Codimension-two two-hard-mode bifurcation and coexistence of multiple attractors in a two-photon laser with an injected signal

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A codimension-two Hopf bifurcation and the coexistence of multiple attractors of a two-photon laser with an injected signal are investigated. The quasiperiodic motion with two incommensurate frequencies (2-torus) may arise directly from a stationary solution through the type-III codimension-two bifurcation mechanism. We analyze conditions and identify the quasiperiodic motion in terms of time-dependent solutions and Poincaré maps. Moreover, the coexistence of multiple attractors in the vicinity of the type-III bifurcation point and the symmetry-breaking bifurcation can be observed as well.

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I. INTRODUCTION

A large number of publications have focused on the study of instabilities, self-pulsations, coexistence of attractors, as well as chaotic motions of lasers and optical bistable (OB) systems with an injected signal (LIS) in the past several decades [1–11]. Recently, a systematic study in multicodimension bifurcation in OB has been also reported [12–14]. These systems are used as a good example to show the rich characteristic behaviors of nonlinear dynamic systems.

When the linear part of the vector field undergoes doubly degenerate bifurcation, there are three basic cases, and the corresponding standard linear matrices of the order-parameter equations read [15]

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \text{type I}; \tag{1.1}$$

$$\begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{type II}; \tag{1.2}$$

$$\begin{bmatrix} 0 & -\omega_1 & \multicolumn{2}{c}{\underline{0}} \\ \omega_1 & 0 & 0 & -\omega_2 \\ \multicolumn{2}{c}{\underline{0}} & \omega_2 & 0 \end{bmatrix}, \quad \text{type III}. \tag{1.3}$$

Type-I and type-II bifurcation have been discussed in LIS and OB [14], but type-III bifurcation has not been analytically revealed in LIS and OB. However, in the case of the two-photon laser and bistable systems with an injected signal (2LIS and 2OB), the codimension-two bifurcation of type III can be observed. Nevertheless, a systematic study in multicodimension bifurcation and dynamic behaviors is still lacking in 2LIS.

The aim of the present paper is to investigate the instability boundary conditions for type-III bifurcation and the coexistence of multiple attractors of 2LIS. In Sec. II, our model and its stationary solution will be present. In Sec. III, distribution of the instability boundary and codimension-two bifurcation of type III on the boundary will be clearly shown. In Sec. IV we reveal the coexistence of the stable stationary solution and a time-dependent solution, and the coexistence of two or more attractors. Section V will give some brief discussion.

II. THE MODEL AND ITS STATIONARY SOLUTION

Our model is an optical undirectional ring cavity filled with an active medium, consisting of homogeneously broadened two-level atoms. The laser system is driven by an external coherent field. We take the plane-wave approximation and the mean-field limit, and consider only the single-mode case. We reduce the Maxwell-Bloch equations to [17]

$$\begin{aligned} \dot{x} &= -\kappa[(1+i\theta)x - Y + 2Cvx^*], \\ \dot{v} &= -(1+i\Delta)v + x^2 m, \\ \dot{m} &= -\gamma\{[v(x^*)^2 + v^* x^2]/2 + m + 1\}, \end{aligned} \tag{2.1}$$

where all the variables and the parameters are dimensionless. x and v, being proportional to the output field and the atomic polarization, and are complex numbers, while the normalized atomic population m is real. Then the equations are essentially five dimensional. The system parameters, C, κ, and γ designate the small-signal gain, the cavity linewidth, and the atomic population decay rate, respectively. Both κ and γ are scaled to the homogeneous linewidth γ. Given the frequencies of the external field, the atoms, and the cavity as ω_0, ω_a, and ω_c, respectively, we scale the atomic detuning and the cavity mistuning as $\Delta=(\omega_a-2\omega_0)/\gamma$ and $\theta=(\omega_c-\omega_0)/(\kappa\gamma)$. The normalized amplitude of the external field Y is assumed to be real and positive.

The stationary solution of (2.1) can be worked out explicitly. It reads

$$\begin{aligned}
Y&=X\{1-2CX^2(1+\Delta^2+X^4)]^2\\
&\quad+[\theta+2C\Delta X^2(1+\Delta^2+X^4)]^2\}^{1/2}\,.\\
v_s&=-(1-i\Delta)x_s^2/(1+\Delta^2+X^4)\,,\\
m_s&=-(1+\Delta^2)/(1+\Delta^2+X^4)\,,
\end{aligned} \tag{2.2}$$

where $X=|x_s|$. The standard way to study the bifurcation set of (2.1) is to linearize (2.1) about the stationary solution x_s, v_s, and m_s, and then to investigate the changes in the sign of the real part of the eigenvalues of the linearized equations. The equations of the linearizations of Eqs. (2.1) about the steady state (2.2) turn out to be

$$\begin{bmatrix} d\delta x/dt\\ d\delta x^*/dt\\ d\delta v/dt\\ d\delta v^*/dt\\ d\delta m/dt\end{bmatrix}=\begin{bmatrix}-\kappa(1+i\theta) & -2C\kappa v_s & -2C\kappa x_s^* & 0 & 0\\ -2C\kappa v_s^* & -\kappa(1-i\theta) & 0 & -2C\kappa x_s & 0\\ 2m_s x_s & 0 & -(1+i\Delta) & 0 & x_s^2\\ 0 & 2m_s x_s^* & 0 & -(1-i\Delta) & (x_s^*)^2\\ -\gamma v_s^* x_s & -\gamma v_s(x_s^*) & -\gamma_s(x_s^*)^2/2 & -\gamma x_s^*/2 & -\gamma\end{bmatrix}\begin{bmatrix}\delta x\\ \delta x^*\\ \delta v\\ \delta v^*\\ \delta m\end{bmatrix}, \tag{2.3}$$

where

$$\delta x=x-x_s\,,\quad \delta v=v-v_s\,,\quad \delta m=m-m_s\,.$$

Equation (2.3) gives rise to the characteristic equation

$$\lambda^5=a_1\lambda^4+a_2\lambda^3+a_3\lambda^2+a_4\lambda+a_5=0 \tag{2.4}$$

with

$$\begin{aligned}
a_1&=2\kappa+2+\gamma\,,\\
a_2&=\kappa^2(1+\theta^2)+2\kappa(\gamma+2)+(2\gamma+1+\Delta^2+\gamma X^4)+8C\kappa m_s X^2+4C^2\kappa^2 m_s X^4(1+\Delta^2+X^4)\,,\\
a_3&=\kappa^2(1+\theta^2)(\gamma+2)+\gamma(1+\Delta^2+X^4)+2\kappa(2\gamma+1+\Delta^2+\gamma X^4)+8C\kappa m_s X^2(1+\gamma+\kappa)\\
&\quad+[4C\kappa\gamma X^6+4C^2\kappa^2(\gamma+2)X^4 m_s]/(1+\Delta^2+X^4)\,,\\
a_4&=2\kappa\gamma(1+\Delta^2+X^4)+\kappa^2(1+\theta^2)(2\gamma+1+\Delta^2+\gamma X^4)+8C\kappa\gamma m_s X^2+8C\kappa^2(1-\theta\Delta+\gamma)m_s X^2\\
&\quad+16C^2\kappa^2 m_s^2 X^4-[8C^2\kappa^2\gamma m_s X^8-4C^2\kappa^2(2\gamma+1+\Delta^2+\gamma X^4)m_s X^4-4C\kappa^2\gamma(1+\theta\Delta)X^6]/(1+\Delta^2+X^2)\,,\\
a_5&=\kappa^2\gamma[(1+\theta^2)(1+\Delta^2+X^4)+8Cm_s(1-\theta\Delta)X^2+4C^2 m_s X^4+16C^2 m_s^2 X^4]\,.
\end{aligned} \tag{2.5}$$

All the coefficients in (2.5) are expressed in terms of the external control parameters C, γ, κ, Δ, θ, and X.

It is, obviously, impossible to solve Eq. (2.4) analytically and to obtain explicit solutions of the eigenvalues. However, a general discussion about the instability of the steady solution (2.2) is possible without seeking a precise solution of λ. We define

$$\begin{aligned}
D_1&=a_1\,,\\
D_2&=a_1a_2-a_3\,,\\
D_3&=D_2a_3-(a_1a_4-a_5)a_1\,,\\
D_4&=D_2(a_3a_4-a_2a_5)-(a_1a_4-a_5)^2\,,\\
D_5&=a_5D_4\,.
\end{aligned} \tag{2.6}$$

According to the Routh-Hurwits criterion, the necessary and sufficient conditions for (2.2) to be stable are

$$D_1,D_2,D_3D_4,D_5>0\,. \tag{2.7}$$

The steady state may lose its stability via a certain kind of codimension bifurcations.

III. TYPE-III CODIMENSION-TWO BIFURCATION

There are two kinds of codimension-one bifurcation. First, class A, a real eigenvalue of (2.4), which is the largest compared with the real part of all the other eigenvalues, changes its sign from negative to positive. The critical condition for the bifurcation of class A (i.e., saddle-mode bifurcation in our case) is

$$a_5=0 . \quad (3.1)$$

Second, class B, a pair of complex conjugate eigenvalues, which have the largest real part, cross the imaginary axis, and their real part becomes positive. Accordingly, the critical condition for the bifurcation of class B, i.e., Hopf bifurcation, reads

$$D_4=0 . \quad (3.2)$$

It is obvious that neither (3.1) nor (3.2) is the sufficient condition for the corresponding bifurcations. However, if we start from a stable region, the necessary and sufficient condition for the bifurcation of class A is the first transversal crossing of the hypersurface (3.1), while for class B, the hypersurface (3.2). By the first crossing we mean that no other surface, (3.1) or (3.2), has been crossed before the given surface is crossed. Transversal crossing of surface A or B becomes the necessary and sufficient condition for the instability of the stationary state. This fact has been discussed in great detail in Refs. [10], [14], and [16].

The codimension-two bifurcations of Eqs. (1.1), (1.2), and (1.3) in our case correspond to the following conditions, respectively:

Type I:

$$a_5=a_4=0 . \quad (3.3)$$

Two distinctive class-A sheets intersect each other and the eigenvalue equation (2.4) has the double-zero eigenvalue degeneracy

$$\lambda_1=\lambda_2=0 . \quad (3.4)$$

Type II:

$$a_5=0 , \quad D_4=0 , \quad a_4\neq 0 . \quad (3.5)$$

One class-A sheet and one class-B sheet intersect each other and at the bifurcation point we have simultaneously

$$\lambda_{1,2}=\pm i\omega , \quad \lambda_3=0 \quad (3.6)$$

with

$$\omega^2=a_1a_4/D_2\neq 0 . \quad (3.7)$$

Type III:

$$D_4=D_2=0 . \quad (3.8)$$

Two distinctive class-B sheets intersect each other and, at the bifurcation set, the eigenvalue equation (2.4) has the solution

$$\lambda_{1,2}=\pm i\omega_1 , \quad \lambda_{3,4}=\pm i\omega_2 , \quad \lambda_5=-a_1 \quad (3.9)$$

with

$$\omega_1^2=[a_2-(a_2^2-4a_4)^{1/2}]/2 , \quad \omega_2^2=[a_2+(a_2^2-4a_4)^{1/2}]/2 . \quad (3.10)$$

Since surface B is defined by the first crossing of the boundary (3.2) from a stable region, the hypersurface defined by Eqs. (3.8) serves as the necessary and sufficient condition for bifurcation of type III. The standard linear matrix of the order-parameter equation takes the form (1.3).

In the one-photon OB system [14], we always have $D_2>0$ and no codimension-two bifurcation of type III can be found, while type-I and type-II bifurcations can be observed very easily. Numerically, it is very difficult to find quasiperiodic motion and the route from quasiperiodicity to chaos in OB. Though $D_2=0$ can be crossed in the case of one-photon LIS, the negative D_2 always appears in the unstable region. Then, the quasiperiodicity can never be predicted in one-photon LIS directly through the instability of the stationary state, though it exists after instability of certain periodic orbits. In our case of 2LIS, the codimension-two bifurcation of type III can be observed. Since quasiperiodic motion and the route from quasiperiodicity to chaos are very interesting for the applications of optical devices and for the theoretical study of nonlinear dynamics, it is of importance to analytically predict the condition for quasiperiodicity. We are very glad to find that in 2LIS systems, quasiperiodicity can be successfully predicted through the type-III instability of the stationary solution.

In Fig. 1 we plot two bifurcation figures of 2LIS in the Δ-X plane by fixing C, κ, γ, and θ. The dash-dotted and dotted lines correspond to Eq. (3.1) (curve A). The dashed and solid curves correspond to Eqs. (3.2) (curve B) but only dash-dotted and solid curves are the instability boundary. All the S regions are stable. The N region surrounded by the dash-dotted and dotted lines is the negative-slope region in which the stationary solution is always unstable. S represents the steady state and N represents the negative-slope region of the state equation $X=X(Y)$.

Some of the interesting features of the bifurcation figure which are very useful for analyzing the global structure of the instability and codimension-two bifurcations in the parameter space are listed below.

(1) The stationary solution may be single valued, triple valued, or five valued for certain combinations of parameters. If $\theta\Delta>0$, no S-shaped solution exists, i.e., the state equation $X=X(Y)$ is always single valued. If $\theta\Delta<0$, the stationary solution may be triple valued for some combination of parameters C, κ, and γ, or five valued for certain other combinations of parameters.

(2) When the S-shaped solution exists, almost the entire lower branch is stable from $Y=0$ to the lower turning point.

(3) Surface B may contain various sheets. Different sheets might intersect each other. In Fig. 1(a), there are two Hopf instability regions which are completely isolated from each other. It is worth noting that a stable island may appear in the instability region surrounded by outer solid curves [see Fig. 1(b)].

(4) We have found codimension-two bifurcations of type II denoted by II and type III denoted by III. We have varied C, κ, γ, and θ in a wide region, and no bifurcation of type I has been found.

As a matter of fact, we have observed the bifurcation of type III in Fig. 1(a) simply by searching the interection of distinctive B sheets given by Eq. (3.2), which occurs in the upper branch of the S-shaped solution at

$\Delta = 19.53$, $X = 5.02$, ($Y = 29.61$). In the close vicinity of bifurcation point, the system finally approaches a stable torus ($2Q$), i.e., a quasiperiodic motion. The phase portraits in the plane of $\mathrm{Re}(x) - \mathrm{Im}(x)$ and the Poincaré surface of section displaying an enlarged view of the $2Q$ torus for $X = 5.012$ ($Y = 29.52$), $\mathrm{Im}(x) = 4.8$ are shown in Fig. 2, where the initial state is taken near the steady state. Actually, type-III codimension-two bifurcations can be found for other parameters, for instance, in Fig. 1(b).

So far, we have shown a codimension-two bifurcation from the stationary solution to a stable torus through both the theoretical analysis and computer simulation of the system. This can be the most important difference between the one-photon processes and two-photon ones.

IV. THE COEXISTENCE OF MULTIPLE ATTRACTORS

In Sec. III, we have revealed that a stable torus ($2Q$) can arise from a stationary solution through the codimension-two bifurcation mechanism. A dynamical system, however, can have more than one attractor for the same combination of parameters and these attractors can be modified by adjusting the control parameters. Since 2LIS is much more complicated than one-photon LIS and OB systems as we will see later, the distribution of the various attractors and the complicated dynamical behavior around these attractors must be of interest. We will present some results based on the understanding of the type-III bifurcation.

In Fig. 3(a) we present various attractors for $C = 20$, $\theta = -15$, $\Delta = 18.5$, $\gamma = \kappa = 2$. To draw the curves in this figure, we numerically solve Eqs. (2.1) for a given control parameter Y and a set of the initial variables x_i, v_i, and m, for a time long enough to ensure that the evolution is well after the transient process, and then plot the maximum values of X in a final time interval as a function of Y. The stationary solution is five valued for certain values of Y. The letters L, R_1, R_2, M, and N represent various turning points. The two bistability loops are apparent. Between Y_M and Y_L there is a coexistence of a stable stationary solution and an oscillation. Increasing Y from below, the stationary solution loses its stability, after Y exceeds Y_L. Between Y_N and Y_{R_1}, there is other coexistence of a stable stationary solution with an oscilla-

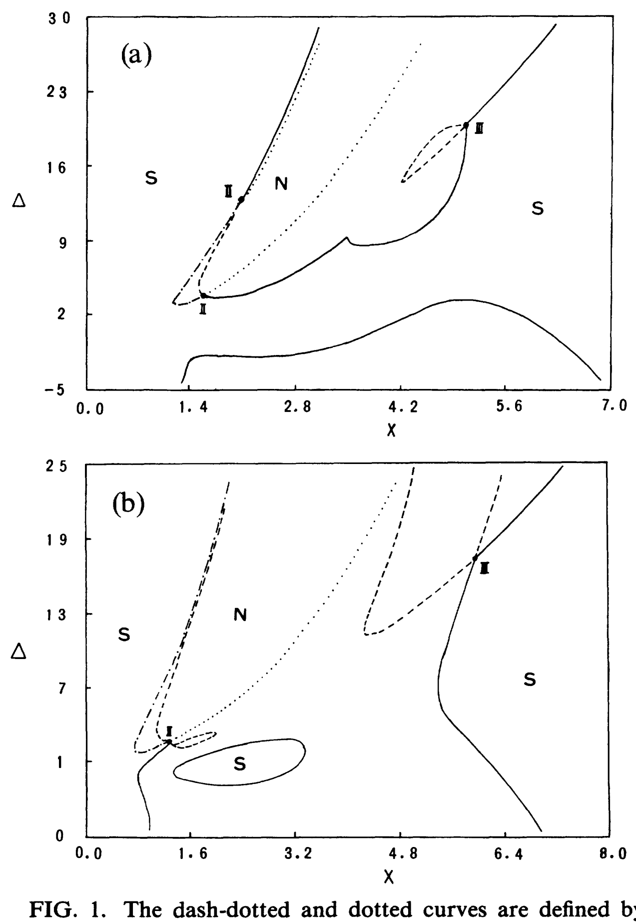

FIG. 1. The dash-dotted and dotted curves are defined by Eqs. (3.1). The dashed and solid curves correspond to Eqs. (3.2). However, only the dashed-dotted and solid curves separating the stable region from the unstable region serve as the instability boundaries of classes A and B. At points II and III, codimension-two bifurcation of types II and III take place, respectively. (a) $C = 10$, $\kappa = \gamma = 2$, $\theta = -15.55$. There are two instability regions completely isolated. (b) The same as in (a) with $C = 12$, $\theta = -13$. A stable island appears in the instability region surrounded by the outer solid curve.

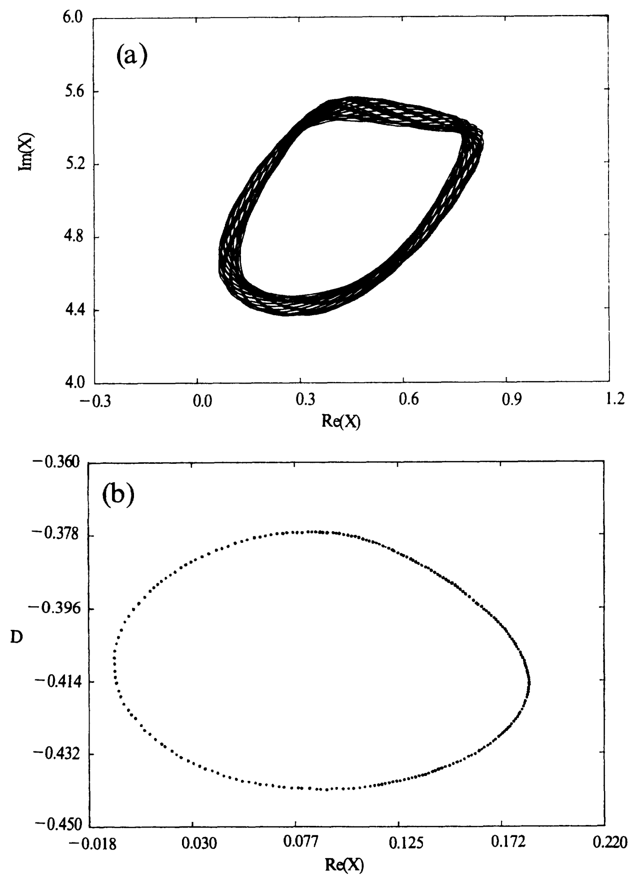

FIG. 2. (a) A two-dimensional phase portrait of $2Q$ in the $\mathrm{Re}(x)$-$\mathrm{Im}(x)$ plane. The parameters chosen are the same as in Fig. 1(a) with $\Delta = 19.53$, $Y = 29.52$ ($X = 5.012$). (b) Poincaré surface of section displaying an enlarged view of the $2Q$ torus for $\mathrm{Im}(x) = 4.8$ in 2(a). This initial state is near the steady state.

tion. Y_{R_1} is the upper boundary of the unstable region. Decreasing Y from $Y > Y_{R_1}$, to $Y < Y_{R_1}$, the stationary solution is replaced by a small-amplitude periodic oscillation when the upper branch is destablized by supercritical Hopf bifurcation. Hence, a coexistence of two attractors of time-dependent motions is identified between Y_{R_1} and Y_{R_2}.

In Fig. 3(b), we take $\Delta = 0.5$. All other parameters are given in Fig. 1(b). In this case, the stationary solution is single valued. A subcritical bifurcation arises at Y_L. Between Y_M and Y_L there is a bistability loop which is the coexistence of a stable stationary solution and a time-dependent solution. This is similar to the OB system (see Ref. [14]), but Hopf bifurcation of the upper branch is supercritical at Y_R. The route to chaos via period-doubling bifurcation can be observed by continuously decreasing the control parameter Y. Between Y_{S_1} and Y_{S_2} we find a stable island of the steady state, where a coexistence of the stable stationary solution and a time-dependent solution exists. Hopf bifurcation on the lower branch is subcritical. The system jumps directly to the upper attractor at Y_{S_2}, while it jumps to another attractor, which is chaotic, at Y_{S_1}. Decreasing Y from the attractor, the system can also jump to the upper attractor at Y_C. Thus, there is a coexistence of two attractors of time-dependent motion between Y_C and Y_{S_1}.

The coexistence of multiple attractors is a typical feature around the type-III codimension-two bifurcation points. In Fig. 4, the parameters are taken in the vicinity of the type-III bifurcation point at $Y = 120.25$. In Fig. 4(a) the initial state is located in the vicinity of the stationary solution. The system finally approaches a stable period motion. In Fig. 4(b), the initial state is located in the vicinity of the stationary solution, however, the asymptotic state the system eventually approaches is a 2-torus. By decreasing Y slightly from Fig. 4 to $Y = 119.57$ we observe four coexisting attractors shown in Fig. 5. The initial states are chosen in the vicinity of the stationary solution in Figs. 5(a)–5(c). In Fig. 5(d), the initial state is far from the stationary solution. The attractor in

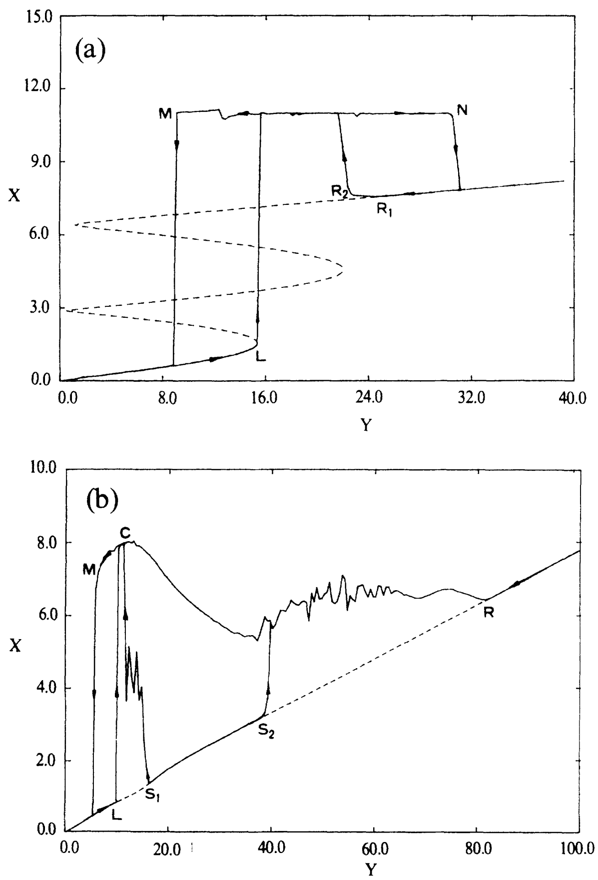

FIG. 3. (a) $C = 20$, $\kappa = \gamma = 2$, $\Delta = 18.5$, and $\theta = -15$. The stationary solution is five valued. After the transient process of Eqs. (2.1), the maximum X in the trajectory is plotted against Y. Two bistability loops are apparent. (b) The parameters are given in Fig. 1(b) with $\Delta = 0.5$. The meanings of the curves are the same as in 3(a). The stationary solution is single valued. Bistability can be found in regions $Y_M Y_L$, $Y_C Y_{S_1}$ and $Y_{S_1} Y_{S_2}$.

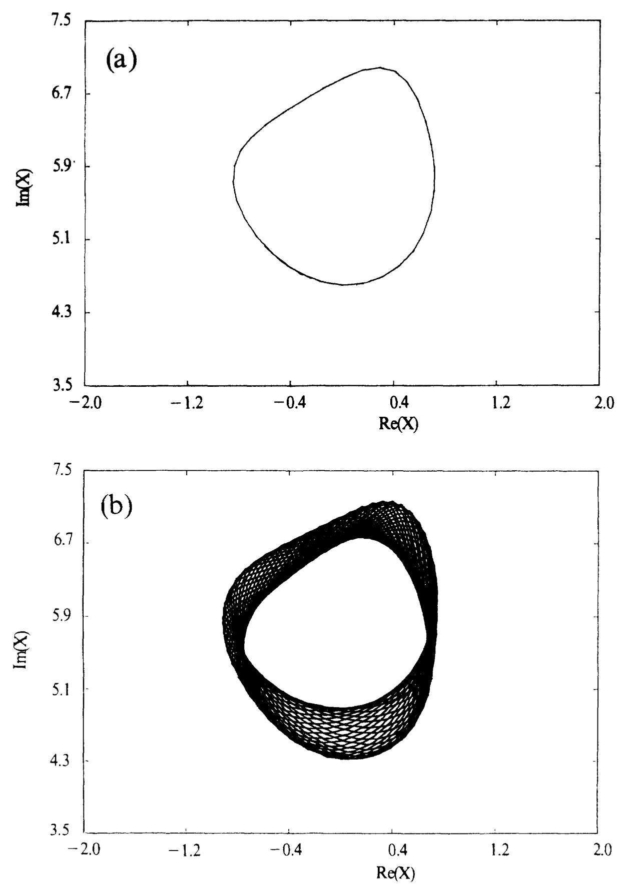

FIG. 4. $C = 30$, $\Delta = 26.5$, $\theta = -50$, $\kappa = 1$, $\gamma = 2$, and $Y = 120.25$. The parameters are taken in close vicinity of a type-III codimension-two bifurcation point. (a) The initial state is located in the vicinity of the stationary solution. The system finally approaches a stable period motion. (b) The initial condition is changed from those of 4(a) while still in the vicinity of the stationary solution. The system finally reaches a stable torus.

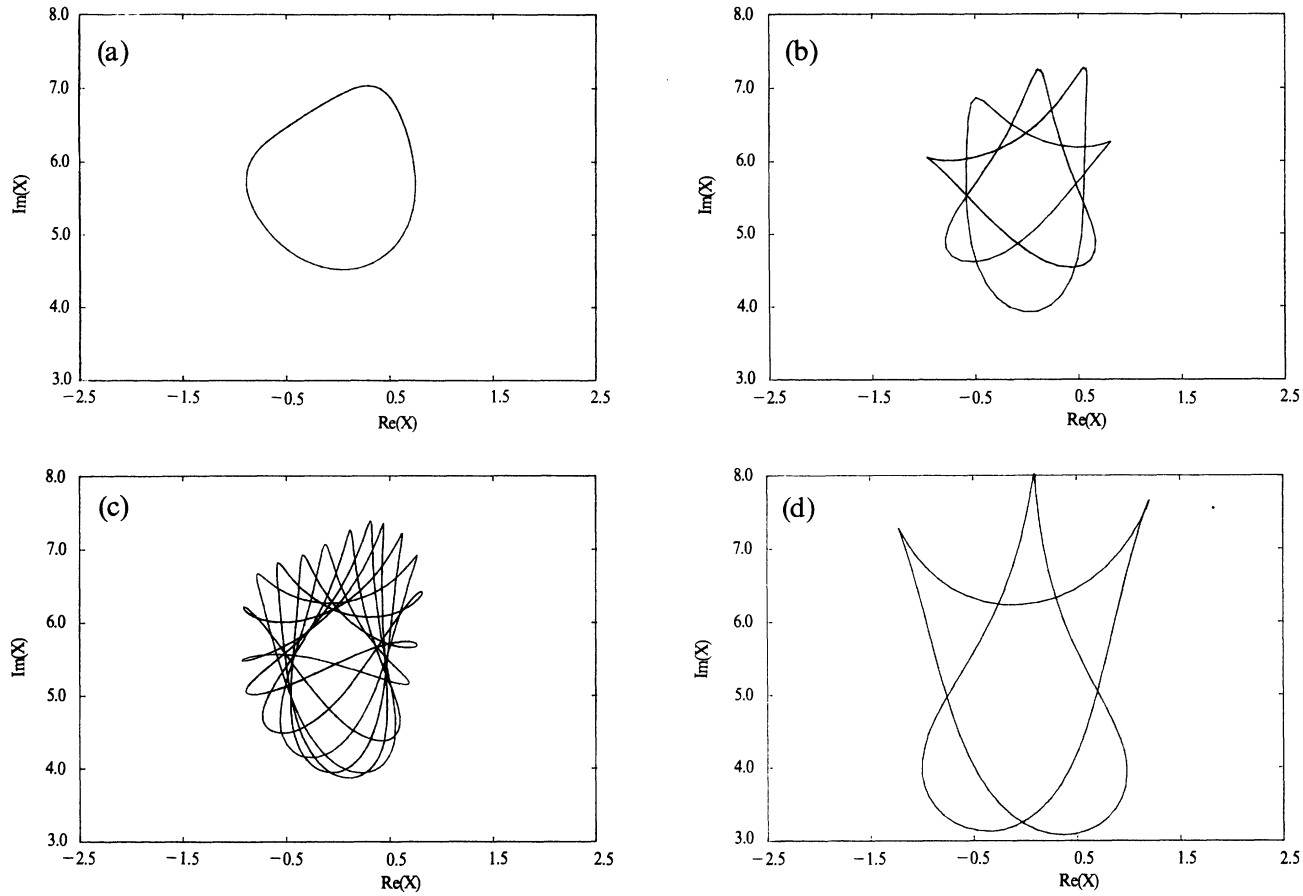

FIG. 5. Four coexisting attractors. The parameters are the same as in Fig. 4 with $Y=119.57$. For (a), (b), and (c) the initial values are taken as different values in the vicinity of the steady solution. For (d) the initial state is far from the stationary solution.

Fig. 5(a) is similar to that in Fig. 4(a). It is obvious that the latter attractor can be reached by adjusting Y continuously from the former attractor. They belong to the same basin. In Figs. 5(b)–5(d), the three attractors coexist. Two of them [5(b) and 5(c)] are bifurcated from the quasiperiodic motion of Fig. 4(b) through frequency lock- An interesting point is that the symmetry-breaking bifurcation (the original basin of attractor splits into two domains of attractor) may exist between $Y=119.57$ and 120.25. This type of bifurcation has not yet been found for one-photon LIS [4]. The basin structures of the two attractors are rather complicated. It is worthwhile investigating the structure and dynamics in these basins in more detail. However, we do not intend to go further here.

V. DISCUSSION

Let us end our presentation by offering the following remarks.

(i) We have detailed the distribution of the instability boundary of the two-photon LIS system and the codimension-two bifurcations modeled by Eqs. (2.1). The type-III codimension-two bifurcation is the most important finding which has never been predicted in one-photon LIS and OB systems. The bifurcation from a stationary solution to a stable torus is analytically predicted and numerically observed. It is well known that the type-III codimension-two bifurcation is associated with many interesting dynamic behaviors such as the route from quasiperiodicity to chaos, coexistence of multiple attractors, symmetry-breaking bifurcations, a 3-torus with two hard modes and one soft (low-frequency) mode and so on [15]. Our prediction can be extended to all these significant features, which is much more complicated than that around type-I and type-II bifurcations.

(ii) We have numerically verified the coexistence of attractors. The multistability structure of 2LIS is much richer than LIS. The coexistence of four attractors of time-dependent orbits has been observed in the parameter region close to the type-III bifurcation point. It is also remarkable that the symmetry-breaking bifurcation may exist in our system. The parameter values used in this presentation are entirely in the experimentally realizable regime [17,18]. The value $\gamma=2$ reaches its upper limit. However, the essential features are not changed by reducing γ.

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