g_{11}(l)|1_l, 1_l\rangle}{1 - \xi_l(t) g_{11}(l)}. \end{aligned} \tag{3.50}$$

Using (3.49) and (3.50), we can get the change in the average number of electrons in the system due to the one-count process at time t,

$$\langle N \rangle_{t+0} - \langle N \rangle_t = -\frac{\displaystyle\sum_k \lambda_k \left[\frac{e^{-\lambda_k t} g_{11}(k)}{1 - \xi_k(t) g_{11}(k)}\right]^2}{\displaystyle\sum_k \lambda_k \frac{e^{-\lambda_k t} g_{11}(k)}{1 - \xi_k(t) g_{11}(k)}}. \tag{3.51}$$

It is easily seen that the relation $-1 \le \langle N \rangle_{t+0} - \langle N \rangle_t \le 0$ is satisfied.

The counting probability distribution $P_m(t)$ that m electrons are registered by the counter during measurement time t is calculated by (2.27) and (3.25). For the noncorrelated state given by (3.47), $P(t; \mu) = \langle\langle 1|\hat{\mathcal{N}}(t; \mu)|\Psi\rangle\rangle$ becomes

$$P(t; \mu) = \prod_k \left[1 + \mu \frac{\xi_k(t) g_{11}(k)}{1 - \xi_k(k) g_{11}(k)}\right] \langle\langle 1|\hat{S}(t)|\Psi\rangle\rangle, \tag{3.52}$$

where $\langle\langle 1|\hat{S}(t)|\Psi\rangle\rangle = \prod_k [1 - \xi_k(t) g_{11}(k)]$. We thus obtain the counting probability $P_m(t)$ from (2.27),

$$\begin{aligned} P_m(t) = {} & \frac{1}{m!} \sum_{k_1} \cdots \sum_{k_m} \delta(k_1, \ldots, k_m) \\ & \times \prod_{j=1}^m \frac{\xi_{k_j}(t) g_{11}(k_j)}{1 - \xi_{k_j}(t) g_{11}(k_j)} \\ & \times \langle\langle 1|\hat{S}(t)|\Psi\rangle\rangle, \end{aligned} \tag{3.53}$$

where $\delta(k_1, \ldots, k_m)$ is defined by $\delta(k) = 1$ for $m = 1$ and

$$\delta(k_1, \ldots, k_m) = \prod_{\langle i,j;m \rangle} (1 - \delta_{k_i k_j}) \qquad \text{for } m \ge 2, \tag{3.54}$$

and $\langle i, j; m \rangle$ indicates that all possible pairs (k_i, k_j) are taken from $(k_1, \ldots, k_m)$, such as

$$\delta(k_1, k_2, k_3) = (1 - \delta_{k_1 k_2})(1 - \delta_{k_1 k_3})(1 - \delta_{k_2 k_3}). \tag{3.55}$$

Next, we consider the counting statistics for the electron system. Since we have $\langle N_k N_l \rangle = \langle N_k \rangle \langle N_l \rangle$ $(k \neq l)$ for the noncorrelated initial state of electrons, it is found from (3.29) that the average and fluctuation of electron number registered by the counter are given by

$$\bar{n} = \sum_k \xi_k(t) g_{11}(k), \quad \Delta n^2 = \bar{n} - \sum_k [\xi_k(t) g_{11}(k)]^2. \tag{3.56}$$

This result shows that the statistics of the electron number registered by the counter is characterized by the sub-Poissonian distribution ($\Delta n^2 < \bar{n}$). It is interesting to note that an arbitrary noncorrelated initial state always leads to the sub-Poissonian counting probability, in contrast with the case of photon-counting processes. In the photon-counting measurement, a coherent state leads to the Poissonian distribution and a thermal state to the super-Poissonian distribution. In these states, we have $\langle n_k n_l \rangle = \langle n_k \rangle \langle n_l \rangle$ ($k \neq l$) for free photons, where n_k is the photon-number operator of the kth mode.

When $\xi_k(k) \ll 1$ or $\lambda_k t \ll 1$, we can obtain the approximate counting probability distribution $P_m(t)$ by setting $\Delta = 0$ in (3.32),

$$P_m(t) = \frac{1}{m!}\bar{n}^m e^{-\bar{n}}\{1 - \tfrac{1}{2}\kappa[\bar{n}^2 - 2\bar{n}m + m(m-1)]\}, \tag{3.57}$$

where κ is given by $\kappa = (\bar{n} - \Delta n^2)/\bar{n}^2 > 0$. This approximate counting probability distribution was first derived by Saito *et al.* [12], who used the Mandel formula in quantum optics to derive (3.57) and assumed the chaotic state of electrons as an initial state.

In two-counter measurement, for the noncorrelated initial state of electrons (3.47), we can obtain the following results by substituting $\Delta_{ij} = 0$ into (3.45):

$$Q_j = \frac{\Delta n_j^2 - \bar{n}_j}{\bar{n}_j} = -\kappa_{jj}\bar{n}_j, \quad g_{12}^{(2)} = \frac{\overline{n_1 n_2}}{\bar{n}_1 \bar{n}_2} = 1 - \kappa_{12}, \tag{3.58}$$

where Q_j is the Mandel Q factor and $g_{12}^{(2)}$ is the normalized second-order correlation function. It is found from (3.58) that the statistics of the electron number registered by each counter obeys the sub-Poissonian distribution. Furthermore, it is seen that the electron number correlation $\overline{n_1 n_2}$ between the two counters is smaller than the noncorrelated value $\bar{n}_1 \bar{n}_2$. This effect is the antibunching correlation of electron numbers in coincidence counting measurement.

When $\bar{\xi}_k(t) \ll 1$, since $\Delta_{ij} = 0$ for the noncorrelated initial state, the probability distribution $P_{m_1 m_2}(t)$ becomes

$$\begin{aligned} P_{m_1 m_2}(t) = \frac{\bar{n}_1^{m_1} \bar{n}_2^{m_2}}{m_1! m_2!} e^{-\bar{n}_1 - \bar{n}_2} \{1 - \tfrac{1}{2}\kappa_{11}[\bar{n}_1^2 - 2\bar{n}_1 m_2 + m_1(m_1 - 1)] \\ -\kappa_{12}(\bar{n}_1 - m_1)(\bar{n}_2 - m_2) - \tfrac{1}{2}\kappa_{22}[\bar{n}_2^2 - 2\bar{n}_2 m_2 + m_2(m_2 - 1)]\}. \end{aligned} \tag{3.59}$$

For the chaotic initial state of electrons, this probability distribution was first derived by Saito *et al.* [12].

IV. SELECTIVE COUNTING PROCESSES FOR ELECTRONS

A. Selective electron counting

In this section, we consider selective counting processes for electrons. By selective electron counting process we mean that the electron counter used in the measurement can register only a certain kind of electron. In the following, we investigate the selective counting processes for electrons with up spin or with down spin. Such counting processes can be achieved using the Mott detectors [44]. For simplicity, we refer to electrons with up spin as A electrons and to electrons with down spin as B electrons, and we set $(a_k, a_k^\dagger) = (c_{k\uparrow}, c_{k\uparrow}^\dagger)$ and $(b_k, b_k^\dagger) = (c_{k\downarrow}, c_{k\downarrow}^\dagger)$. Index k represents a momentum of electron. In this counting process, we have to consider the one-count process for both A and B electrons. According to the general theory of quantum counting processes based on the quantum Markov theory explained in Sec. II, it is reasonable to assume that the one-count processes of A and B electrons can be described respectively by

$$\hat{J}_A = -\sum_k \lambda(k, A) a_k \tilde{a}_k, \quad \hat{J}_B = -\sum_k \lambda(k, B) b_k \tilde{b}_k, \tag{4.1}$$

where $\{\lambda(k, A)\}$ and $\{\lambda(k, B)\}$ characterize the measurement performed by the electron counters for A and B electrons. Furthermore, corresponding to $\hat{J}_A$ and $\hat{J}_B$, the nonunitary time-evolution operators without counting electrons during time t become

$$\hat{S}_A(t) = \exp\left[-\tfrac{1}{2}\sum_k \lambda(k, A)(a_k^\dagger a_k + \tilde{a}_k^\dagger \tilde{a}_k)\right], \tag{4.2a}$$

$$\hat{S}_B(t) = \exp\left[-\tfrac{1}{2}\sum_k \lambda(k, B)(b_k^\dagger b_k + \tilde{b}_k^\dagger \tilde{b}_k)\right]. \tag{4.2b}$$

We have ignored here the free Hamiltonian of electrons.

Now we will consider the change in the state of the system when one A electron is registered by the counter at time t. After one A electron is registered, the state of the system can be expressed as

$$|\Psi(t+0)\rangle\rangle_A = \frac{\hat{J}_A |\Psi(t)\rangle\rangle}{\langle\langle 1|\hat{J}_A|\Psi(t)\rangle\rangle} = -\frac{\sum_k \lambda(k, A) a_k \tilde{a}_k |\Psi(t)\rangle\rangle}{\sum_k \lambda(k, A)\langle a_k^\dagger a_k \rangle}, \tag{4.3}$$

where $|\Psi(t)\rangle\rangle$ is the state of the system before the A electron is registered and we set $\langle\ \rangle = \langle\langle 1|\,|\Psi(t)\rangle\rangle$. Using this state, we can calculate the changes in the average number of A and B electrons due to the one-count process for the A electron,

$$\langle N_A\rangle_A - \langle N_A\rangle = \frac{\sum_k \sum_l \lambda(l,A)[\langle A_k A_l\rangle - \langle A_k\rangle\langle A_l\rangle]}{\sum_k \lambda(k,A)\langle A_k\rangle} - 1, \tag{4.4a}$$

$$\langle N_B\rangle_A - \langle N_B\rangle = \frac{\sum_k \sum_l \lambda(l,A)[\langle B_k A_l\rangle - \langle B_k\rangle\langle A_l\rangle]}{\sum_k \lambda(k,A)\langle A_k\rangle}, \tag{4.4b}$$

where we have defined $\langle\,\rangle_A = \langle\langle 1|\,|\Psi(t+0)\rangle\rangle_A$, and we set

$$A_k = a_k^\dagger a_k, \quad N_A = \sum_k A_k, \quad B_k = b_k^\dagger b_k, \quad N_B = \sum_k B_k. \tag{4.5}$$

These notations are used frequently in this section. It should be noted that the average number of B electrons does change, though only A electrons are registered. This change is caused through the correlation between A and B electrons. Indeed, if there is no correlation, so that $\langle A_k B_l\rangle = \langle A_k\rangle\langle B_l\rangle$, we have $\langle N_B\rangle_A = \langle N_B\rangle$. When $\lambda(k,A)$ is independent of k, (4.4a) and (4.4b) can be simplified to

$$\langle N_A\rangle_A - \langle N_A\rangle = \frac{\langle N_A^2\rangle - \langle N_A\rangle^2}{\langle N_A\rangle} - 1, \tag{4.6a}$$

$$\langle N_B\rangle_A - \langle N_B\rangle = \frac{\langle N_A N_B\rangle - \langle N_A\rangle\langle N_B\rangle}{\langle N_A\rangle}. \tag{4.6b}$$

When we perform the two-counter measurement described above, we have several measurement setups. To describe these setups, we introduce three electron counters: the A counter, the B counter, and the AB counter, where the A counter registers only A electrons (up-spin electrons), the B counter registers only B electrons (down-spin electrons), and the AB counter registers both kinds of electrons. The AB counter is insensitive to the spin of the electron and so it does not distinguish between A and B electrons. The four measurement setups are as follows. First, the two A counters are used to observe the intensity correlation function for the A electrons only. Next, the two B counters are used to observe the intensity correlation function for the B electrons only. Then, one A counter and one B counter are used to determine the intensity correlation function between A and B electrons. Finally, the two AB counters are used to measure the correlation of both kinds of electrons as a whole. Since the first and second measurement setups give equivalent results, we will discuss the electron counting processes for the three differing cases in the following subsections. It will be seen that another measurement setup can be handled in the same way developed in this section.

B. Counting process for A electrons

Consider the electron-counting process for two-counter measurement, where two A counters are used, so that only A electrons are registered. The one-count processes of the two A counters are expressed by

$$\hat{J}_1 = -\sum_k \lambda_1(k,A) a_k \tilde{a}_k, \quad \hat{J}_2 = -\sum_k \lambda_2(k,A) a_k \tilde{a}_k, \tag{4.7}$$

where $\{\lambda_1(k,A)\}$ and $\{\lambda_2(k,A)\}$ are parameters of the two A counters. The nonunitary time evolution with no-count during measurement time t can then be obtained from (2.20),

$$\hat{S}(t) = \exp\left\{-\tfrac{1}{2}t\sum_k [\lambda_1(k,A) + \lambda_2(k,A)](a_k^\dagger a_k + \tilde{a}^\dagger \tilde{a}_k)\right\}, \tag{4.8}$$

where the free Hamiltonian of electrons has been neglected. Equations (4.7) and (4.8) completely describe this electron-counting process.

Using the same procedure as that used to derive (3.35) and (3.36), we can obtain the counting probability distribution $P_{m_1 m_2}(t)$ that the A counter registers m_1 A electrons and the other A counter registers m_2 A electrons during measurement time t. The result is given by

$$P_{m_1 m_2}(t) = \frac{1}{m_1! m_2!}\frac{\partial^{m_1+m_2}}{\partial \mu_1^{m_1}\partial\mu_2^{m_2}} \times \langle\langle 1|\hat{\mathcal{N}}_A(t;\mu_1,\mu_2)|\Psi\rangle\rangle\Big|_{\mu_1=\mu_2=0}, \tag{4.9a}$$

$$\hat{\mathcal{N}}_A(t;\mu_1,\mu_2) = \exp\left[-\tfrac{1}{2}t\sum_k \bar{\lambda}(k,A)(a_k^\dagger a_k + \tilde{a}_k^\dagger \tilde{a}_k)\right] \times \exp\left[-\sum_k \bar{\mu}(k,A)\bar{\xi}_k(t,A) a_k \tilde{a}_k\right], \tag{4.9b}$$

where $\bar{\lambda}(k,A)$, $\bar{\mu}(k,A)$, and $\bar{\xi}_k(t,A)$ are defined by

$$\bar{\lambda}(k,A) = \lambda_1(k,A) + \lambda_2(k,A),$$
$$\bar{\xi}_k(t,A) = 1 - \exp[-\bar{\lambda}(t,A)t], \tag{4.10a}$$

$$\bar{\mu}(k,A) = \frac{\mu_1\lambda_1(k,A) + \mu_2\lambda_2(k,A)}{\lambda_1(k,A) + \lambda_2(k,A)}. \tag{4.10b}$$

In (4.9a), the state $\langle\langle 1|$ is given by

$$\langle\langle 1| = \prod_k \{[\langle A_k;0,0| + \langle A_k;1,1|] \otimes [\langle B_k;0,0| + \langle B_k;1,1|]\}, \tag{4.11}$$

where $|1,1;A_k\rangle = a_k^\dagger \tilde{a}_k^\dagger |0,0;A_k\rangle$, $|1,1;B_k\rangle = b_k^\dagger \tilde{b}_k^\dagger |0,0;B_k\rangle$, $|0,0;A_k\rangle$, and $|0,0;B_k\rangle$ are the vacuum states of the A and B electrons with momentum k.

Using (3.40) and (4.9), we can calculate the average values, fluctuations, and cross correlation function of the electron numbers registered by the counters. Thus $\bar{n}_j$,

Δn_j^2 ($j=1,2$), and $\overline{n_1 n_2}$ are given by

$$\bar{n}_j = \sum_k \bar{\lambda}_j(k,A)\bar{\xi}_k(t)\langle A_k\rangle, \quad \Delta n_j^2 = \bar{n}_j + \bar{n}_j^2[\Delta_{jj} - \kappa_{jj}] \quad \text{for } j=1,2 \tag{4.12a}$$

$$\overline{n_1 n_2} = \bar{n}_1\bar{n}_2 + \bar{n}_1\bar{n}_2[\Delta_{12} - \kappa_{12}], \tag{4.12b}$$

where Δ_{ij} and κ_{ij} are defined by

$$\Delta_{ij} = \frac{1}{\bar{n}_i\bar{n}_j}\sum_k \sum_{\substack{l\\(l\neq k)}} \bar{\lambda}_i(k,A)\bar{\lambda}_j(l,A)\bar{\xi}_k(t,A)\bar{\xi}_l(t,A) \times [\langle A_k A_l\rangle - \langle A_k\rangle\langle A_l\rangle], \tag{4.13a}$$

$$\kappa_{ij} = \frac{1}{\bar{n}_i\bar{n}_j}\sum_k \bar{\lambda}_i(k,A)\bar{\lambda}_j(k,A)[\bar{\xi}_k(t,A)\langle A_k\rangle]^2, \tag{4.13b}$$

with $\bar{\lambda}_j(k,A) = \lambda_j(k,A)/[\lambda_1(k,A)+\lambda_2(k,A)]$. We find from (4.12a) that according to $\Delta_{jj} > \kappa_{jj}$, $\Delta_{jj} = \kappa_{jj}$, or $\Delta_{jj} < \kappa_{jj}$, the statistics of the electron number registered by each A counter is characterized by a super-Poissonian, Poissonian, or sub-Poissonian distribution, respectively. Furthermore, we can see from (4.12b) that according to $\Delta_{12} > \kappa_{12}$, $\Delta_{12} = \kappa_{12}$, or $\Delta_{12} < \kappa_{12}$, the electron number correlation $\overline{n_1 n_2}$ becomes bunching, independent, or antibunching, respectively. When $\bar{\xi}_k(t,A)$ is sufficiently small, the counting probability distribution $P_{m_1 m_2}(t)$ becomes

$$P_{m_1m_2}(t) = \frac{\bar{n}_1^{m_1}\bar{n}_2^{m_2}}{m_1!m_2!}e^{-\bar{n}_1-\bar{n}_2}\{1+\tfrac{1}{2}(\Delta_{11}-\kappa_{11})[\bar{n}_1^2 - 2\bar{n}_1 m_1 + m_1(m_1-1)] + (\Delta_{12}-\kappa_{12})(\bar{n}_1 - m_1)(\bar{n}_2 - m_2) + \tfrac{1}{2}(\Delta_{22}-\kappa_{22})[\bar{n}_2^2 - 2\bar{n}_2 m_2 + m_2(m_2-1)]\}. \tag{4.14}$$

C. Selective counting process for A and B electrons

Now we consider the electron-counting process with two counters, A counter and B counter, to investigate the correlation between A and B electrons. In this case, the one-count processes by the A and B counter are described, respectively, as

$$\hat{J}_1 = -\sum_k \lambda(k,A)a_k\tilde{a}_k, \quad \hat{J}_2 = -\sum_k \lambda(k,B)b_k\tilde{b}_k, \tag{4.15}$$

where we have assumed that counter 1 is the A counter and counter 2 is the B counter. The nonunitary time evolution with no counts during time t is described by

$$\hat{S}(t) = \exp\left[-\tfrac{1}{2}t\sum_k \lambda(k,A)(a_k^\dagger a_k + \tilde{a}_k^\dagger\tilde{a}_k) - \tfrac{1}{2}t\sum_k \lambda(k,B)(b_k^\dagger b_k + \tilde{b}_k^\dagger\tilde{b}_k)\right]. \tag{4.16}$$

The selective counting process for A and B electrons can therefore be completely determined by (4.15) and (4.16).

We find from (4.15) and (4.16) that the counting probability distribution $P_{m_1m_2}(t)$ that counter 1 registers m_1 A electrons and counter 2 registers m_2 B electrons during measurement time t is given by

$$P_{m_1m_2}(t) = \frac{1}{m_1!m_2!}\frac{\partial^{m_1+m_2}}{\partial\mu_1^{m_1}\partial\mu_2^{m_2}} \times \langle\langle 1|\hat{\mathcal{N}}_A(t;\mu_1)\hat{\mathcal{N}}_B(t;\mu_2)|\Psi\rangle\rangle\Big|_{\mu_1=\mu_2=0}, \tag{4.17a}$$

$$\hat{\mathcal{N}}_A(t;\mu) = \exp\left[-\tfrac{1}{2}t\sum_k \lambda(k,A)(a_k^\dagger a_k + \tilde{a}_k^\dagger\tilde{a}_k)\right] \times \exp\left[-\mu_1\sum_k \xi_k(t,A)a_k\tilde{a}_k\right], \tag{4.17b}$$

$$\hat{\mathcal{N}}_B(t;\mu) = \exp\left[-\tfrac{1}{2}t\sum_k \lambda(k,B)(b_k^\dagger b_k + \tilde{b}_k^\dagger\tilde{b})\right] \times \exp\left[-\mu_2\sum_k \xi_k(t,B)b_k\tilde{b}_k\right], \tag{4.17c}$$

where $\xi_k(t,A)$ and $\xi_k(t,B)$ are defined by

$$\xi_k(t,A) = 1 - \exp[-\lambda(k,A)t], \quad \xi_k(t,B) = 1 - \exp[-\lambda(k,B)t]. \tag{4.18}$$

According to the procedure used to derive (4.12) and (4.13a) in Sec. IV B, the average values $\bar{n}_j$, the fluctuations Δn_j^2, and the cross correlation function $\overline{n_1n_2}$ of the electron numbers registered by the two counters are given by

$$\bar{n}_1 = \sum_k \xi_k(t,A)\langle A_k\rangle, \quad \bar{n}_2 = \sum_k \xi_k(t,B)\langle B_k\rangle, \tag{4.19a}$$

$$\Delta n_j^2 = \bar{n}_j + \bar{n}_j^2[\Delta_{jj} - \kappa_{jj}] \quad \text{for } j=1,2 \tag{4.19b}$$

$$\overline{n_1n_2} = \bar{n}_1\bar{n}_2 + \bar{n}_1\bar{n}_2[\Delta_{12} + \kappa_{12}], \tag{4.19c}$$

where Δ_{ij} and κ_{ij} are defined by

$$\Delta_{11} = \frac{1}{\bar{n}_1^2}\sum_k\sum_{\substack{l\\(l\neq k)}} \xi_k(t,A)\xi_l(t,A)[\langle A_kA_l\rangle - \langle A_k\rangle\langle A_l\rangle], \tag{4.20a}$$

$$\Delta_{22} = \frac{1}{\bar{n}_2^2}\sum_k\sum_{\substack{l\\(l\neq k)}} \xi_k(t,B)\xi_l(t,B)[\langle B_kB_l\rangle - \langle B_k\rangle\langle B_l\rangle], \tag{4.20b}$$

$$\Delta_{12} = \frac{1}{\bar{n}_1\bar{n}_2}\sum_k\sum_{\substack{l\\(l\neq k)}} \xi_k(t,A)\xi_l(t,B)[\langle A_kB_l\rangle - \langle A_k\rangle\langle B_l\rangle], \tag{4.20c}$$

$$\kappa_{11} = \frac{1}{\bar{n}_1^2}\sum_k[\xi_k(t,A)\langle A_k\rangle]^2, \quad \kappa_{22} = \frac{1}{\bar{n}_2^2}\sum_k[\xi_k(t,B)\langle B_k\rangle]^2, \tag{4.20d}$$

$$\kappa_{12} = \frac{1}{\bar{n}_1\bar{n}_2}\sum_k \xi_k(t,A)\xi_k(t,B)[\langle A_kB_k\rangle - \langle A_k\rangle\langle B_k\rangle]. \tag{4.20e}$$

The cross correlation function (4.19c) with (4.20c) and (4.20e) reflects the correlation between the A and B electrons. When $\xi_k(t,A)$ and $\xi_k(t,B)$ are sufficiently small, the counting probability distribution $P_{m_1m_2}(t)$ becomes

$$P_{m_1m_2}(t) = \frac{\bar{n}_1^{m_1}\bar{n}_2^{m_2}}{m_1!m_2!}e^{-\bar{n}_1-\bar{n}_2}\{1+\tfrac{1}{2}(\Delta_{11}-\kappa_{11})[\bar{n}_1^2-2\bar{n}_1m_1+m_1(m_1-1)]+(\Delta_{12}+\kappa_{12})(\bar{n}_1-m_1)(\bar{n}_2-m_2) +\tfrac{1}{2}(\Delta_{22}-\kappa_{22})[\bar{n}_2^2-2\bar{n}_2m_2+m_2(m_2-1)]\}. \tag{4.21}$$

D. Nonselective counting process for A and B electrons

In this subsection, we investigate the electron counting process using two AB counters, each of which can register both A and B electrons. The total correlation of electrons can thus be measured. In this case, the one-count processes by the two AB counters are given by

$$\hat{J}_1 = -\sum_k \lambda_1(k,AB)(a_k\tilde{a}_k + b_k\tilde{b}_k), \quad \hat{J}_2 = -\sum_k \lambda_2(k,AB)(a_k\tilde{a}_k + b_k\tilde{b}_k), \tag{4.22}$$

where $\{\lambda_1(k,AB)\}$ and $\{\lambda_2(k,AB)\}$ characterize the measurement performed by the two AB counters. The nonunitary time-evolution generator without counting electrons during measurement time t can then be obtained from (2.20),

$$\hat{S}(t) = \exp\left\{-\tfrac{1}{2}t\sum_k[\lambda_1(k,AB)+\lambda_2(k,AB)](a_k^\dagger a_k + \tilde{a}_k^\dagger\tilde{a}_k + b_k^\dagger b_k + \tilde{b}_k^\dagger\tilde{b}_k)\right\}. \tag{4.23}$$

Thus the counting probability distribution $P_{m_1m_2}(t)$ that m_1 electrons are registered by counter 1 and m_2 electrons by counter 2 during time t is calculated by

$$P_{m_1m_2}(t) = \frac{1}{m_1!m_2!}\frac{\partial^{m_1+m_2}}{\partial\mu_1^{m_1}\partial\mu_2^{m_2}}\langle\langle 1|\hat{\mathcal{N}}'_{AB}(t;\mu_1,\mu_2)|\Psi\rangle\rangle\Big|_{\mu_1=\mu_2=0}, \tag{4.24a}$$

$$\hat{\mathcal{N}}'_{AB}(t;\mu_1,\mu_2) = \exp\left[-\tfrac{1}{2}t\sum_k\bar{\lambda}(k,AB)(a_k^\dagger a_k + \tilde{a}_k^\dagger\tilde{a}_k + b_k^\dagger b_k + \tilde{b}_k^\dagger\tilde{b}_k)\right]\exp\left[-\sum_k\bar{\mu}(k,AB)\xi_k(t,AB)(a_k\tilde{a}_k + b_k\tilde{b}_k)\right], \tag{4.24b}$$

where $\bar{\lambda}(k,AB)$, $\bar{\mu}(k,AB)$, and $\xi_k(t,AB)$ are defined by

$$\bar{\lambda}(k,AB) = \lambda_1(k,AB)+\lambda_2(k,AB), \quad \xi_k(t,AB) = 1-\exp[-\bar{\lambda}(k,AB)t], \tag{4.25a}$$

$$\bar{\mu}(k,AB) = \frac{\mu_1\lambda_1(k,AB)+\mu_2\lambda_2(k,AB)}{\lambda_1(k,AB)+\lambda_2(k,AB)}. \tag{4.25b}$$

We can thus obtain the average values $\bar{n}_j$, fluctuations Δn_j^2, and cross correlation function $\overline{n_1n_2}$ of the electron numbers registered by the two counters as follows:

$$\bar{n}_j = \sum_k \bar{\lambda}_j(k,AB)\xi_k(t,AB)\langle N_k\rangle, \quad \Delta n_j^2 = \bar{n}_j + \bar{n}_j^2(\Delta_{jj}-\kappa_{jj}) \quad \text{for } j=1,2 \tag{4.26a}$$

$$\overline{n_1n_2} = \bar{n}_1\bar{n}_2 + \bar{n}_1\bar{n}_2(\Delta_{12}-\kappa_{12}), \tag{4.26b}$$

where $N_k = A_k + B_k$, and Δ_{ij} and κ_{ij} are defined by

$$\Delta_{ij} = \frac{1}{\bar{n}_i\bar{n}_j}\sum_k\sum_{\substack{l\\(l\neq k)}}\bar{\lambda}_i(k,AB)\bar{\lambda}_j(l,AB)\bar{\xi}_k(t,AB)\bar{\xi}_l(t,AB)[\langle N_kN_l\rangle - \langle N_k\rangle\langle N_l\rangle], \tag{4.27a}$$

$$\kappa_{ij} = \frac{1}{\bar{n}_i\bar{n}_j}\sum_k\bar{\lambda}_i(k,AB)\bar{\lambda}_j(l,AB)\bar{\xi}_k(t,AB)^2[\langle N_k\rangle^2 - 2\langle A_kB_k\rangle], \tag{4.27b}$$

with $\bar{\lambda}(k,AB) = \lambda_j(k,AB)/[\lambda_1(k,AB)+\lambda_2(k,AB)]$. When $\xi_k(t,AB) \ll 1$, the counting probability distribution $P_{m_1m_2}(t)$ becomes

$$P_{m_1m_2}(t) = \frac{\bar{n}_1^{m_1}\bar{n}_2^{m_2}}{m_1!m_2!}e^{-\bar{n}_1-\bar{n}_2}\{1+\tfrac{1}{2}(\Delta_{11}-\kappa_{11})[m_1(m_1-1)-2m_1\bar{n}_1+\bar{n}_1^2]+(\Delta_{12}-\kappa_{12})(m_1-\bar{n}_1)(m_2-\bar{n}_2) +\tfrac{1}{2}(\Delta_{22}-\kappa_{22})[m_2(m_2-1)-2m_2\bar{n}_2+\bar{n}_2^2]\}. \tag{4.28}$$

We have considered the three kinds of electron counting measurements. The results are summarized in Table II.

E. Simple model for the correlated state of electrons

In this subsection, we explicitly consider a correlated state of electrons. We assume that the correlated state of electrons in the Hilbert space $\mathcal{H}$ is given by

$$|\psi\rangle = \prod_k [u_k + v_k a_k b_k]|0;A\rangle \otimes |0;B\rangle, \qquad (4.29)$$

where $|0;A_k\rangle$ and $|0;B_k\rangle$ are the vacuum states defined by $a_k|0;A_k\rangle = b_k|0;B_k\rangle = 0$. We also assume that u_k and v_k are real numbers which satisfy $u_k^2 + v_k^2 = 1$. This state is similar to the BCS state in superconductivity. In the Liouville space, since we have $|\Psi\rangle\rangle = |\psi\rangle \otimes |\tilde{\psi}\rangle$, we can express the correlated initial state of electrons corresponding to (4.29) as

$$\begin{aligned}|\Psi\rangle\rangle = \prod_k \{ & u_k^2|0,0;A_k\rangle \otimes |0,0;B_k\rangle \\ & + v_k^2|1,1;A_k\rangle \otimes |1,1;B_k\rangle \\ & + u_k v_k(|1,0;A_k\rangle \otimes |1,0;B_k\rangle \\ & + |0,1;A_k\rangle \otimes |0,1;B_k\rangle)\}, \end{aligned} \qquad (4.30)$$

where we set

$$\begin{aligned}|1,0;A_k\rangle &= a_k^\dagger|0,0;A_k\rangle, \\ |0,1;A_k\rangle &= \tilde{a}_k^\dagger|0,0;A_k\rangle, \\ |1,1;A_k\rangle &= a_k^\dagger\tilde{a}_k^\dagger|0,0;A_k\rangle,\end{aligned} \qquad (4.31a)$$

$$\begin{aligned}|1,0;B_k\rangle &= b_k^\dagger|0,0;B_k\rangle, \\ |0,1;B_k\rangle &= \tilde{b}_k^\dagger|0,0;B_k\rangle, \\ |1,1;B_k\rangle &= b_k^\dagger\tilde{b}_k^\dagger|0,0;B_k\rangle,\end{aligned} \qquad (4.31b)$$

with $|0,0;A_k\rangle = |0;A_k\rangle \otimes |\tilde{0};A_k\rangle$ and $|0,0;B_k\rangle = |0;B_k\rangle \otimes |\tilde{0};B_k\rangle$. The normalization in the Liouville space is satisfied since $\langle\langle 1|\Psi\rangle\rangle = \prod_k (u_k^2 + v_k^2) = 1$.

We first consider the counting process for only A electrons as discussed in Sec. IV B. Since Δ_{ij} defined by (4.13a) vanishes for (4.30), the average numbers, fluctuations, and cross correlation function of the electron numbers registered by the two A counters can be obtained from (4.12)

$$\begin{aligned}\bar{n}_j &= \sum_k \xi_k(t,A)\bar{\lambda}_j(k,A)v_k^2, \\ \Delta n_j^2 &= \bar{n}_j - \kappa_{jj}\bar{n}_j^2, \\ \overline{n_1 n_2} &= \bar{n}_1\bar{n}_2 - \kappa_{12}\bar{n}_1\bar{n}_2,\end{aligned} \qquad (4.32)$$

where the positive quantity κ_{ij} is defined by

$$\kappa_{ij} = \frac{1}{\bar{n}_i\bar{n}_j}\sum_k \bar{\lambda}_i(k,A)\bar{\lambda}_j(k,A)[\xi_k(t,A)v_k^2]^2. \qquad (4.33)$$

When $\xi_k(t,A) \ll 1$, the counting probability distribution $P_{m_1 m_2}(t)$ is obtained by substituting $\Delta_{ij} = 0$ and κ_{ij} given by (4.33) into (4.14). It is found from (4.33) that since $\Delta n_j^2 < \bar{n}_j$, the statistics of the electron number registered by each A counter is characterized by a sub-Poissonian distribution. Furthermore, it is seen that since $\kappa_{12} > 0$, the electron number correlation $\overline{n_1 n_2}$ between the two counters is smaller than the noncorrelated value $\bar{n}_1\bar{n}_2$. The antibunching correlation of electrons is thus obtained.

Next, we consider the selective counting process for the A and B electrons discussed in Sec. IV C. Since Δ_{ij} given by (4.20a)–(4.20c) vanishes, it is found from (4.19) that $\bar{n}_j$, Δn_j^2 $(j=1,2)$, and $\overline{n_1 n_2}$ become

$$\bar{n}_1 = \sum_k \xi_k(t,A)v_k^2, \quad \bar{n}_2 = \sum_k \xi_k(t,B)v_k^2, \qquad (4.34a)$$

$$\Delta n_j^2 = \bar{n}_j - \kappa_{jj}\bar{n}_j^2, \quad \overline{n_1 n_2} = \bar{n}_1\bar{n}_2 + \kappa_{12}\bar{n}_1\bar{n}_2, \qquad (4.34b)$$

where the κ_{ij}'s are defined by

$$\begin{aligned}\kappa_{11} &= \frac{1}{\bar{n}_1^2}\sum_k [\xi_k(t,A)v_k^2]^2, \\ \kappa_{22} &= \frac{1}{\bar{n}_2^2}\sum_k [\xi_k(t,B)v_k^2]^2,\end{aligned} \qquad (4.35a)$$

$$\kappa_{12} = \frac{1}{\bar{n}_1\bar{n}_2}\sum_k \xi_k(t,A)\xi_k(t,B)(u_k v_k)^2. \qquad (4.35b)$$

In deriving (4.35b), we used the relation $u_k^2 + v_k^2 = 1$. When $\xi_k(t,A) \ll 1$ and $\xi_k(t,B) \ll 1$, the counting probability distribution $P_{m_1 m_2}(t)$ can be obtained by substituting $\Delta_{ij} = 0$ and (4.35) into (4.21). From the above results, the statistics of the electron number registered by each counter is characterized by a sub-Poissonian distribution. It is also seen from (4.34b) that the electron number correlation $\overline{n_1 n_2}$ is greater than the noncorrelated value $\bar{n}_1\bar{n}_2$. This effect is the bunching correlation between the A and B electrons. It should be noted that the correlation among the same kind of electrons becomes antibunching while the correlation between A and B electrons becomes bunching. These electron-counting measurement results reflect the characteristics of the initial state of electrons.

Finally, in the nonselective counting process for the A and B electrons, the average numbers, fluctuations, and cross correlation function of the electron numbers registered by the counters are obtained from (4.26):

TABLE II. List of the results of the selective electron-counting measurements.

	A-A (B-B) counting	A-B counting	Nonselective counting
Counting probability	Eqs. (4.9) and (4.14)	Eqs. (4.17a) and (4.21)	Eqs. (4.24a) and (4.28)
Fluctuation and correlation	Eqs. (4.12) and (4.13)	Eqs. (4.19) and (4.20)	Eqs. (4.26) and (4.27)

$$\bar{n}_j = 2\sum_k \bar{\lambda}_j(k,AB)\xi_k(t,AB)v_k^2, \quad \Delta n_j^2 = \bar{n}_j - \kappa_{jj}\bar{n}_j^2, \tag{4.36a}$$

$$\overline{n_1 n_2} = \bar{n}_1\bar{n}_2 - \kappa_{12}\bar{n}_1\bar{n}_2, \tag{4.36b}$$

where the κ_{ij}'s are defined by

$$\kappa_{ij} = \frac{2}{\bar{n}_i\bar{n}_j}\sum_k \bar{\lambda}_i(k,AB)\bar{\lambda}_j(k,AB)\xi_k(t,AB)^2(2v_k^2-1)v_k^2. \tag{4.37}$$

If $\xi_k(t,AB) \ll 1$, the counting probability distribution $P_{m_1m_2}(t)$ is obtained by substituting $\Delta_{ij}=0$ and (4.37) into (4.28). It is important to note that κ_{ij} defined by (4.37) is not always positive. This is due to the synthetic effect of the two correlations: the bunching correlation between the A and B electrons and the antibunching correlation among the same kind of electrons. It is clear that κ_{ij} becomes negative if $v_k^2 < 1/2$ for all k. In this case, the statistics of the electron number registered by each counter obeys a super-Poissonian distribution, and the electron number correlation $\overline{n_1n_2}$ becomes bunching. Furthermore, under certain conditions it may be possible that $\kappa_{ij}=0$. In this case, we have $\Delta n_j^2 = \bar{n}_j$ and $\overline{n_1n_2} = \bar{n}_1\bar{n}_2$, and so obtain Poisson-like statistics. We would like to remark that correlation among electrons is indispensable for getting Poisson-like statistics, in contrast with the photon-counting process.

V. COUNTING PROCESS WITH CHAOTIC ELECTRON SOURCE

A. Chaotic electron source

Up to now, we have treated only the relevant system (the cavity) of electrons and the counters, but have not considered the source of electrons which is an important consideration in electron counting experiments. Without an electron source, the electron number in the system decreases in time as counting measurement proceeds, since the counter removes electrons from the system as it registers them. As a result, the state of the system becomes a vacuum as $t \to \infty$, and the electron number registered by the counter approaches the total electron number in the initial state of the system. In this section, we show that an electron-counting process including an electron source can be formulated in the same manner as developed in Sec. III. We assume a chaotic electron source, which is a model for a thermal or field emission of electrons, since this seems to be the most important case in a real experiment.

We first consider the following time-evolution equation for state $|\Psi(t)\rangle\rangle$ of the electron system in the Liouville space:

$$\frac{\partial}{\partial t}|\Psi(t)\rangle\rangle = \hat{\Pi}|\Psi(t)\rangle\rangle, \tag{5.1}$$

with

$$\hat{\Pi} = -\sum_k 2\kappa_k[(1-2\bar{n}_k)J_0(k) + (1-\bar{n}_k)J_-(k) + \bar{n}_kJ_+(k) + \tfrac{1}{2}], \tag{5.2}$$

where $J_\pm(k)$ and $J_0(k)$ are the generators of the su(2) Lie algebra defined by (3.6), κ_k is a positive constant, and $\bar{n}_k$ is a certain distribution function. Using the Baker-Campbell-Hausdorff formula, (5.1) can be solved as follows:

$$\begin{aligned}|\Psi(t)\rangle\rangle = \prod_k\{&e^{-\kappa_k t}\exp[A_+(k;t)J_+(k)] \\ &\times\exp[\ln A_0(k;t)J_0(k)] \\ &\times\exp[A_-(k;t)J_-(k)]\}|\Psi\rangle\rangle,\end{aligned} \tag{5.3}$$

where $A_\pm(k;t)$ and $A_0(k;t)$ are given by

$$A_+(k;t) = -\frac{\bar{n}_k(1-e^{-2\kappa_k t})}{1-\bar{n}_k(1-e^{-2\kappa_k t})},$$
$$A_-(k;t) = -\frac{(1-\bar{n}_k)(1-e^{-2\kappa_k t})}{1-\bar{n}_k(1-e^{-2\kappa_k t})}, \tag{5.4}$$

$$A_0(k;t) = \left[\frac{e^{-\kappa_k t}}{1-\bar{n}_k(1-e^{-2\kappa_k t})}\right]^2. \tag{5.5}$$

Let us assume the initial state of electrons, for simplicity, to be

$$|\Psi\rangle\rangle = \prod_k[(1-m_k)|0_k,0_k\rangle + m_k|1_k,1_k\rangle]. \tag{5.6}$$

Substituting (5.6) into (5.3), we obtain

$$|\Psi(t)\rangle\rangle = \prod_k\{[1-n_k(t)]|0_k,0_k\rangle + n_k(t)|1_k,1_k\rangle\}, \tag{5.7}$$

with $n_k(t) = \bar{n}_k + (m_k - \bar{n})e^{-2\kappa_k t}$. For $\kappa_k t \gg 1$, we obtain the chaotic state of an electron,

$$|\Psi(\infty)\rangle\rangle = \prod_k[(1-\bar{n}_k)|0_k,0_k\rangle + \bar{n}_k|1_k,1_k\rangle]. \tag{5.8}$$

This result is independent of the initial state of electrons. Indeed, we can show that an arbitrary state of electrons approaches (5.8) through the time-evolution generator (5.2). Thus (5.1) can be considered to describe the time evolution of the state of the system caused by the chaotic electron source.

B. Electron-counting process with a chaotic source

In this subsection, we consider the time evolution of the system interacting with the chaotic electron source and the electron counter and investigate the electron-counting statistics. In this setup, the one-count process $\hat{J}$ is given by (3.1) and the generator $\hat{Y}$ of time evolution without detecting electrons defined by (2.19) becomes

$$\hat{Y} = \hat{\Pi} + \hat{Y}_0, \quad \hat{Y}_0 = -\tfrac{1}{2}\sum_k \lambda_k(c_k^\dagger c_k + \tilde{c}_k^\dagger\tilde{c}_k), \tag{5.9}$$

where $\hat{\Pi}$ is given by (5.2). It is easily seen that relation (2.20) is satisfied since $\langle\langle 1|\hat{\Pi} = 0$. In (5.9), we have ignored the free time-evolution generator $H_0 - \tilde{H}_0$ of electrons, since $H_0 - \tilde{H}_0$ commutes with $\hat{\Pi}$ and $\hat{Y}_0$, thus giving unimportant phase factor. Here H_0 is the free Hamiltonian of electrons and $\tilde{H}_0$ is its tilde conjugate.

We first consider the time evolution of the electron system without referring to the result indicated by the counter. In this case, by using the operator $\hat{\mathcal{T}}(t)$ defined by (2.9) and (2.21b), the state of the system during time t is given by

$$|\Psi(t)\rangle\rangle = \hat{\mathcal{T}}(t)|\Psi(0)\rangle\rangle = \exp[t(\hat{Y}_0 + \hat{\Pi} + \hat{J})]|\Psi(0)\rangle\rangle. \tag{5.10}$$

Using the generators of su(2) Lie algebra defined by (3.6), we have

$$\hat{Y}_0 + \hat{\Pi} + \hat{J} = -\sum_k [\kappa_k + \tfrac{1}{2}\lambda_k + a_+(k)J_+(k) + a_0(k)J_0(k) + a_-(k)J_-(k)], \tag{5.11}$$

with $a_+(k) = 2\kappa_k\bar{n}_k$, $a_0(k) = 2\kappa_k(1-2\bar{n}_k) + \lambda_k$, and $a_-(k) = 2\kappa_k(1-\bar{n}_k) + \lambda_k$. When we assume that the initial state of the system is given by (5.6) and we use the Baker-Campbell-Hausdorff formula of su(2) Lie algebra, we obtain the following result:

$$|\Psi(t)\rangle\rangle = \prod_k \{[1 - \tilde{n}_k(t)]|0_k, 0_k\rangle + \tilde{n}_k(t)|1_k, 1_k\rangle\}, \tag{5.12}$$

where $\tilde{n}(t) = m_k + [\kappa_k\bar{n}_k/\gamma_k - m_k](1 - e^{-2\gamma_k t})$ and $\gamma_k = \kappa_k + \frac{1}{2}\lambda_k$.

It is found from (5.7) and (5.12) that the interaction with the electron counter changes the distribution function from $n_k(t)$ to $\tilde{n}_k(t)$. The stationary state of the system then becomes

$$|\Psi_{\text{stationary}}\rangle\rangle = \prod_k \left[\frac{\lambda_k\bar{n}_k}{2\gamma_k}|0_k, 0_k\rangle + \frac{\kappa_k\bar{n}_k}{\gamma_k}|1_k, 1_k\rangle\right] \tag{5.13}$$

and the equilibrium value of the average electron number in the system interacting with the electron source and the counter is given by $\langle N\rangle_{\text{stationary}} = \sum_k \kappa_k\bar{n}_k/\gamma_k$.

Next, we consider electron counting measurement using one electron counter. For a counting process with a chaotic electron source, $P(t;\mu)$ is given by $P(t;\mu) = \langle\langle 1|\hat{\mathcal{N}}_s(t;\mu)|\Psi\rangle\rangle$ and $\hat{\mathcal{N}}_s(t;\mu)$ is expressed in terms of the su(2) generators $J_\pm(k)$ and $J_0(k)$,

$$\hat{\mathcal{N}}_s(t;\mu) = \exp\left\{-t\sum_k [f_0(k)J_0(k) + f_-(k)J_-(k) + f_+(k)J_+(k) + \gamma_k]\right\}, \tag{5.14}$$

with $f_0(k) = 2\kappa_k(1-2\bar{n}_k) + \lambda_k$, $f_-(k) = 2\kappa_k(1-\bar{n}_k) + \mu\lambda_k$, and $f_+(k) = 2\kappa_k\bar{n}_k$. When we assume that the initial state is given by (5.6) and use the Baker-Campbell-Hausdorff formula, we obtain

$$\hat{\mathcal{N}}_s(t;\mu)|\Psi(0)\rangle\rangle = \prod_k \{e^{-\gamma_k t}[(A_k + m_k B_k)|0_k, 0_k\rangle + (C_k + m_k D_k)|1_k, 1_k\rangle]\}, \tag{5.15}$$

where A_k, B_k, C_k, and D_k are given by

$$A_k = \cosh\left[\gamma_k t\sqrt{1+\alpha_k(\mu-1)}\right] + \frac{\gamma_k - 2\bar{n}_k\kappa_k}{\gamma_k\sqrt{1+\alpha_k(\mu-1)}} \times \sinh\left[\gamma_k t\sqrt{1+\alpha_k(\mu-1)}\right], \tag{5.16a}$$

$$B_k = -\cosh\left[\gamma_k t\sqrt{1+\alpha_k(\mu-1)}\right] + \frac{\gamma_k + (\mu-1)\lambda_k}{\gamma_k\sqrt{1+\alpha_k(\mu-1)}} \times \sinh\left[\gamma_k t\sqrt{1+\alpha_k(\mu-1)}\right], \tag{5.16b}$$

$$C_k = \frac{2\bar{n}_k\kappa_k}{\gamma_k\sqrt{1+\alpha_k(\mu-1)}} \sinh\left[\gamma_k t\sqrt{1+\alpha_k(\mu-1)}\right], \tag{5.16c}$$

$$D_k = \cosh\left[\gamma_k t\sqrt{1+\alpha_k(\mu-1)}\right] - \frac{1}{\sqrt{1+\alpha_k(\mu-1)}} \times \sinh\left[\gamma_k t\sqrt{1+\alpha_k(\mu-1)}\right], \tag{5.16d}$$

with $\alpha_k = 2\bar{n}_k\lambda_k\kappa_k/\gamma_k^2$. Using (5.15), we can calculate the state of the system after m electrons are registered. The quantity $P(t;\mu+1)$ then becomes

$$P(t;\mu+1) = \prod_k e^{-\gamma_k t}\left\{\cosh\left[\gamma_k t\sqrt{1+\alpha_k\mu}\right] + \left(1 + \frac{m_k\lambda_k\mu}{\gamma_k}\right) \times \frac{\sinh\left[\gamma_k t\sqrt{1+\alpha_k\mu}\right]}{\sqrt{1+\alpha_k\mu}}\right\}. \tag{5.17}$$

Let us assume that we begin recording the results indicated by the counter after the system reaches its stationary state given by (5.13) and that we set $t = 0$ when recording begins. In this case, $P(t;\mu+1)$ is obtained by replacing m_k in (5.17) with a stationary value $\kappa_k\bar{n}_k/\gamma_k$. We thus get

$$P(t;\mu+1) = \prod_k e^{-\gamma_k t}\left\{\cosh\left[\gamma_k t\sqrt{1+\alpha_k\mu}\right] + (1 + \tfrac{1}{2}\alpha_k\mu) \times \frac{\sinh\left[\gamma_k t\sqrt{1+\alpha_k\mu}\right]}{\sqrt{1+\alpha_k\mu}}\right\}. \tag{5.18}$$

To calculate the average value $\bar{n}$ and fluctuation Δn of the electron number registered by the counter, we expand (5.18) up to the second order with respect to μ, and we use (2.31). We then obtain the following results:

$$\bar{n} = \sum_k \frac{\kappa_k \bar{n}_k}{\gamma_k} \lambda_k t,$$
$$\Delta n^2 = \bar{n} - \sum_k \left(\frac{\kappa_k \lambda_k \bar{n}_k}{\gamma_k^2} \right)^2 \left[\gamma_k t - \frac{1 - e^{-2\gamma_k t}}{2} \right]. \tag{5.19}$$

These expressions are quite different from those obtained for counting measurement without an electron source. The average of the electron number registered by the counter per unit time is given by $\sum_k \kappa_k \lambda_k \bar{n}_k / \gamma_k$. It is easily seen from (5.19) that the relation $\Delta n^2 < \bar{n}$ is established for all times $t \geq 0$. We thus obtain the sub-Poissonian statistics for the counting measurement, even though there is a chaotic electron source.

We now compare the results (5.19) with those obtained for counting measurement without a chaotic electron source. In the absence of an electron source, the average value $\bar{n}_0$ and fluctuation Δn_0^2 are obtained from (3.56) by substituting the equilibrium value $\kappa_k \bar{n}_k / \gamma_k$ into $g_{11}(k)$,

$$\bar{n}_0 = \sum_k \frac{\kappa_k \bar{n}_k}{\gamma_k} (1 - e^{-\lambda_k t}),$$
$$\Delta n_0^2 = \bar{n}_0 - \sum_k \left[\frac{\kappa_k \bar{n}_k}{\gamma_k} (1 - e^{-\lambda_k t}) \right]^2. \tag{5.20}$$

Let us consider the result for a very short time region ($1 \gg \kappa_k t, \lambda_k t$). Thus we have from (5.19) and (5.20)

$$\bar{n}_0 \approx \bar{n} = \sum_k \frac{\kappa_k \bar{n}_k}{\gamma_k} \lambda_k t,$$
$$\Delta n_0^2 - \bar{n}_0 \approx \Delta n^2 - \bar{n} \approx - \sum_k \left(\frac{\kappa_k \bar{n}_k}{\gamma_k} \lambda_k t \right)^2. \tag{5.21}$$

It is found from these expressions that in an extremely short time region, the average value and the fluctuation of the electron number registered by the counter with an electron source are equal to those obtained without an electron source. This result is reasonable since in such a short time region, the change in the initial state of electrons due to counting measurement is negligible and the role of the electron source is not so important. When we perform measurement using a highly sensitive electron counter, where the parameter λ_k takes a large value, the difference becomes remarkable even for a short time region.

C. Electron counting with two counters

In this subsection, we consider electron-counting measurement using two electron counters with a chaotic electron source. This is done by taking into account the time-evolution generator $\hat{\Pi}$ due to the chaotic electron source in (3.35) and (3.36). Consequently, we obtain

$$P_{m_1 m_2}(t) = \frac{1}{m_1! m_2!} \frac{\partial^{m_1+m_2}}{\partial \mu_1^{m_1} \partial \mu_2^{m_2}} P(t; \mu_1, \mu_2) \bigg|_{\mu_1=\mu_2=0}, \tag{5.22a}$$

$$P(t; \mu_1, \mu_2) = \langle\langle 1 | \exp[t \hat{Y}_s(\mu_1, \mu_2)] | \Psi \rangle\rangle, \tag{5.22b}$$

where $\hat{Y}_s(\mu_1, \mu_2)$ is given by

$$\hat{Y}_s(\mu_1, \mu_2) = - \sum_k \bigg[g_0(k) J_0(k) + g_-(k) J_-(k) + g_+(k) J_+(k) + \kappa_k + \frac{\lambda_1(k) + \lambda_2(k)}{2} \bigg], \tag{5.23}$$

with

$$g_-(k) = 2\kappa_k (1 - \bar{n}_k) + \mu_1 \lambda_1(k) + \mu_2 \lambda_2(k), \tag{5.24a}$$
$$g_+(k) = 2\kappa_k \bar{n}_k,$$
$$g_0(k) = 2\kappa_k (1 - 2\bar{n}_k) + \lambda_1(k) + \lambda_2(k), \tag{5.24b}$$

and where $\lambda_1(k)$ and $\lambda_2(k)$ are the parameters which characterize the two electron counters. Furthermore, since we investigate the counting process in a stationary situation, we can assume that the initial state of the electron system is given by (5.6) without a loss of generality.

Before considering the counting measurement, we first investigate the time evolution of the system when we do not refer to the result indicated by the electron counters. In this case, the state of the system at time t is given by $|\Psi(t)\rangle\rangle = \hat{\mathcal{N}}_s(t; \mu_1 = 1, \mu_2 = 1) |\Psi\rangle\rangle$. By using the decomposition formula of su(2) Lie algebra, this state is calculated to be

$$|\Psi(t)\rangle\rangle = \prod_k \{ [1 - \tilde{n}_k(t)] |0_k, 0_k\rangle + \tilde{n}_k(t) |1_k, 1_k\rangle \}, \tag{5.25}$$

where $\tilde{n}_k(t) = m_k + [\kappa_k \bar{n}_k / \tilde{\gamma}_k - m_k](1 - e^{-2\tilde{\gamma}_k t})$ with $\tilde{\lambda}_k = \lambda_1(k) + \lambda_2(k)$ and $\tilde{\gamma}_k = \kappa_k + \frac{1}{2}\tilde{\lambda}_k$.

Using the same procedure as that used in Sec. V B, $P(t; \mu_1 + 1, \mu_2 + 1)$ is calculated as follows:

$$P(t; \mu_1 + 1, \mu_2 + 1) = \prod_k e^{-\tilde{\gamma}_k t} \bigg\{ \cosh\left[\tilde{\gamma}_k t \sqrt{1 + \tilde{\alpha}_k \mu_k} \right] + \frac{1}{\sqrt{1 + \tilde{\alpha}_k \mu_k}} \sinh\left[\tilde{\gamma}_k t \sqrt{1 + \tilde{\alpha}_k \mu_k} \right] + \frac{m_k \tilde{\lambda}_k}{\tilde{\gamma}_k} \frac{\mu_k}{\sqrt{1 + \tilde{\alpha}_k \mu_k}} \sinh\left[\tilde{\gamma}_k t \sqrt{1 + \tilde{\alpha}_k \mu_k} \right] \bigg\}, \tag{5.26}$$

where $\mu_k = [\mu_1\lambda_1(k) + \mu_2\lambda_2(k)]/\tilde{\gamma}_k$ and $\tilde{\alpha}_k = 2\kappa_k\tilde{\lambda}_k\bar{n}_k/\tilde{\gamma}_k^2$. Since we consider the counting process in the stationary state, we can substitute $m_k = \tilde{n}_k(\infty) = \kappa_k\bar{n}_k/\tilde{\gamma}_k$ into (5.26) and obtain

$$P(t;\mu_1+1,\mu_2+1) = \prod_k e^{-\tilde{\gamma}_k t}\left\{ \cosh\left[\tilde{\gamma}_k t\sqrt{1+\tilde{\alpha}_k\mu_k}\right] + (1+\tfrac{1}{2}\tilde{\alpha}_k\mu_k) \times \frac{\sinh\left[\tilde{\gamma}_k t\sqrt{1+\tilde{\alpha}_k\mu_k}\right]}{\sqrt{1+\tilde{\alpha}_k\mu_k}}\right\}. \qquad (5.27)$$

We can thus obtain the average values $\bar{n}_j$, fluctuations Δn_j^2, and cross correlation function $\overline{n_1n_2}$ of the electron numbers registered by the counters,

$$\bar{n}_j = \sum_k \beta_j(k)\frac{\kappa_k\bar{n}_k}{\tilde{\gamma}_k}\tilde{\lambda}_k t, \qquad (5.28a)$$

$$\Delta n_j^2 = \bar{n}_j - \sum_k\left(\beta_j(k)\frac{\kappa_k\tilde{\lambda}_k\bar{n}_k}{\tilde{\gamma}_k^2}\right)^2\left[\tilde{\gamma}_k t - \frac{1-e^{-2\tilde{\gamma}_k t}}{2}\right], \qquad (5.28b)$$

$$\overline{n_1n_2} = \bar{n}_1\bar{n}_2 - \sum_k \beta_1(k)\beta_2(k)\left(\frac{\kappa_k\tilde{\lambda}_k\bar{n}_k}{\tilde{\gamma}_k^2}\right)^2 \times\left[\tilde{\gamma}_k t - \frac{1-e^{-2\tilde{\gamma}_k t}}{2}\right], \qquad (5.28c)$$

with $\beta_j(k) = \lambda_j(k)/\tilde{\lambda}_k$. Therefore, we can obtain the sub-Poisson statistics and the antibunching correlation.

VI. SUMMARY

We have investigated the electron-counting processes in terms of the Liouville space formulation. The model of the electron-counting processes considered here is based on the quantum Markov processes developed by Davies and Srinivas. After presenting the general theory of the electron-counting process in the Liouville space, we considered the counting statistics for the two initial states of electrons: the noncorrelated state and the state having a correlation between up-spin and down-spin electrons. The time evolution of the electron system interacting with the counter and the statistics for the electron number registered by the counter were calculated. For a noncorrelated initial state, the results show a sub-Poissonian counting probability and antibunching correlation among the electrons. It seems that these results are explained by the Pauli exclusion principle [24]. For a correlated initial state of electrons, we used the two-counter measurement, somewhat similar to the Hanbury-Brawn and Twiss setup, and calculated the average values, fluctuations, and cross correlation functions for the electron numbers registered by the counters. In this case, depending on the properties of the initial correlation among the electrons and of the electron counters used in the measurement, the statistics for the electron numbers registered by the counter obeys a sub-Poissonian, Poissonian, or super-Poissonian distribution and the intensity correlation becomes antibunching, independent, or bunching. In particular, for a BCS-like state given by (4.29), the correlation between up-spin and down-spin electron numbers registered by the counters becomes bunching. Finally, we investigated the counting process with a chaotic electron source and obtained the sub-Poissonian statistics and the antibunching correlation, even though there is a chaotic electron source. In this paper, we confined ourselves to investigating a homogeneous system, so position dependence does not appear in our results. To investigate an inhomogeneous system, we would have to use the electron field operators $\psi(x)$ and $\psi^\dagger(x)$ instead of c_k and $c_k^\dagger$. We could then apply the method developed in this paper.

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