PHYSICAL REVIEW A VOLUME 49, NUMBER 5 MAY 1994

All-optical versus electro-optical quantum-limited feedback

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(Received 22 September 1993)

All-optical feedback can be effected by putting the output of a source cavity through a Faraday isolator and into a second cavity which is coupled to the source cavity by a nonlinear crystal. If the driven cavity is heavily damped, then it can be adiabatically eliminated and a master equation or quantum Langevin equation derived for the first cavity alone. This is done for an input bath in an arbitrary state, and for an arbitrary nonlinear coupling. If the intercavity coupling involves only the intensity (or one quadrature) of the driven cavity, then the effect on the source cavity is identical to that which can be obtained from electro-optical feedback using direct (or homodyne) detection. If the coupling involves both quadratures, this equivalence no longer holds and a coupling linear in the source amplitude can produce a nonclassical state in the source cavity. The analogous electro-optic scheme using heterodyne detection introduces extra noise which prevents the production of nonclassical light. Unlike the electro-optical case, the all-optical feedback loop has an output beam (reflected from the second cavity). We show that this may be squeezed, even if the source cavity remains in a classical state.

PACS number(s): 42.50.Lc, 42.50.Dv, 03.65.Bz

I. INTRODUCTION

Electro-optical feedback is the use of a photocurrent to control the source of the light incident on the detector producing that current. Such feedback has long been used to control noise in optical systems such as lasers. In many cases, the noise which is being controlled is classical noise. That is, noise which is well above the shot-noise limit (also known as the quantum limit). However, technological advance now enables experimentalists to work with light which is at, or even (thanks to squeezing [1]) below, the quantum limit. Thus the quantum limits to electro-optical feedback have been of considerable interest in the past decade [2–5]. Our interest lies with feedback onto the dynamics of the source cavity [6–8]. We have described such feedback using quantum measurement theory, in the form of stochastic quantum trajectories [9–12]. In the limit that the time delay in the feedback loop is much less than the cavity lifetime, a master equation describing the feedback can be derived. This is a simple and elegant way to treat feedback. There is also a corresponding Langevin equation approach [8], which is more convenient for some purposes. The clearest result of our theory is that, unless the system without feedback has nonclassical dynamics, then controlling its dynamics via a photocurrent cannot produce nonclassical light. That is, feedback based on external photodetection cannot produce squeezing.

Given these limitations on the noise-reduction abilities of electro-optical feedback, it is natural to ask, what about all-optical feedback? That is, instead of detecting the emitted light, it is reflected around a loop back into the source cavity, or, more fruitfully, into another cavity which is coupled to the first cavity in some way. Of course, for this mechanism to be considered feedback, the feedback loop must be one way, otherwise we would simply be describing a pair of doubly coupled cavities. The general configuration under consideration is shown schematically in Fig. 1. One mechanism for achieving the required unidirectionality is the Faraday isolator which utilizes Faraday rotation and polarization-sensitive beam splitters. A quantum theoretical treatment which incorporates this spatial symmetry breaking at the level of the Hamiltonian has been given recently by Gardiner [13]. If the propagation time between the source system and the driven system is ignored, then a master equation for both systems may be derived. This result was obtained simultaneously by Carmichael [14], who introduced the term "cascaded systems." The new feature which we consider here is for the driven system to be coupled back to the source system via some interaction Hamiltonian, giving rise to all-optical feedback.

The quantum theory of cascaded systems is summarized in Sec. II, and the simplest example of feedback

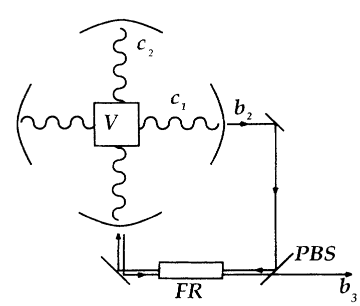

FIG. 1. Diagram of the general all-optical feedback scheme considered. The annihilation operators for the source and driven cavities are denoted c_1 and c_2, respectively, while b_2 and b_3 represent traveling waves. The nonlinear coupling between the cavities is indicated by V, FR denotes a Faraday rotator, and PBS a polarization-sensitive beam splitter.

1050-2947/94/49(5)/4110(16)/$06.00 49 4110

via this method is considered. In Sec. III we consider a more complex feedback model utilizing a nonlinear crystal to influence the source cavity, with the strength of the feedback being proportional to the intensity of the source cavity. Both a master equation and a quantum Langevin equation are derived, and are shown to be approximately the same as those pertaining to electro-optical feedback via direct detection. Similarly, the quadrature-sensitive all-optical feedback scheme considered in Sec. IV is equivalent to homodyne-mediated feedback in its effect on the source cavity. It is thus not possible to produce a nonclassical state in the source cavity via a coupling which is linear in the source cavity amplitude or intensity (which correspond to classical driving or detuning). Unlike the electro-optical case, the all-optical feedback loop has an output beam of light (reflected off the driven cavity), and we show that this may be squeezed even though the source cavity is not. Section V treats an all-optical feedback scheme which is sensitive to both quadratures of the source cavity. This has no direct electro-optical equivalent, and may produce a nonclassical source state even with a coupling linear in the source amplitude. An electro-optical analog can be defined, and the extra noise which it introduces seen explicitly. The two schemes are contrasted using the simplest feedback example introduced in Sec. II.

II. CASCADED SYSTEMS

A. Unidirectional coupling

In this section, we present a summary of the theory of cascaded open quantum systems. For different presentations, see Refs. [13,14]. Our starting point is the input-output theory of open quantum systems [15,16]. This theory describes a system interacting locally with a bath consisting of a continuum of harmonic oscillators. Physically, the system may be an optical cavity, and the bath the external electromagnetic field modes with momentum aligned to the cavity axis. The electric field (or rather, one polarization component) at a particular point in space time (parametrized by z, t) is represented approximately by the Heisenberg picture operator [16]

$$E(z,t) = \sqrt{\frac{\hbar k}{2\epsilon_0 A}}\left[b(z,t) + b^\dagger(z,t)\right]. \qquad (2.1)$$

Here, A is the cross sectional area of the beam, and only frequencies near the central wave number k are assumed to be of interest. The canonical commutation relations for the complex amplitudes $b(z,t)$ are

$$[b(z,t), b^\dagger(z',t)] = c\delta(z-z'), \qquad (2.2)$$

where c is the speed of light, and for regions where the field propagates freely,

$$b(z,t+\tau) = b(z-c\tau,t). \qquad (2.3)$$

Let the external field be coupled to the cavity by a very good mirror at $z=0$. The field with $z<0$ then represents an incoming field and that with $z>0$ an outgoing one. We assume a linear coupling of the form

$$H_1(t) = i\hbar\sqrt{\gamma_1}[b^\dagger(0,t)c_1(t) - c_1^\dagger(t)b(0,t)], \qquad (2.4)$$

where γ_1 is the coupling constant having the dimension of inverse time, and $c_1(t)$ is the annihilation operator of the cavity tuned to the frequency ck. Ignoring other dynamics, the evolution of an arbitrary Heisenberg operator $a(t)$ is

$$\dot{a}(t) = -[b^\dagger(0,t)c_1(t) - c_1^\dagger(t)b(0,t), a(t)]. \qquad (2.5)$$

Now because of the singularity of the canonical commutation relations (2.2), it is necessary to be careful in dealing with this evolution equation. A convenient method is to use quantum Ito stochastic differential calculus [15,17]. Define an input field, representing the field just before it interacts with the cavity at time t, by

$$b_1(t) = b(0^-,t). \qquad (2.6)$$

This can be thought of as a white-noise term, independent of the state of the cavity at time t. The analog of the Weiner increment in the Ito calculus is then

$$dB_1(t) = b_1(t)dt, \qquad (2.7)$$

which satisfies

$$[dB_1(t), dB_1^\dagger(t)] = dt. \qquad (2.8)$$

The evolution of an arbitrary operator is then given by

$$a(t+dt) = U_1^\dagger(t,t+dt)a(t)U_1(t,t+dt)\ , \qquad (2.9)$$

where

$$U_1(t,t+dt) = \exp\{\sqrt{\gamma_1}[dB_1^\dagger(t)c_1(t) - dB_1(t)c_1(t)]\}. \qquad (2.10)$$

In Eq. (2.9), the bath operators $dB_1(t)$ and $dB_1^\dagger(t)$ are independent of the system operator $a(t)$, which makes it an easy equation to solve. However, the price which must be paid by this simplification of Ito calculus is that $U_1(t,t+dt)$ must be expanded to second order in the increment $dB_1(t)$. Now if $b_1(t)$ is to be thought of as a bath, it should be specifiable simply by its moments. We need only the first and second order moments. The first order moment corresponds to a coherent amplitude

$$\langle dB_1(t)\rangle = \beta(t)dt, \qquad (2.11)$$

while the second order moments indicate white noise:

$$\langle dB_1^\dagger(t)dB_1(t)\rangle = Ndt = \langle dB_1(t)dB_1^\dagger(t)\rangle - 1\ , \qquad (2.12)$$

$$\langle dB_1(t)dB_1(t)\rangle = Mdt = \langle dB_1^\dagger(t)dB^\dagger(t)\rangle^*. \qquad (2.13)$$

These equations include the cases of thermal noise ($M=0$), and perfectly squeezed white noise [$|M|^2 = N(N+1)$]. It is nonclassical only if $|M| > N$. Note that even if $N=0$, it is necessary to expand $U_1(t,t+dt)$ to second order because then $dB_1(t)dB_1^\dagger(t) = dt$, which could be regarded as vacuum noise. The full result (with time arguments omitted for convenience) is the quantum Langevin equation

$$da = \frac{\gamma_1}{2}\{(N+1)(2c_1^\dagger a c_1 - a c_1^\dagger c_1 - c_1^\dagger c_1 a) + N(2c_1 a c_1^\dagger - a c_1 c_1^\dagger - c_1 c_1^\dagger a) + M[c_1^\dagger,[c_1^\dagger,a]] + M^*[c_1,[c_1,a]]\}dt - \sqrt{\gamma_1}[dB_1^\dagger c_1 - dB_1 c_1^\dagger, a]. \quad (2.14)$$

The final, stochastic term in this equation is essential to preserve canonical commutation relations. However, the stochastic terms can be ignored when changing from the Heisenberg to the Schrödinger picture and deriving the evolution of the density operator for the cavity mode alone. This is found from the relation

$$\langle da(t)\rangle = \mathrm{Tr}[d\rho(t)\, a], \quad (2.15)$$

where the picture (Schrödinger or Heisenberg) is specified by the placement of the time argument. The resulting master equation is

$$\dot\rho(t) = \gamma_1\{(N+1)\mathcal{D}[c_1]\rho + N\mathcal{D}[c_1^\dagger]\rho + \frac{M}{2}[c_1^\dagger,[c_1^\dagger,\rho]] + \frac{M^*}{2}[c_1,[c_1,\rho]]\} + \sqrt{\gamma_1}[\beta^*(t)c_1 - \beta(t)c_1^\dagger, \rho], \quad (2.16)$$

where we have defined a superoperator $\mathcal{D}$ taking an arbitrary operator as its argument by

$$\mathcal{D}[a]\rho = a\rho a^\dagger - \tfrac{1}{2}a^\dagger a\rho - \tfrac{1}{2}\rho a^\dagger a. \quad (2.17)$$

It might be thought that this master equation is the only product of the above theory which is of any significance, since it generates the evolution of the cavity mode. However, experimentally, it is often the light which leaves the cavity through the output mirror, rather than the internal state of the cavity, which is of interest. This is specifically so for this paper, in which the output of the cavity will be used in the feedback loop. Thus we need an expression for the field leaving the cavity. From Eq. (2.3), this is evidently given by

$$b_2(t) \equiv b(0^+,t) = U_1(t,t+dt)^\dagger b_1(t) U_1(t,t+dt). \quad (2.18)$$

To lowest order in dt, this is

$$b_2(t) = b_1(t) + \sqrt{\gamma_1}c_1(t). \quad (2.19)$$

Just as $b_1(t)$ is independent of, and so commutes with, an arbitrary system operator $a(t')$ at an earlier time $t' < t$, the output field commutes with all system operators at a later time.

Now let this field be the input into another cavity with annihilation operator $c_2(t)$. If the damping rate for this cavity is γ_2, then the Hamiltonian coupling is

$$H_2(t) = i\hbar\sqrt{\gamma_2}[b^\dagger(c\tau,t)c_2(t) - c_2^\dagger(t)b(c\tau,t)], \quad (2.20)$$

where $c\tau$ is the path length between the two cavities. Proceeding as above, the unitary evolution generated by this Hamiltonian is

$$U_2(t,t+dt) = \exp\{\sqrt{\gamma_2}[dB_2^\dagger(t-\tau)c_2(t) - dB_2(t-\tau)c_2^\dagger(t)]\}, \quad (2.21)$$

where $dB_2(t) = b_2(t)dt$. An arbitrary operator in the source or driven cavity obeys the equation

$$a(t+dt) = U_1^\dagger(t,t+dt)U_2^\dagger(t,t+dt)a(t) \times U_2(t,t+dt)U_1(t,t+dt). \quad (2.22)$$

Note that $U_2(t,t+dt)$ commutes with $U_1(t,t+dt)$ because of the finite τ, so the ordering in the above equation is not significant. Expanding the terms as above gives

$$da = \frac{\gamma_1}{2}\{(N+1)(2c_1^\dagger a c_1 - a c_1^\dagger c_1 - c_1^\dagger c_1 a) + N(2c_1 a c_1^\dagger - a c_1 c_1^\dagger - c_1 c_1^\dagger a) + M[c_1^\dagger,[c_1^\dagger,a]] + M^*[c_1,[c_1,a]]\}dt$$
$$+\frac{\gamma_2}{2}\{(N+1)(2c_2^\dagger a c_2 - a c_2^\dagger c_2 - c_2^\dagger c_2 a) + N(2c_2 a c_2^\dagger - a c_2 c_2^\dagger - c_2 c_2^\dagger a)$$
$$+M[c_2^\dagger,[c_2^\dagger,a]] + M^*[c_2,[c_2,a]]\}dt - \sqrt{\gamma_1}[dB_1^\dagger c_1 - dB_1 c_1^\dagger, a] - \sqrt{\gamma_2}[dB_2^\dagger c_2 - dB_2 c_2^\dagger, a]. \quad (2.23)$$

Here, the implicit time argument of a, c_1, and c_2 is t, while that of $dB_2 = dB_1 + \sqrt{\gamma_1}c_1 dt$ is $t-\tau$.

Equation (2.23) is an Ito equation in that it gives an explicit algorithm for calculating an infinitesimal increment in some operator, given all of the other operators at the start of the time interval. However, it is not an Ito equation in the sense that the noise terms are not independent of the other operators. Although the noise input $dB_1(t)$ is independent, $dB_1(t-\tau)$ in $dB_2(t-\tau)$ is not independent of an arbitrary system operator $a(t)$, as can be seen as follows:

$$a(t) = a(t-\tau) - \sqrt{\gamma_1}[dB_1^\dagger(t-\tau)c_1(t-\tau) - dB_1(t-\tau)c_1^\dagger(t-\tau), a(t-\tau)] + O(\tau). \quad (2.24)$$

To derive a master equation, it is necessary to take the formal limit $\tau \to 0$. The basic physics of the problem is independent of τ because, as yet, we are not considering feedback. This means that corrections of order τ in Eq. (2.24) can be ignored, after calculating the second order Ito corrections. By using this equation carefully, Eq. (2.23) can be converted to an Ito equation in which the noise terms are independent:

$$
\begin{aligned}
da = {} & (N+1)\left[\frac{\gamma_1}{2}(2c_1^\dagger a c_1 - a c_1^\dagger c_1 - c_1^\dagger c_1 a) + \frac{\gamma_2}{2}(2c_2^\dagger a c_2 - a c_2^\dagger c_2 - c_2^\dagger c_2 a)\right. \\
& \left. + \sqrt{\gamma_1\gamma_2}(c_2^\dagger a c_1 - a c_2^\dagger c_1 + c_1^\dagger a c_2 - c_1^\dagger c_2 a)\right] \\
& + N\left[\frac{\gamma_1}{2}(2c_1 a c_1^\dagger - a c_1 c_1^\dagger - c_1 c_1^\dagger a) + \frac{\gamma_2}{2}(2c_2 a c_2^\dagger - a c_2 c_2^\dagger - c_2 c_2^\dagger a)\right. \\
& \left. + \sqrt{\gamma_1\gamma_2}(c_2 a c_1^\dagger - a c_2 c_1^\dagger + c_1 a c_2^\dagger - c_1 c_2^\dagger a)\right] \\
& + M\left[\frac{\gamma_1}{2}[c_1^\dagger,[c_1^\dagger,a]] + \frac{\gamma_2}{2}[c_2^\dagger,[c_2^\dagger,a]] + \sqrt{\gamma_1\gamma_2}[c_1^\dagger,[c_2^\dagger,a]]\right] \\
& + M^*\left[\frac{\gamma_1}{2}[c_1,[c_1,a]] + \frac{\gamma_2}{2}[c_2,[c_2,a]] + \sqrt{\gamma_1\gamma_2}[c_1,[c_2,a]]\right] \\
& - \sqrt{\gamma_1}[dB_1^\dagger c_1 - dB_1 c_1^\dagger, a] - \sqrt{\gamma_2}[dB_1^\dagger c_2 - dB_1 c_2^\dagger, a]. \qquad (2.25)
\end{aligned}
$$

In this equation, all operators have the same time argument. It agrees with the results of Refs. [13,14] for the case $N = M = 0$, but it should be noted that the details of the derivation in both of these papers differ from ours.

It is now a simple matter to convert this stochastic Heisenberg equation into a master equation for the density operator of both systems W,

$$
\begin{aligned}
\dot{W} = {} & (N+1)\{\gamma_1\mathcal{D}[c_1]W + \gamma_2\mathcal{D}[c_2]W + \sqrt{\gamma_1\gamma_2}([c_1 W, c_2^\dagger] + [c_2, W c_1^\dagger])\} \\
& + N\{\gamma_1\mathcal{D}[c_1^\dagger]W + \gamma_2\mathcal{D}[c_2^\dagger]W + \sqrt{\gamma_1\gamma_2}([c_1^\dagger W, c_2] + [c_2^\dagger, W c_1])\} \\
& + M\left[\frac{\gamma_1}{2}[c_1^\dagger,[c_1^\dagger,W]] + \frac{\gamma_2}{2}[c_2^\dagger,[c_2^\dagger,W]] + \sqrt{\gamma_1\gamma_2}[c_2^\dagger,[c_1^\dagger,W]]\right] \\
& + M^*\left[\frac{\gamma_1}{2}[c_1,[c_1,W]] + \frac{\gamma_2}{2}[c_2,[c_2,W]] + \sqrt{\gamma_1\gamma_2}[c_2,[c_1,W]]\right] \\
& + \sqrt{\gamma_1}[\beta^*(t)c_1 - \beta(t)c_1^\dagger, W] + \sqrt{\gamma_2}[\beta^*(t)c_2 - \beta(t)c_2^\dagger, W] - i[H,W]\ . \qquad (2.26)
\end{aligned}
$$

Here we have finally included the intrinsic evolution for the two systems, generated by the Hamiltonian $\hbar H$. This is the general equation for two open quantum systems, linked unidirectionally by a bath of harmonic oscillators with an optical frequency coherent amplitude contaminated by white noise. (In the following sections, the coherent amplitude will be ignored.) It is a generalization of the equations published by Gardiner [13] and Carmichael [14], which treated only a bath in the vacuum state. It has the necessary property that, if H is the sum of Hamiltonians operating in the Hilbert subspaces of the two systems, then the source system (c_1) is unaffected by the driven system (c_2). That is to say, the density operator ρ for the source system (obtained by tracing over the driven system) obeys the original master equation (2.16). This is evident from the fact that the only terms in Eq. (2.26) containing operators from both systems involve an exterior commutator with a driven system operator, which gives zero when traced over the driven subspace. It is not possible in general to derive a master equation for the second system alone. Its evolution is literally driven by the source system.

B. Feedback

The simplest imaginable all-optical feedback scheme would be to take the light from one end of a cavity and reflect it back into the other, using a Faraday isolator to prevent interference from reflections in the opposite directions. For simplicity, we assume a bath in the vacuum state and a cavity with equal transmittivities at both end mirrors. Denote the input vacuum state by $b_1(t) = \nu(t)$, the cavity mode annihilation operator by a, and the damping rate by γ. Then the evolution of the cavity mode due to the first mirror is

$$\dot{a}(t) = -\frac{\gamma}{2}a(t) - \sqrt{\gamma}\nu(t), \qquad (2.27)$$

and the output field is $\sqrt{\gamma}a + \nu(t)$. If the time delay in the loop is τ, then the effect of the feedback is

$$\dot{a}_{\text{fb}}(t) = -\frac{\gamma}{2}a(t) - \sqrt{\gamma}[\sqrt{\gamma}a(t-\tau) + \nu(t-\tau)]. \qquad (2.28)$$

Provided that $\gamma\tau \ll 1$, the effect of the time delay can be ignored, apart from introducing an arbitrary phase factor depending on the optical path length of the loop. Because the bath is in a vacuum state, it is not necessary to worry about the Ito corrections used in going from Eq. (2.23) to Eq. (2.25). Thus the total evolution for a is

$$\dot{a}(t) = -(1+e^{i\phi})[\gamma a(t) + \sqrt{\gamma}\nu(t)]. \qquad (2.29)$$

The master equation equivalent to this quantum Langevin equation is

$$\dot{\rho} = 2\gamma(1+\cos\phi)\mathcal{D}[a]\rho - i\gamma\sin\phi[a^\dagger a, \rho]. \qquad (2.30)$$

Evidently, if $\phi = \pi$, the damping through the mirrors can be completely eliminated. Of course, this is an approximation only. In reality, the lifetime of the cavity is enhanced by a factor of order $1/\gamma\tau \gg 1$. Losses in the loop would also decrease the effectiveness of the feedback. For $\phi = 0$, the damping rate is doubled, and for other values of ϕ, the cavity becomes detuned, as well as having its damping rate altered. In Sec. V we will show that these features can be reproduced by feedback using a heterodyne detection photocurrent to control a driving field at the second mirror. However, electro-optical feedback will necessarily introduce extra noise. The all-optical model here obviously does not introduce noise; its results are a consequence of classical geometrical optics.

III. INTENSITY FEEDBACK

In this section we consider feedback of the form described in the Introduction. The light from the source cavity is used to drive a second cavity, which coincides spatially at least in part with the source cavity (see Fig. 1). The two modes interact via a nonlinear crystal, allowing the driven cavity to control the source cavity. The form of the interaction is assumed to depend only on the intensity of the driven cavity. It is thus analogous to feedback of the photon flux output of the source cavity. We analyze the feedback system using both master and quantum Langevin equations.

A. Master equation

The form of our feedback master equation is that of the general cascaded systems equation (2.26) derived in the preceding section. What distinguishes it as feedback is that the Hamiltonian $\hbar H$ is assumed to consist of a source cavity Hamiltonian $\hbar H_0$ plus an interaction between the source and driven cavity of the form

$$V = \hbar c_2^\dagger c_2 K \ , \tag{3.1}$$

where K is a Hermitian operator on the source cavity. In this, and all following sections, we will measure time in inverse units of the decay constant γ_1 for the source cavity. The decay constant γ_2 for the driven cavity will be denoted γ. Because of the interaction term, the dynamics of the source cavity is no longer independent of that of the driven cavity. Thus we must consider the master equation for the density operator of both modes W. According to Eq. (2.26), this obeys the master equation

$$\begin{aligned}\dot{W} = {}& -i[H_0 + c_2^\dagger c_2 K, W] + \mathcal{D}[c_1]W + \gamma\mathcal{D}[c_2]W \\ & + \sqrt{\gamma}([c_1 W, c_2^\dagger] + [c_2, W c_1^\dagger]).\end{aligned} \tag{3.2}$$

Note that here we have assumed that the bath is in the vacuum state ($N = 0$). The reason for this will become apparent later.

In deriving Eq. (3.2), it is of course necessary to set $\tau = 0$. However, this is no longer a formal mathematical limit. Rather, it is a physical condition that the time delay in the feedback loop should be negligible. This is usually desirable in feedback loops. A substantial time delay may lead to instabilities and chaos, rather than control. Similarly, for the feedback to be effective, the second cavity should respond much faster than the first. This ensures that the state of c_2 is effectively slaved to that of c_1, and the interaction effects instantaneous feedback as far as the source is concerned. Thus, in the limit $\gamma \gg 1$, it should be possible to derive a master equation including feedback for the source density operator ρ alone. To do this we note that since $\gamma \gg 1$ and the bath is in the vacuum state, the driven cavity will be very close to being in the vacuum state also. This enables the adiabatic elimination procedure we have used previously [18] to be employed again.

We expand W in powers of $1/\sqrt{\gamma}$ as

$$W = \rho_0 \otimes |0\rangle\langle 0| + [\rho_1 \otimes |1\rangle\langle 0| + \text{H.c.}] + \rho_2 \otimes |1\rangle\langle 1|, \tag{3.3}$$

where the ρ's exist in the source subspace and the other operators in the driven subspace. There is another term of second order in $1/\sqrt{\gamma}$, but it plays no part in the procedure which follows. Substituting the above expansion into the master equation (3.2) gives the following coupled equations:

$$\dot{\rho}_0 = \gamma\rho_2 + \sqrt{\gamma}(\rho_1 c_1^\dagger + c_1\rho_1^\dagger) + \mathcal{L}_0\rho_0 \ , \tag{3.4a}$$

$$\dot{\rho}_1 = -\tfrac{1}{2}\gamma\rho_1 - iK\rho_1 - \sqrt{\gamma}[c_1\rho_0 + O(1/\gamma)] + \mathcal{L}_0\rho_1 \ , \tag{3.4b}$$

$$\dot{\rho}_2 = -\gamma\rho_2 - i[K, \rho_2] - \sqrt{\gamma}(\rho_1 c_1^\dagger + c_1\rho_1^\dagger) + \mathcal{L}_0\rho_2. \tag{3.4c}$$

Here $\mathcal{L}_0\rho \equiv \mathcal{D}[c_1]\rho - i[H_0, \rho]$. In this approximation, the source density operator is $\rho = \rho_0 + \rho_2$ which evidently obeys

$$\dot{\rho} = -i[K, \rho_2] + \mathcal{L}_0\rho. \tag{3.5}$$

To turn this into a master equation, we require an expression for ρ_2 in terms of $\rho \simeq \rho_0$. It is now obvious why we assumed a zero temperature bath. For N finite, ρ_2 would have a finite size irrespective of γ. Thus the signal due to the driving from the source, which is of order $1/\gamma$, would be swamped by the noise, and the feedback would not work.

To obtain ρ_2 it is first necessary to obtain ρ_1. Since almost all of the probability is in ρ_0, it is evident from Eq. (3.4b) that ρ_1 relaxes much more rapidly than $\rho_0 \simeq \rho$. It is thus permissible to set ρ_1 equal to its steady-state value of

$$\rho_1 = \left(1 + i\frac{2K}{\gamma}\right)^{-1} \frac{-2}{\sqrt{\gamma}} c_1\rho. \tag{3.6}$$

The expression on the right hand side will be a well-defined operator if we assume that $|K| \ll \gamma$ in some sense. This assumption will be valid in practice, as single-photon nonlinearities are typically much smaller than damping rates. It allows the denominator to be expanded to first order in K/γ. Since ρ_1 is now slaved to ρ_0, it will evolve at a rate much smaller than γ. Thus, from Eq. (3.4c), ρ_2 will relax to a steady state determined by the slaved value of ρ_1:

$$\dot{\rho}_2 = -\gamma\rho_2 - i[K,\rho_2] + 4c_1\rho c_1^\dagger - 4i\gamma^{-1}[K, c_1\rho c_1^\dagger] + \mathcal{L}_0\rho_2. \tag{3.7}$$

The slaved value of ρ_2, again to first order in K/γ, is

$$\rho_2 = \frac{4c_1\rho c_1^\dagger}{\gamma} - \frac{4i}{2\gamma}\left[\frac{4K}{\gamma}, c_1\rho c_1^\dagger\right]. \tag{3.8}$$

Substituting this into Eq. (3.5) gives the master equation

$$\dot{\rho} = -i[Z, c_1\rho c_1^\dagger] - \tfrac{1}{2}[Z,[Z,c_1\rho c_1^\dagger]] + \mathcal{D}[c_1]\rho - i[H_0,\rho], \tag{3.9}$$

where we have defined

$$Z = 4K/\gamma. \tag{3.10}$$

This master equation is the general equation for Markovian, intensity-dependent feedback in the small Z limit. Unfortunately, it is not a valid master equation [19] in the sense that it cannot be written in the form

$$\dot{\rho} = -i[H,\rho] + \sum_\mu \mathcal{D}[c_\mu]\rho, \tag{3.11}$$

where the c_μ are arbitrary operators. However, there is an equation of this form which is equal to Eq. (3.9) when expanded to second order in Z. That equation is

$$\dot{\rho} = -i[H_0,\rho] + \mathcal{D}[e^{-iZ}c_1]\rho. \tag{3.12}$$

This equation has been previously derived as a feedback master equation appropriate to electro-optical feedback [8]. This can be seen more clearly by rewriting Eq. (3.12) as

$$\begin{aligned}\rho(t+dt) = {} & \exp[(-iH_0 - \tfrac{1}{2}c_1^\dagger c_1)dt]\rho(t) \\ & \times \exp[(iH_0 - \tfrac{1}{2}c_1^\dagger c_1)dt] \\ & + e^{-iZ}c_1\rho(t)c_1^\dagger e^{iZ}dt.\end{aligned} \tag{3.13}$$

The two terms in this equation can be given an interpretation in terms of density operators conditioned on possible photon detections. The norm of each term gives the probability for the associated event. The first term represents the conditioned density operator when no photon is detected [12]. Note that it is only infinitesimally changed, and that the probability for this event is thus very close to unity. The second term represents the evolved density operator if a photon is detected at the time t. This does not happen very often, but when it does, the state of the system jumps (changes discontinuously) via the application of the operator c_1 [12]. The effect of the feedback is to cause some finite unitary evolution immediately following the detection, via the operator e^{-iZ}. Of course this does not change the norm of this term. The physical interpretation of the operator Z is that it can be derived from a time-dependent feedback Hamiltonian

$$H_{\rm fb} = \hbar Z I(t), \tag{3.14}$$

where $I(t)$ is the photocurrent derived from the detector, measured in units of detections per second. This could be produced by a variety of electro-optic devices.

We thus see that there is a strong correspondence between electro-optical feedback using direct detection and all-optical feedback with the intensity-dependent coupling (3.1). One conclusion which can be drawn from this analogy is that, as was previously established for electro-optical feedback [7], the all-optical feedback considered in this section cannot produce a nonclassical state in the source cavity if K is a classical operator. By "classical" in this context we mean a linear combination of $c_1, c_1^\dagger$, and $c_1^\dagger c_1$. (K is of course Hermitian.) Given that the nonlinear interaction Hamiltonian (3.1) is already of second order in the field of the driven (c_2) cavity, any coupling with a nonclassical K would require at least a five-wave mixing interaction (including an auxiliary pump field). This seems exceedingly impractical. Thus we can conclude that intensity-dependent all-optical feedback is not a practical way to produce a nonclassical source state.

B. Langevin equation

The above derivations for the all-optical and electro-optical intensity feedback were done using a master equation for the density operator. It is possible to use the Langevin approach for both types of feedback, and this yields some extra information. We begin with the all-optical feedback. The quantum Langevin equation equivalent to the master equation (3.2) is

$$\begin{aligned}\dot{a} = {} & \frac{1}{2}(2c_1^\dagger a c_1 - a c_1^\dagger c_1 - c_1^\dagger c_1 a) \\ & + \frac{\gamma}{2}(2c_2^\dagger a c_2 - a c_2^\dagger c_2 - c_2^\dagger c_2 a) - [b_1^\dagger c_1 - b_1 c_1^\dagger, a] \\ & - \sqrt{\gamma}[b_2^\dagger c_2 - b_2 c_2^\dagger, a] + i[H_0 + c_2^\dagger c_2 K, a].\end{aligned} \tag{3.15}$$

Here b_1 is the input vacuum field, and $b_2 = b_1 + c_1$ is the output field from the first cavity. This field which is fed back is to be understood to be at a slightly earlier time than all of the other operators in the above equation. For this reason, it commutes with all other operators. If b_2 is moved to the rear (far right) of any operator expression, the vacuum noise of b_2 will not contribute to any average, as the annihilation operator will act directly on the bath in the vacuum state. Similar remarks hold for moving $b_2^\dagger$ to the front of any expression. Thus, if we always put the bath operators in normal order as described here, then the fact that it is at a slightly earlier time can be ignored. Of course, this technique only works because the bath is in the vacuum state. In general, the time delay causes the corrections derived in Sec. II. In the remainder of this section, we will always put b_2 in normal order without mentioning this explicitly.

We wish to adiabatically eliminate mode c_2. This obeys

$$\dot{c}_2 = -\frac{\gamma}{2}c_2 - \sqrt{\gamma}b_2 - iKc_2. \tag{3.16}$$

Now the relaxation rate γ of mode c_2 cannot be much greater than the bandwidth of the vacuum fluctuations, which are assumed infinite. Hence, it is not strictly possible to slave c_2 to the vacuum fluctuations in b_2. However, as far as mode c_1 is concerned, vacuum fluctuations restricted to a bandwidth of γ are still effectively white, and so there is no harm in pretending that c_2 is slaved to the original vacuum fluctuations. This allows us to write the slaved value of c_2 as

$$c_2 = \left[1 - \frac{2iK}{\gamma}\right] \frac{-2}{\sqrt{\gamma}} b_2 \ . \tag{3.17}$$

Substituting this into the equation for a source cavity operator a gives

$$\begin{aligned}\dot{a} = &\frac{1}{2}(2c_1^\dagger a c_1 - a c_1^\dagger c_1 - c_1^\dagger c_1 a) - \nu^\dagger [c_1, a] + [c_1^\dagger, a]\nu \\ &+ i\frac{4}{\gamma} b_2^\dagger [K, a] b_2 - \frac{8}{\gamma^2} b_2^\dagger [K, [K, a]] b_2 + i[H_0, a]. \end{aligned}\tag{3.18}$$

Here we have set $b_1 = \nu$ because the equation is only valid when the input is a vacuum. Substituting in the expression for b_2 gives

$$\begin{aligned}\dot{a} = &c_1^\dagger a c_1 - \tfrac{1}{2} a c_1^\dagger c_1 - \tfrac{1}{2} c_1^\dagger c_1 a - \nu^\dagger [c_1, a] + [c_1^\dagger, a]\nu \\ &+ (c_1^\dagger + \nu^\dagger)\left(i[Z, a] - \tfrac{1}{2}[Z, [Z, a]]\right)(c_1 + \nu) \\ &+ i[H_0, a], \end{aligned}\tag{3.19}$$

where Z is as in Eq. (3.10). It is easy to verify by tracing over the bath that this equation is equivalent to the master equation (3.9).

Just as master equation (3.9) was an approximation to a strictly valid master equation, so is the Langevin equation (3.19). The details of the following are to be found in Ref. [8]. The exact Langevin equation can be derived from the Hamiltonian

$$H_{\text{fb}} = \hbar Z b_2^\dagger b_2. \tag{3.20}$$

This turns out to be equivalent to the photocurrent-dependent Hamiltonian (3.14). That is, simply by replacing the photocurrent with the output photon flux $b_2^\dagger b_2$, it is possible to derive a Langevin picture equation equivalent to the full master equation (ME) (3.12):

$$\begin{aligned}\dot{a} = &c_1^\dagger a c_1 - \tfrac{1}{2} a c_1^\dagger c_1 - \tfrac{1}{2} c_1^\dagger c_1 a - \nu^\dagger [c_1, a] + [c_1^\dagger, a]\nu \\ &+ (c_1^\dagger + \nu^\dagger)\left(e^{iZ} a e^{-iZ} - a\right)(c_1 + \nu) + i[H_0, a]. \end{aligned}\tag{3.21}$$

To second order in Z, this is equivalent to the all-optical Langevin equation derived above. Unlike Eq. (3.19), however, Eq. (3.21) is a completely valid Langevin equation so that if a_1 and a_2 are two arbitrary operators, then the equation of motion it generates for the product operator $a_1 a_2$ is equal to what would be obtained from the two operators separately, using the Ito rules for quantum stochastic calculus

$$d(a_1 a_2) = (da_1) a_2 + a_1 (da_2) + (da_1)(da_2). \tag{3.22}$$

Thus it is also possible to find a correspondence between all-optical and electro-optical feedback using quantum Langevin equations.

One property that a Langevin equation has which the master equation lacks is that it simply gives an expression for the field reflected from the driven cavity (see Fig. 1). Calling this field b_3, we have

$$b_3 = b_2 + \sqrt{\gamma} c_2. \tag{3.23}$$

From the adiabatic expression for c_2, we have

$$b_3 = -\left[1 - \frac{4iK}{\gamma}\right] b_2. \tag{3.24}$$

Apart from a change of sign (due to reflection), this is equal to what would be obtained from the idealized Hamiltonian (3.20),

$$b_3 = e^{-iZ} b_2 \tag{3.25}$$

to first order in Z. The statements which were made above regarding the inability of a classical feedback operator Z to produce a nonclassical source state do not necessarily apply to the output operator b_3. It may exhibit nonclassical features even if the source state is classical. We will not pursue the properties of this output field further in this section, but we will in the next section where we consider quadrature-dependent feedback.

IV. QUADRATURE FEEDBACK

A. Master equation

We begin this section on quadrature feedback by reproducing the calculation of Sec. III A, but with the interaction between the source and driven cavity

$$V = \hbar(c_2 + c_2^\dagger) J, \tag{4.1}$$

which is linear in the real quadrature of the driven cavity. Here J is a Hermitian operator in the source cavity. In practice, conservation of energy would require at least one other auxiliary field which may be treated classically. With the input to the source cavity in the vacuum state as before, and with the the damping rate γ of the second cavity much greater than that of the first (set to unity), it is again possible to expand the combined density operator in powers of $1/\sqrt{\gamma}$ as in Eq. (3.3). In this case, the source cavity operators obey

$$\dot{\rho}_0 = \gamma \rho_2 + \sqrt{\gamma}(\rho_1 c_1^\dagger + c_1 \rho_1^\dagger) - i[J\rho_1 - \rho_1^\dagger J] + \mathcal{L}_0 \rho_0 \ , \tag{4.2a}$$

$$\begin{aligned}\dot{\rho}_1 = &-\tfrac{1}{2}\gamma \rho_1 - i[J\rho_0 + O(1/\gamma)] \\ &- \sqrt{\gamma}[c_1 \rho_0 + O(1/\gamma)] + \mathcal{L}_0 \rho_1 \ , \end{aligned}\tag{4.2b}$$

$$\dot{\rho}_2 = -\gamma \rho_2 - i[J\rho_1^\dagger - \rho_1 J] - \sqrt{\gamma}(\rho_1 c_1^\dagger + c_1 \rho_1^\dagger) + \mathcal{L}_0 \rho_2 \ . \tag{4.2c}$$

The approximate source density operator $\rho = \rho_0 + \rho_2$ obeys

$$\dot{\rho} = -i[J, \rho_1 + \rho_1^\dagger] + \mathcal{L}_0 \rho. \tag{4.3}$$

Evidently, to derive a ME for ρ in this case it is only necessary for ρ_1 to be slaved to $\rho = \rho_0 + O(1/\gamma)$. From Eq. (4.2b), this is obviously true for large γ, with the slaved value

$$\rho_1 = \frac{2}{\gamma}[-\sqrt{\gamma}c_1\rho - iJ\rho]. \tag{4.4}$$

Here we are keeping only the lowest order in $1/\gamma$, and assuming that J is of order $\sqrt{\gamma}$. Substituting this expression into Eq. (4.3) gives the master equation

$$\dot{\rho} = -i[Y, c_1\rho + \rho c_1^\dagger] - \tfrac{1}{2}[Y,[Y,\rho]] + \mathcal{D}[c_1]\rho - i[H_0,\rho], \tag{4.5}$$

where we have defined

$$Y = \frac{-2J}{\sqrt{\gamma}}, \tag{4.6}$$

which is of order unity. This master equation can be put into the form required of master equations thus:

$$\dot{\rho} = \mathcal{D}[c_1 - iY]\rho - i[H_0 + \tfrac{1}{2}(c_1^\dagger Y + Yc_1),\rho]. \tag{4.7}$$

This feedback master equation has been previously derived from a model of electro-optical feedback using homodyne detection [6,7]. If the output of the cavity is subject to unit efficiency homodyne detection of the real quadrature, then the state of the cavity conditioned on the measured photocurrent obeys

$$d\rho_c(t) = dt\,\{-i[H_0,\rho_c(t)] + \mathcal{D}[c_1]\rho_c(t)\} + dW(t)\mathcal{H}[c_1]\rho_c(t), \tag{4.8}$$

where the nonlinear superoperator $\mathcal{H}$ is defined by

$$\mathcal{H}[a]\rho = a\rho + \rho a^\dagger - \mathrm{Tr}[(a\rho + \rho a^\dagger)]\rho, \tag{4.9}$$

and where dW is an infinitesimal real Weiner increment satisfying $dW(t)^2 = dt$. The subscript c indicates that the state is conditioned on the signal photocurrent, which is given by

$$I_c(t) = \langle c_1 + c_1^\dagger\rangle_c(t) + \xi(t), \tag{4.10}$$

where $\xi(t)dt = dW(t)$ represents Gaussian white noise [20]. These equations can be derived from the quantum jumps mentioned in Sec. III A, in the limit that the local oscillator amplitude used in the homodyne detection becomes infinitely large [9,18]. The feedback is then effected by introducing the time-dependent Hamiltonian

$$H_{\text{fb}}(t) = \hbar I_c(t)Y. \tag{4.11}$$

Taking into account that the photocurrent in (4.11) is in reality smooth [unlike the mathematical expression (4.10)], and that its time argument must include a delay, one can derive the master equation (4.7) relatively straightforwardly.

B. Langevin equation

If the input to the source cavity is not a vacuum, then the technique of adiabatic elimination of Sec. IV A cannot be used. Unlike the intensity-dependent feedback of Sec. III, it is nevertheless possible to obtain a sensible result for nonvacuum input using a Langevin equation approach. The difference arises because the quadrature flux of a broadband squeezed or thermal state is a well-defined stochastic quantity, whereas the photon flux is not well defined (it tends to infinity in the limit of white noise). Because of the thermal and squeezed terms, it is necessary to use the explicit quantum equation (2.23) also including the time delay τ. Adding the intercavity coupling gives the feedback Langevin equation for an arbitrary operator a,

$$\begin{aligned} da = {}& \tfrac{1}{2}\{(N+1)(2c_1^\dagger a c_1 - a c_1^\dagger c_1 - c_1^\dagger c_1 a) + N(2c_1 a c_1^\dagger - a c_1 c_1^\dagger - c_1 c_1^\dagger a) + M[c_1^\dagger,[c_1^\dagger,a]] + M^*[c_1,[c_1,a]]\}dt \\ & + \frac{\gamma}{2}\{(N+1)(2c_2^\dagger a c_2 - a c_2^\dagger c_2 - c_2^\dagger c_2 a) + N(2c_2 a c_2^\dagger - a c_2 c_2^\dagger - c_2 c_2^\dagger a) \\ & + M[c_2^\dagger,[c_2^\dagger,a]] + M^*[c_2,[c_2,a]]\}dt \\ & - [dB_1^\dagger c_1 - dB_1 c_1^\dagger, a] - \sqrt{\gamma}[dB_2^\dagger c_2 - dB_2 c_2^\dagger, a] + i[H_0 + (c_2 + c_2^\dagger)J, a]dt, \end{aligned} \tag{4.12}$$

where the time argument of $dB_2 = dB_1 + c_1 dt$ is $t - \tau$, and all other operators are at time t. An operator a for the source cavity will obey the equation

$$\begin{aligned} da = {}& \tfrac{1}{2}\{(N+1)(2c_1^\dagger a c_1 - a c_1^\dagger c_1 - c_1^\dagger c_1 a) + N(2c_1 a c_1^\dagger - a c_1 c_1^\dagger - c_1 c_1^\dagger a) \\ & + M[c_1^\dagger,[c_1^\dagger,a]] + M^*[c_1,[c_1,a]]\}dt \\ & - [dB_1^\dagger c_1 - dB_1 c_1^\dagger, a] + i[H_0 + (c_2 + c_2^\dagger)J, a]dt. \end{aligned} \tag{4.13}$$

Evidently, to obtain a Langevin equation for a source cavity operator involving no driven cavity operators, it is necessary only to adiabatically eliminate c_2. From Eq. (4.12), this obeys

$$\dot{c_2}(t) = -\frac{\gamma}{2}c_2(t) - \sqrt{\gamma}b_2(t-\tau) - -iJ(t). \tag{4.14}$$

As before, if $\gamma \gg 1$, then there is no harm in replacing c_2 by the slaved value

$$c_2(t) = \frac{-2}{\sqrt{\gamma}}\left[b_2(t-\tau) + \frac{iJ(t)}{\sqrt{\gamma}}\right], \tag{4.15}$$

providing that the resulting term in the Langevin equation is treated in the Stratonovich sense. That is to say, the effective feedback Hamiltonian

$$H_{\rm fb}(t) = \hbar(c_2 + c_2^\dagger)J = i\hbar\left[b_2^\dagger(t-\tau)\left(\frac{2iJ(t)}{\sqrt{\gamma}}\right) - \left(\frac{-2iJ(t)}{\sqrt{\gamma}}\right)b_2(t-\tau)\right] \tag{4.16}$$

is to be treated in the same manner as the coupling Hamiltonian (2.20). The effective Hamiltonian (4.16) has the same form as Eq. (2.20), but for the replacement of $\sqrt{\gamma_2}c_2$ by $-iY$ [as defined above (4.6)]. Thus it is possible to use the same procedure to derive a Langevin equation for a source cavity operator a in the limit $\tau \to 0$. From Eq. (2.25), this is immediately seen to be

$$\begin{aligned} da = {}& (N+1)\{\tfrac{1}{2}(2c_1^\dagger a c_1 - a c_1^\dagger c_1 - c_1^\dagger c_1 a) - \tfrac{1}{2}[Y,[Y,a]] + (i[Y,a]c_1 + ic_1^\dagger[Y,a])\} \\ & + N\{\tfrac{1}{2}(2c_1 a c_1^\dagger - a c_1 c_1^\dagger - c_1 c_1^\dagger a) - \tfrac{1}{2}[Y,[Y,a]] + (-i[Y,a]c_1^\dagger - ic_1[Y,a])\} \\ & + M\tfrac{1}{2}\{[c_1^\dagger,[c_1^\dagger,a]] - [Y,[Y,a]] + 2i[c_1^\dagger,[Y,a]]\} + M^*\tfrac{1}{2}\{[c_1,[c_1,a]] - [Y,[Y,a]] - 2i[c_1,[Y,a]]\} \\ & - [dB_1^\dagger(c_1 - iY) - dB_1(c_1^\dagger + iY), a] + i[H_0, a]dt, \end{aligned} \tag{4.17}$$

where dB_1 is a true Ito increment.

The master equation corresponding to this Langevin equation is now easy to derive. It is

$$\begin{aligned} \dot{\rho} = {}& (N+1)\{\mathcal{D}[c_1]\rho - i[Y, c_1\rho + \rho c_1^\dagger]\} + N\{\mathcal{D}[c_1^\dagger]\rho + i[Y, c_1^\dagger\rho + \rho c_1]\} \\ & + M\{\tfrac{1}{2}[c_1^\dagger,[c_1^\dagger,\rho]] + i[Y,[c_1^\dagger,\rho]]\} + M^*\{\tfrac{1}{2}[c_1,[c_1,\rho]] - i[Y,[c_1,\rho]]\} \\ & + (2N+1+M+M^*)\mathcal{D}[Y]\rho - i[H_0,\rho]. \end{aligned} \tag{4.18}$$

This is the general master equation for quadrature-based feedback in the presence of quantum white noise. For a vacuum input, it reduces to the master equation derived in the preceding section (4.7). Recall that there are two derivations of Eq. (4.7), from the all-optical model which we have just generalized for a nonvacuum input, and from an electro-optical model of feedback based on homodyne detection [6,7]. The correspondence between the two models is again close, as they can both be derived directly from a time-dependent Hamiltonian. In the case of all-optical feedback, we have just seen that the effective feedback Hamiltonian (4.16) is

$$H_{\rm fb} = \hbar(b_2 + b_2^\dagger)Y. \tag{4.19}$$

This is identical to the current controlled Hamiltonian (4.11) if the homodyne signal is identified with the quadrature flux output of the cavity.

This close relationship suggests that it should be possible to derive the full master equation (4.18) from a measurement-theory approach to homodyne measurement in the presence of noise. Such a theory has not been published before, to our knowledge. The difficulty in formulating the theory is that any white noise will, in theory, cause an infinite photon flux at the detectors. Thus, for any finite local oscillator strength, the signal will be swamped, just as it is for direct detection. An alternative approach is needed, in which the limit of infinite local oscillator strength is taken before determining the effect of the measurement on the system. After some effort, it can be shown that the generalization of the homodyne photocurrent expression (4.10) is

$$I(t) = \langle c_1 + c_1^\dagger\rangle(t) + \sqrt{L}\,\xi(t), \tag{4.20}$$

where we have defined

$$L = 2N + 1 + M + M^*, \tag{4.21}$$

and $\xi(t)$ is δ-function normalized real Gaussian noise as before. Note that L can be less than one (its vacuum value) in the case of squeezed input noise. The stochastic evolution of the system given this photocurrent is

$$\begin{aligned} d\rho(t) = {}& dt\left\{(N+1)\mathcal{D}[c_1]\rho + N\mathcal{D}[c_1^\dagger]\rho + \frac{M}{2}[c_1^\dagger,[c_1^\dagger,\rho]] + \frac{M^*}{2}[c_1,[c_1,\rho]] - i[H_0,\rho]\right\} \\ & + \frac{1}{\sqrt{L}}dW(t)\,\mathcal{H}\left[(N+M+1)c_1 - (N+M^*)c_1^\dagger\right]\rho, \end{aligned} \tag{4.22}$$

where the nonlinear superoperator $\mathcal{H}$ is as defined in (4.9). If the current (4.20) is fed back in the same manner as before [Eq. (4.11)], then a master equation can be derived using the same procedure as in the case of a vacuum input. The result is, not surprisingly, Eq. (4.18). Thus there is a complete correspondence between electro-optical and all-optical feedback in the case of quadrature-dependent feedback.

C. In-loop and output squeezing

As with intensity-dependent feedback, the all-optical quadrature-dependent feedback scheme described above is not a good way to try to produce nonclassical light in the source cavity. This is evident from the master equation (4.5). Ignoring H_0, this will only produce squeezing if Y is a nonclassical operator (see preceding section). Given the origin of Y in the original coupling (4.1), it is evident that at least a $\chi^{(3)}$ nonlinearity would be required to produce nonclassical source light. However, as noted in the preceding section, this requirement does not hold for the production of squeezing in the output light from the second cavity. Of course, this beam of light does not exist in the electro-optical case; it is only present for an all-optical feedback system. It turns out that this light can exhibit complete squeezing on resonance, even though the source cavity is in a classical state.

It is easiest to examine the in-loop and output fields in the Langevin equation approach. The output field operator b_3 from the driven cavity is related to the in-loop field (denoted b_2 as above) by

$$b_3 = b_2 + \sqrt{\gamma} c_2. \tag{4.23}$$

Using the slaved value of c_2 (4.15), this gives

$$b_3 = -(b_2 - iY) = -(b_1 + c_1 - iY). \tag{4.24}$$

Apart from the unimportant sign change, this is just what would be expected from the effective Hamiltonian (4.19).

Now consider the simplest possible case, where b_1 is in a vacuum state and Y is proportional to the imaginary quadrature of the source cavity

$$Y = -\frac{\lambda}{2}(-ic_1 + ic_1^\dagger). \tag{4.25}$$

This gives linear equations of motion by which the feedback alters the statistics of the real quadrature. Specifically, using $x = c_1 + c_1^\dagger$ and $y = -ic_1 + ic_1^\dagger$, we get from Eq. (4.17)

$$\dot{x} = -\left(\tfrac{1}{2}x + \nu_x\right) - \lambda(x + \nu_x) \ , \tag{4.26}$$

$$\dot{y} = -\left(\tfrac{1}{2}y + \nu_y\right) . \tag{4.27}$$

Here ν_x and ν_y are vacuum quadrature noise operators which obey

$$\langle \nu_x(t)\nu_x(t')\rangle = \langle \nu_y(t)\nu_y(t')\rangle = \delta(t - t') \ , \tag{4.28}$$

$$[\nu_x(t), \nu_y(t')] = 2i\delta(t - t'). \tag{4.29}$$

Since the equations for the two quadratures are uncoupled, the operator nature of ν_x and ν_y [as evidenced by Eq. (4.29)] is mostly unimportant. Hence it is easy to derive the steady state variance in x to be

$$V(x)_{\rm ss} = \frac{(1+\lambda)^2}{1+2\lambda}. \tag{4.30}$$

This is always greater than one for any nonzero λ, which is as expected since this sort of feedback cannot produce a squeezed intracavity state.

To examine the extracavity fields, it is necessary to consider the noise at different frequencies. Defining the Fourier transform of a quantity by

$$\tilde{a}(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} a(t) dt, \tag{4.31}$$

Eqs. (4.26) and (4.27) become

$$\tilde{x}(\omega) = -\frac{1+\lambda}{\frac{1}{2} + \lambda - i\omega}\tilde{\nu}_x(\omega) \ , \tag{4.32}$$

$$\tilde{y}(\omega) = -\frac{1}{\frac{1}{2} - i\omega}\tilde{\nu}_y(\omega). \tag{4.33}$$

The frequency domain counterparts to the time domain relationships (4.28) and (4.29) are identical but for the replacement of $\delta(t - t')$ by $2\pi\delta(\omega + \omega')$. The in-loop quadratures of b_2 are

$$\tilde{x}_2(\omega) = -\frac{\frac{1}{2} + i\omega}{\frac{1}{2} + \lambda - i\omega}\tilde{\nu}_x(\omega) \ , \tag{4.34}$$

$$\tilde{y}_2(\omega) = -\frac{\frac{1}{2} + i\omega}{\frac{1}{2} - i\omega}\tilde{\nu}_y(\omega). \tag{4.35}$$

These obey the commutation relations

$$[\tilde{x}_2(\omega), \tilde{y}_2(\omega')] = 4\pi i\delta(\omega + \omega')\frac{\frac{1}{4} + \omega^2}{\frac{1}{4} + \omega^2 + \lambda\left(\frac{1}{2} + i\omega\right)}. \tag{4.36}$$

At first sight, the presence of the second factor here would seem to be a flaw in the theory, as it shows that the in-loop field does not obey the usual commutation relations for a free field. The commutator in Eq. (4.36) vanishes for low frequencies in the limit $\lambda \to \infty$. However, it must be remembered that the canonical commutation relations (2.2) for the electromagnetic field are defined between the field at different points in space, but at the same time. If they are defined in terms of frequency components, as here, then these are strictly speaking spatial, not temporal, frequencies. It is only for a field which is free to propagate over an infinite space that the distinction vanishes. In our case, the in-loop field is spatially confined to a length $c\tau$, and we have let τ go to 0. If we put τ back in the equations, then it is not difficult to see that the only modification is to replace λ in the frequency domain by $\lambda e^{i\omega\tau}$. With this replacement in the above frequency commutation relations, it is possible to show [4] that the time commutation relations are only changed for times greater than τ. That is to say, the canonical (spatial) commutation relations are never violated. The field with operator $b_2(t)$ does not persist for longer than the time τ as it travels from the first to the second cavity, and so there are never in existence two values of the field which violate the canonical commutation relations.

Returning to the output field, we find from Eq. (4.24) that the quadratures of the output field b_3 are (including the time delay τ)

$$\tilde{x}_3(\omega) = \frac{e^{i\omega\tau}\left(\frac{1}{2} + i\omega\right)}{\frac{1}{2} + \lambda e^{i\omega\tau} - i\omega}\tilde{\nu}_x(\omega) \ , \tag{4.37}$$

$$\tilde{y}_3(\omega) = \frac{e^{i\omega\tau}\left(\frac{1}{2} + i\omega\right) + \lambda}{\frac{1}{2} - i\omega}\tilde{\nu}_y(\omega). \tag{4.38}$$

That is to say, the x quadrature of the output is unchanged from the b_2 value (apart from the phase change due to the time delay), but the y quadrature has picked up an extra term. This extra term ensures that

$$[\tilde{x}_3(\omega), \tilde{y}_3(\omega')] = 4\pi i\delta(\omega + \omega'), \qquad (4.39)$$

as required because these fields may propagate to infinity. The spectrum for the real quadrature (which is equal to the spectrum of the photocurrent from a homodyne measurement) is defined by

$$S^x(\omega) = \int_{-\infty}^{\infty} \langle \tilde{x}(\omega)\tilde{x}(-\omega')\rangle d\omega', \qquad (4.40)$$

and similarly for y. For the output field b_3 we get

$$S_3^x(\omega) = \frac{\frac{1}{4} + \omega^2}{|\frac{1}{2} + \lambda e^{i\omega\tau} - i\omega|^2}, \qquad (4.41)$$

$$S_3^y(\omega) = 1/S_3^x(\omega). \qquad (4.42)$$

Note that for $\lambda \to \infty$, we get perfect squeezing at low frequencies for the x quadrature, and infinite noise in the y quadrature, as required by Heisenberg's uncertainty relations.

Is there a simple way to understand this result, that the output may be perfectly squeezed, even though the cavity variance in x is unbounded [see Eq. (4.30)]? We look for an answer by an analogy with electro-optical feedback. Obviously the above analogy with electro-optical feedback mediated by homodyne detection will not suffice, because such detection destroys the output beam. What would be necessary would be a QND (quantum-nondemolition) [21] measurement of the output field. Feedback based on a QND measurement of output quadrature would have the same effect on the intracavity field as homodyne detection, provided it was efficient. It is not able to produce intracavity squeezing because an extracavity measurement is a poor measurement of the intracavity quadrature. However, if the output field x quadrature emerges from the QND device unchanged (or relatively little changed), then this quantity (the output field quadrature) can be well controlled by feedback. The fact that the feedback acts on the source cavity is relevant only so far as it affects the output. Such QND feedback schemes have been considered by Yamamoto and co-workers [2,3] and Shapiro *et al.* [4].

In the all-optical quadrature feedback scheme, the second cavity can be considered as a QND apparatus for the x quadrature output of the source cavity. As shown above, it does not alter the statistics of the x quadrature which reflects from it. However, it increases the variance of the output y quadrature, as required for a QND apparatus. The nonlinearity used in this case (coupling the quadrature of one mode to that of another) is precisely what has been used to model ideal QND quadrature measurements [21,7]. Rather than giving a current out as its measurement result, the output is directly coupled into the dynamics of the source cavity via the interaction Hamiltonian (4.1). This enables fluctuations in the quantity it is monitoring (the output x quadrature) to be suppressed to an arbitrary degree. The Langevin equations for the all-optical case are identical to those of the electro-optical case, once the intervening current in the latter case has been eliminated. Thus we see that the correspondence between all-optical and electro-optical feedback can be made complete by using QND detectors, rather than normal (quantum-demolition) photodetectors.

V. COMPLEX AMPLITUDE FEEDBACK

The two all-optical feedback schemes analyzed so far have both had an equivalent electro-optical scheme. The reason for this is that the couplings between driven and source cavity were QND couplings for the driven cavity. Since the driven cavity is slaved to the output of the source cavity, that means that the coupling effectively acts as a measurement of the output field of the source cavity, the result of which directly acts on the source cavity. In this section, we consider a Hamiltonian coupling between the two cavities which does not factorize as the direct product of a Hermitian operator in each cavity. We shall see that the feedback master equation obtained cannot be derived from any electro-optical scheme.

A. General equations

Consider the following Hamiltonian coupling between the two cavities

$$V = \hbar(c_2 B^\dagger + c_2^\dagger B), \qquad (5.1)$$

where B is an operator on the source cavity. If B is Hermitian (up to a phase factor), then this coupling is equivalent to that considered in the preceding section on quadrature-dependent feedback. In general, however, this feedback is sensitive to both quadratures simultaneously and hence we have dubbed it complex amplitude feedback. Nevertheless, the analysis is identical to that used in the case of quadrature feedback. We will quote the main results.

The master equation arising from complex amplitude feedback, including a squeezed or thermal input bath to the source cavity, is

$$\begin{aligned}\dot{\rho} = &(N+1)\left\{\mathcal{D}[c_1 + A]\rho - -i\left[\tfrac{i}{2}(c_1^\dagger A - A^\dagger c_1), \rho\right]\right\} + N\left\{\mathcal{D}[c_1^\dagger + A^\dagger]\rho - i\left[\tfrac{i}{2}(c_1 A^\dagger - A c_1^\dagger), \rho\right]\right\} \\ &+M\left\{\tfrac{1}{2}[c_1^\dagger + A^\dagger, [c_1^\dagger + A^\dagger, \rho]] + i\left[\tfrac{i}{2}[c_1^\dagger, A^\dagger], \rho\right]\right\} \\ &+M^*\left\{\tfrac{1}{2}[c_1 + A, [c_1 + A, \rho]] + i[\tfrac{i}{2}[c_1, A], \rho]\right\} - i[H_0, \rho], \end{aligned} \qquad (5.2)$$

where

$$A = \frac{2iB}{\sqrt{\gamma}}, \tag{5.3}$$

where γ is the large damping rate of the driven cavity as previously. It can be verified that in the case $A = -iY$, where Y is Hermitian, this equation is equal to Eq. (4.18) for quadrature feedback.

One feature which distinguishes Eq. (5.2) from the quadrature feedback equation (4.18) is that it can produce a nonclassical state in the source cavity even if A is linear in c_1 and $c_1^\dagger$. To see this, consider the case $N = 0$, $H_0 = 0$ to prevent any obscuring effects. The feedback master equation can be rewritten as

$$\dot{\rho} = \mathcal{D}[c_1]\rho + \mathcal{D}[A]\rho + (A\rho c_1^\dagger + c_1\rho A^\dagger - A^\dagger c_1\rho - \rho c_1^\dagger A). \tag{5.4}$$

For A linear, this equation can be converted into a Fokker-Planck equation for the Glauber-Sudarshan P function representation of the density operator [19]. The condition for an initially positive P function (representing a classical state) to remain so is that the diffusion matrix be positive semidefinite. The first term (damping) and third term (enclosed in parentheses above) will only give first order derivatives. Thus we need consider only the second term. Let

$$A = \frac{\lambda}{2}(c_1 + \mu c_1^\dagger). \tag{5.5}$$

Then it can be readily shown that the eigenvalues of the diffusion matrix are proportional to $|\mu|^2 \pm |\mu|$. That is to say, if $0 < |\mu| < 1$, then this complex amplitude feedback will produce a nonclassical state in the cavity. The $|\mu| = 1$ limit gives the case of quadrature feedback, which, as we showed in Sec. IV C, cannot produce a nonclassical intracavity state. The other limit at $|\mu| = 0$ corresponds to the simple feedback considered in Sec. II B, which does not even require a nonlinear crystal (it is a classical geometrical optics problem). The property of nonclassicality distinguishes this all-optical feedback from any form of electro-optical feedback, because the latter is known not to produce nonclassical light from a linear feedback operator.

The effective feedback operator (5.5) can be achieved from the interaction Hamiltonian

$$V = -i\hbar g[(c_2c_1^\dagger + c_2^\dagger c_1) + \mu(c_2c_1 + c_2^\dagger c_1^\dagger)], \tag{5.6}$$

where we have set λ and μ real for simplicity, and where $g = \sqrt{\gamma}\lambda/4$. Let the second mode c_2 be an orthogonal polarization mode of the cavity containing the first mode c_1. That is to say, the second cavity is physically the same as the first, unlike the diagrammatic representation in Fig. 1. Then the first term in the Hamiltonian (5.6) could describe mode conversion, via a polarization rotator. The coupling constant g would be proportional to the (small) proportion of light converted at each pass, divided by the round-trip time of the cavity. The second term, with strength $g\mu$, could only be produced by a nonlinear medium, such as a $\chi^{(2)}$ crystal. The two polarization modes would be the signal and idler, and the second harmonic would have to be strongly driven and heavily damped so that it could be adiabatically eliminated. There are no obvious bars to setting up this scheme experimentally.

The Heisenberg picture Ito equation equivalent to the master equation (5.2) is

$$\begin{aligned} da = {} & \frac{(N+1)}{2}(2\bar{c}_1^\dagger a\bar{c}_1 - a\bar{c}_1^\dagger\bar{c}_1 - \bar{c}_1^\dagger\bar{c}_1 a - [c_1^\dagger A - A^\dagger c_1, a])dt + \frac{N}{2}(2\bar{c}_1 a\bar{c}_1^\dagger - a\bar{c}_1\bar{c}_1^\dagger - \bar{c}_1\bar{c}_1^\dagger a - [c_1A^\dagger - Ac_1^\dagger, a])dt \\ & + \frac{M}{2}([\bar{c}_1^\dagger, [\bar{c}_1^\dagger, a]] + [[c_1^\dagger, A^\dagger], a])dt + \frac{M^*}{2}([\bar{c}_1, [\bar{c}_1, a]] + [[c_1, A], a])\,dt \\ & -[dB_1^\dagger\bar{c}_1 - dB_1\bar{c}_1^\dagger, a] + i[H_0, a]dt, \end{aligned} \tag{5.7}$$

where we have defined

$$\bar{c}_1 = c_1 + A. \tag{5.8}$$

This equation can be derived from the effective feedback Hamiltonian

$$H_{\text{fb}} = i\hbar(b_2^\dagger A - A^\dagger b_2). \tag{5.9}$$

The output field, reflected off the mirror of the second cavity, is

$$b_3 = -(b_2 + A) = -(b_1 + c_1 + A). \tag{5.10}$$

Again, these equations are equivalent to the quadrature feedback equations when $A = -iY$.

Consider the case where A is given by Eq. (5.5), with λ real and positive and μ real. This gives independent linear equations for the quadratures x, y of the field defined in Sec. IV C. Specifically, from Eq. (5.7) with $N = M = 0$, one obtains

$$\begin{aligned} \dot{x} = {} & -\frac{1}{2}\left[1 + \lambda(1-\mu) + \left(\frac{\lambda}{2}\right)^2(1-\mu^2)\right]x \\ & -\left[1 + \frac{\lambda}{2}(1-\mu)\right]\nu_x, \end{aligned} \tag{5.11}$$

where ν_x is as defined in Sec. IV C. The equation for the y quadrature is identical, but for the replacement of x by y, ν_x by ν_y, and μ by $-\mu$. Note that for $\mu = -1$, these equations agree with Eqs. (4.26) and (4.27), as required. The intracavity steady-state variance for x is

$$V(x)_{\text{ss}} = \frac{1 + \lambda(1-\mu) + \left(\frac{\lambda}{2}\right)^2(1-\mu)^2}{1 + \lambda(1-\mu) + \left(\frac{\lambda}{2}\right)^2(1-\mu^2)}. \tag{5.12}$$

Note that for $0 < \mu < 1$, this implies a variance in x less than the unit variance of a coherent state. For $0 < -\mu < 1$, the variance in x will be greater than one, but that for y [obtained by replacing μ by $-\mu$ in Eq. (5.12)] will be less than one. This is in accord with the result stated above, that for $0 < |\mu| < 1$, this linear all-optical feedback will produce a nonclassical intracavity state. Furthermore, in the limit where λ is very large, and $\epsilon = 1 - \mu$ is very small (but not as small as λ^{-1}), then

$$V(x)_{\rm ss} \to \epsilon/2. \tag{5.13}$$

That is to say, the intracavity state can be arbitrarily squeezed. This ideal result could presumably be obtained from electro-optical feedback, with some form of nonlinear feedback Hamiltonian. However, the nonlinear nature of such feedback is quite different from the linear all-optical feedback considered here.

Now consider the output squeezing. From the method of Sec. IV C, the output x spectrum can be calculated to be

$$S_3^x(\omega) = \frac{\frac{1}{4}(\sigma + \lambda\mu)^2 + \omega^2}{\frac{1}{4}(\sigma - \lambda\mu)^2 + \omega^2}, \tag{5.14}$$

where

$$\sigma = 1 + \lambda + \left(\frac{\lambda}{2}\right)^2 (1 - \mu^2). \tag{5.15}$$

The spectrum $S_3^y(\omega)$ can be found by replacing μ by $-\mu$, as before. As shown in Sec. IV C, the output of the feedback loop can show perfect squeezing even for "classical" feedback with $|\mu| = 1$. Consider $0 < |\mu| < 1$, so that σ is always positive. For low frequencies, the output x quadrature is squeezed, with $S_3^x(0) < 1$, for $\mu < 0$. Note that this is the sign of μ which produces intracavity squeezing in the y quadrature. For $\mu > 0$, the output y quadrature is squeezed, while inside the cavity the x quadrature exhibits the nonclassical statistics. In understanding these counterintuitive results, it must be remembered that the output beam b_3 is not simply the output of the source cavity; the statistics of both quadratures are changed by its action as the feedback control beam.

B. Electro-optical analog

As stated above, all-optical complex amplitude feedback has no electro-optical counterpart in general. This is because it is not possible to measure both the real and imaginary quadratures of the output field simultaneously with unit efficiency. Even a QND measurement of one quadrature would introduce noise into the other and so prevent a measurement of both. However, it is possible to do two inefficient measurements of both quadrature with the two efficiencies adding to one (or less than one in practice). It is simplest to consider heterodyne detection, which is equivalent to a homodyne measurement of each quadrature, each with efficiency of one half. The effect of inefficient measurement is to increase the noise introduced by the feedback by a factor inversely proportional to the efficiency. The precise meaning of this statement will become clear shortly.

In order to compare this all-optical complex amplitude feedback to feedback from heterodyne detection, it is convenient to rewrite Eq. (5.2) in terms of the Hermitian operators X, Y defined by

$$A = X - iY. \tag{5.16}$$

We obtain

$$\begin{aligned}\dot\rho = &-i[H_0,\rho] + (N+1)\{\mathcal{D}[c_1]\rho - i[Y, c_1\rho + \rho c_1^\dagger] - i[X, -ic_1\rho + i\rho c_1^\dagger] + \mathcal{D}[X - iY]\rho\} \\ &+N\{\mathcal{D}[c_1^\dagger]\rho + i[Y, c_1^\dagger\rho + \rho c_1] - i[X, ic_1^\dagger\rho - i\rho c_1] + \mathcal{D}[X + iY]\rho\} \\ &+M\{\tfrac{1}{2}[c_1^\dagger,[c_1^\dagger,\rho]] + i[Y,[c_1^\dagger,\rho]] - i[X,[ic_1^\dagger,\rho]] + \tfrac{1}{2}[X+iY,[X+iY,\rho]]\} \\ &+M^*\left\{\tfrac{1}{2}[c_1,[c_1,\rho]] - i[Y,[c_1,\rho]] + i[X,[-ic_1,\rho]] + \tfrac{1}{2}[X-iY,[X-iY,\rho]]\right\}.\end{aligned} \tag{5.17}$$

If A is linear in the field amplitude, then X and Y are proportional to two orthogonal quadratures of the field. For controlling noise, it would be sensible for these to be proportional to the x and y quadratures, respectively.

The terms in this equation linear in the feedback operators X and Y are reminiscent of the terms in the quadrature-dependent feedback master equation (4.18). However, the terms, bilinear in X, Y, which we will refer to as noise terms, for reasons which will become evident later, are different. As noted in the preceding section, quadrature feedback can be effected by either an all-optical scheme or a homodyne detection scheme. In the latter case, the master equation was derived from the stochastic evolution equation describing the effect of homodyne detection on the system. The equivalent equation for heterodyne detection is found to be

$$\begin{aligned}d\rho(t) = dt &\left\{(N+1)\mathcal{D}[c_1]\rho + N\mathcal{D}[c_1^\dagger]\rho + \frac{M}{2}[c_1^\dagger,[c_1^\dagger,\rho]] + \frac{M^*}{2}[c_1,[c_1,\rho]] - i[H_0,\rho]\right\} \\ &+ \frac{1}{\sqrt{2L_x}}\, dW_x(t)\,\mathcal{H}\left[(N+M+1)c_1 - (N+M^*)c_1^\dagger\right]\rho \\ &+ \frac{1}{\sqrt{2L_y}}\, dW_y(t)\,\mathcal{H}\left[(N-M+1)(-ic_1) - (N-M^*)(ic_1^\dagger)\right]\rho,\end{aligned} \tag{5.18}$$

where

$$L_x = 2N + 1 + M + M^* \ , \qquad (5.19)$$

$$L_y = 2N + 1 - M - M^*. \qquad (5.20)$$

Here, $dW_x(t)$ and $dW_y(t)$ are real infinitesimal Weiner increments. Physically, they are related to the noise $\xi(t)$ in the two photocurrents which arise from the heterodyne detection by $dW_q(t) = \xi_q(t)dt$ $(q = x, y)$. These currents are given by

$$I_x(t) = \langle c_1 + c_1^\dagger \rangle + \sqrt{2L_x}\,\xi_x(t) \ , \qquad (5.21)$$

$$I_y(t) = \langle -ic_1 + ic_1^\dagger \rangle + \sqrt{2L_y}\,\xi_y(t), \qquad (5.22)$$

where the normalization has been chosen to make the deterministic term the same as in the homodyne case. Note that the noise is greater in this case, with L_q multiplied by 2. This is because each measurement is of efficiency half. In general, if the efficiency of the extra-cavity measurement of the quadrature q is η_q (such that $\eta_x + \eta_y \leq 1$), then $2L_q$ is replaced by L_q/η_q in the above equations.

Now the stochastic equation (5.18) can be used to derive a feedback master equation analogous to the all-optical Eq. (5.17). The currents control the time-dependent feedback Hamiltonian

$$H_{\text{fb}}(t) = \hbar[I_x(t)Y + I_y(t)X]. \qquad (5.23)$$

This should be compared with Eq. (5.9). It is not equivalent to that equation, because there is more noise in the currents than there is in the quadratures. If one wished to represent the currents by operators as was done for the case of homodyne detection, then this extra noise would enter at the beam splitter necessary to split the output into two separate homodyne devices. By adding the evolution from the Hamiltonian (5.23) to that of Eq. (5.18), we obtain the heterodyne feedback master equation

$$\begin{aligned}\dot{\rho} = {} & (N+1)\{\mathcal{D}[c_1]\rho - i[Y, c_1\rho + \rho c_1^\dagger] - i[X, -ic_1\rho + i\rho c_1^\dagger]\} \\ & + N\{\mathcal{D}[c_1^\dagger]\rho + i[Y, c_1^\dagger\rho + \rho c_1] - i[X, ic_1^\dagger\rho - i\rho c_1]\} \\ & + M\{\tfrac{1}{2}[c_1^\dagger, [c_1^\dagger, \rho]] + i[Y, [c_1^\dagger, \rho]] - i[X, [ic_1^\dagger, \rho]]\} \\ & + M^*\{\tfrac{1}{2}[c_1, [c_1, \rho]] - i[Y, [c_1, \rho]] + i[X, [-ic_1, \rho]]\} \\ & + 2L_x\mathcal{D}[Y]\rho + 2L_y\mathcal{D}[X]\rho - i[H_0, \rho]. \end{aligned} \qquad (5.24)$$

Note that the desired feedback terms (linear in X and Y) are the same in this equation as in the all-optical Eq. (5.17), but the diffusion terms are different.

To elucidate this difference, we return to the most basic example of all-optical feedback, considered in Sec. II B. The output of the source cavity is simply fed back into another mirror of that cavity. Such feedback is covered by the master equation derived in this section, with

$$A = \sqrt{\gamma}e^{i\phi}a. \qquad (5.25)$$

Here, a is the annihilation operator for the source cavity with decay rate γ. In this simple case, there is no need for a second cavity. As derived in Sec. II, the master equation for the cavity is

$$\dot{\rho} = 2\gamma(1 + \cos\phi)\mathcal{D}[a]\rho - i\gamma\sin\phi[a^\dagger a, \rho], \qquad (5.26)$$

where the input is in the vacuum state. Attempting to replicate this feedback by using heterodyne detection yields the master equation

$$\begin{aligned}\dot{\rho} = {} & 2\gamma(1 + \cos\phi)\mathcal{D}[a]\rho - i\gamma\sin\phi[a^\dagger a, \rho] \\ & + \gamma(\mathcal{D}[a] + \mathcal{D}[a^\dagger])\rho. \end{aligned} \qquad (5.27)$$

The equation of motion for the mean field from this equation is identical to that of Eq. (5.26). However, the presence of the extra term introduces noise into both quadratures equally. If $\phi = \pi$, so that the deterministic dynamics are eliminated, then the variance in each quadrature will simply grow linearly. This clearly shows the effect of the noise introduced by attempting to measure both quadratures in electro-optical feedback, as opposed to the coherent back coupling of both quadratures in all-optical feedback.

VI. CONCLUSION

At the simplest level, feedback is a process by which a system influences itself through its action on a second system. Usually, the second system would be a complicated feedback apparatus, consisting of measurement devices, signal processors and the like. However, it is always possible to conceive of more direct schemes, in which the feedback loop is treated on the same level as the system. In this paper we have examined feedback on optical cavities. The usual implementation of optical feedback is electro-optical feedback. The light emitted by the cavity enters a detector, and the photocurrent produced is used to control the dynamics of the cavity by some electro-optical devices. The more direct method is all-optical feedback. We have considered turning the output beam from the source cavity onto a second cavity which is directly coupled to the first by some nonlinear crystal. To compare these two methods at the quantum limit, we assumed that the feedback could be approximated as a Markovian process. In the electro-optical case, this corresponds to assuming that the overall time delay in the feedback loop is much smaller than the lifetime of the source cavity, while in the all-optical case, we also need the linewidth of the second cavity to be much greater than that of the first. Under these conditions, a master equation, or quantum Langevin equation, can be derived for the source cavity alone.

Our main conclusion is that there is a strong relationship between all-optical and electro-optical quantum-limited feedback. In fact, if the direct-coupling Hamiltonian V between the two cavities factorizes as the product of Hermitian operators in each cavity, then the all-optical feedback has an exact electro-optical feedback counterpart, at least as far as the effect on the source cavity is concerned. If V depends on the intensity of the driven cavity, then the result can be reproduced by feedback based on direct photodetection, while quadrature-dependent all-optical feedback can be reproduced by homodyne detection. The explanation for this is that the nonlinear interaction acts as an effective measurement of the driven cavity intensity or quadrature, which (because it is heavily damped) is a measurement of the intensity or quadrature of the output of the first cavity. Unlike the electro-optical case, the measuring device then directly influences the source cavity through the same coupling, instead of having all of the intermediate equipment. On the other hand, a purely optical feedback scheme which is sensitive to both quadratures of the driven cavity cannot be replicated by currents. That is because such an interaction is not like a simultaneous measurement of both quadratures of the driven cavity. Rather, the complex amplitude of the driven cavity interacts coherently with another non-Hermitian quantity in the source cavity. An analogous electro-optical scheme using heterodyne detection can be defined, which has the same semiclassical effect as the complex amplitude feedback. However, the two independent measurements of the two quadratures make the feedback incoherent and introduce extra diffusion terms into the master equation.

The equivalence of electro-optical and all-optical feedback for the intensity and quadrature feedback means that the latter is subject to the same restrictions as the former as far as the generation of nonclassical light is concerned. Specifically, if the nonlinear interaction Hamiltonian V is linear in the *source* cavity amplitude or intensity, then the feedback cannot produce a nonclassical state in the source cavity. The electro-optical equivalent of this theorem is that feedback based on extra-cavity detection cannot produce nonclassical light by driving or detuning the cavity. However, there is a property of all-optical feedback not present in electro-optical schemes which makes the situation less clearcut: an output beam. This is the feedback loop beam reflected from the second cavity. It turns out that this beam may be arbitrarily squeezed, even though the source cavity remains classical.

The way to understand this is using the measurement analogy explained above. The driven cavity is a QND measurement device for, say, the x quadrature of the output of the source cavity. This device also may control the dynamics of the source cavity such that the measured fluctuations in the x quadrature are suppressed. Thus the x quadrature of the output of the source cavity can be squeezed; its statistics are unchanged upon reflection at the second cavity. However, it is not until the output reflects off the second cavity (the QND device) that the extra fluctuations in the y quadrature are put in, due to the effect of the measurement. This would seem to indicate that in the loop, Heisenberg's uncertainty relations fail, even though they are satisfied in the output loop (which is all that is observable). However, this is not the case. The canonical commutation relations, which are actually between different spatial points of the field at the same time, are never violated. The anticorrelations in the output field which result in squeezing are only present between parts of the field which are separated in time by more than the time delay in the feedback loop. There is no contradiction.

Thus it would seem that there is one potential application for all-optical feedback: producing squeezed light. However, in order to build such a device producing well-squeezed light, the coupling would have to use a frequency- (but not polarization-) degenerate $\chi^{(2)}$ crystal, as well as a polarization converter. It would seem easier to use the traditional squeezer, a $\chi^{(2)}$ crystal acting as a degenerate parametric oscillator. Intensity-dependent all-optical feedback is even less practical, requiring a low-loss $\chi^{(3)}$ nonlinearity to operate. The smallness of higher order nonlinearities is sufficient justification as to why we have not considered all-optical feedback with a coupling dependent on higher order field moments of the driven cavity. In fact, such higher order feedback does not produce any new results. At least in the regime where the second cavity can be adiabatically eliminated, the higher order terms either give a vanishing contribution, or reproduce the results of amplitude or intensity feedback. Thus we can conclude that all-optical feedback is probably not a practical way of controlling quantum noise, although there may be other applications. Nevertheless, the predicted results are interesting, and some experiments should be feasible with current technology. The similarities with and differences from electro-optic feedback yield important insights about the nature of feedback in general.

ACKNOWLEDGMENTS

We wish to acknowledge fruitful discussions with W.J. Munro, C.W. Gardiner, and H.J. Carmichael. Part of this work was done while G.J.M. was visiting the Department of Physics, University of Oregon, with the support of a University of Queensland International Collaborative Research Grant.

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