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Atomic Bose gas with a negative scattering length

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We derive the equation of state of a dilute atomic Bose gas with an interatomic interaction that has a negative scattering length and argue that two continuous phase transitions, occurring in the gas due to quantum degeneracy effects, are preempted by a first-order gas-liquid or gas-solid transition, depending on the details of the interaction potential. We also discuss the consequences of this result for future experiments with magnetically trapped spin-polarized atomic gases such as lithium and cesium.

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I. INTRODUCTION

Apart from the intrinsic interest in cooling and trapping atomic gas samples by means of electromagnetic fields [1] and their potential use in various high-precision experiments, the possibility of achieving Bose-Einstein condensation is also an important motivation for studying ultracold atomic gases in a static magnetic trap. The first steps towards this goal were made with atomic hydrogen [2,3] and recent progress in lowering the temperature of the trapped gas using both conventional [4,5] and light-induced [6] evaporative cooling shows that atomic hydrogen is still a very promising candidate for the ultimate achievement of Bose-Einstein condensation.

Due to their excellent optical properties a substantial experimental effort is presently also directed to the alkali-metal vapors cesium [7] and lithium [8]. However, to ensure the stability of the condensed phase it is usually (but see below) required that the interaction between the atoms is effectively repulsive or more precisely that its scattering length a is positive [9]. There is no doubt that the above requirement is fulfilled for atomic hydrogen, but in the case of atomic lithium the state-of-the-art triplet potential leads to a negative scattering length for the doubly spin-polarized $|F=2,M_F=2\rangle$ state [10]. In contrast, the situation for atomic cesium is less straightforward. It has recently been shown that with the present knowledge of the interaction potentials it is not possible to determine the sign of a for the doubly spin-polarized $|F=4,M_F=4\rangle$ state, but that the scattering length for the also low-field seeking $|F=3,M_F=-3\rangle$ state of the lower hyperfine manifold has a pronounced resonance structure, which offers the exciting possibility of controlling the sign of a by an appropriate choice of the magnetic-field strength [11].

In view of these circumstances it is of considerable interest to investigate the properties of a dilute atomic Bose gas with negative scattering length and to predict what one might observe in experiments with magnetically trapped atomic lithium and cesium. In particular, it is interesting to see if also for these gases quantum degeneracy leads to Bose-Einstein condensation at sufficiently high densities. However, before we can begin with a detailed discussion of these issues we must be somewhat more precise and realize that besides the scattering length a, an interatomic potential is also characterized by the hard-core radius r_C, the finite range r_V, and the well depth ϵ_W. In terms of these quantities the main difference between spin-polarized atomic hydrogen and spin-polarized atomic lithium and cesium is that in the former case the potential well is so shallow that it admits no bound states, whereas in the latter case $\epsilon_W \gg \hbar^2/mr_V^2$ and many rovibrational states are possible.

As a result the relationship between the pressure p and the inverse density $1/n$ at a fixed and nonzero temperature $T \ll \epsilon_W/k_B$ is qualitatively different and as shown schematically in Fig. 1 [12]. At these temperatures atomic lithium and cesium may thus undergo a first-order transition from a dilute ($nr_V^3 \ll 1$) gaseous phase to a high density ($nr_C^3 \simeq 1$) phase, which is either liquid or solid depending on the details of the interaction potential. The coexistence line of these two phases is obtained from a Maxwell construction and is therefore such that the areas of regions I and II are equal [13]. Note that the critical pressure $p_c(T)$ found in this manner is always larger than zero, because the area of re-

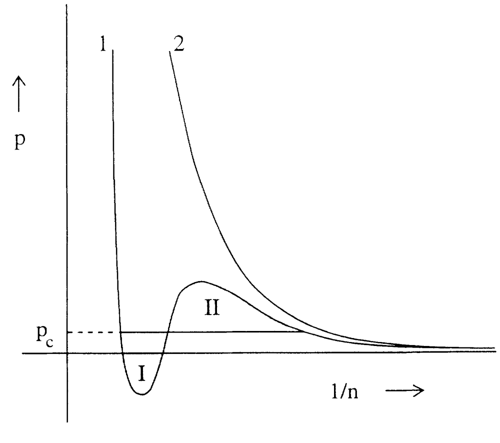

FIG. 1. Qualitative picture of the relationship between pressure and inverse density at a fixed and sufficiently low temperature for (1) atomic lithium or cesium and (2) atomic hydrogen. Note that possible cusps in the curves due to symmetry-breaking phase transitions are not shown here.

gion I is bounded and the area of region II can be made arbitrarily large since for low densities ($n\Lambda^3 \ll 1$ where $\Lambda = \sqrt{2\pi\hbar^2/mk_BT}$ is the thermal de Broglie wavelength) the equation of state reduces to $p = nk_BT$. Hence, for sufficiently low densities the gaseous phase is stable and we are allowed to incorporate the influence of the interaction by means of perturbation theory around the ideal Bose gas. Moreover, at the experimental densities of interest, we are justified in using the T-matrix or ladder approximation because $nr_V^3 \ll 1$ and we only need to include all two-body processes in our description of the dilute phase.

For an adequate determination of the coexistence line, however, a more advanced theory is needed which is capable of describing a strongly interacting system at high densities. Fortunately, in the case of magnetically trapped atomic gases we are dealing with a system that is expected to have a large free-energy barrier for the formation of a critical bubble of the dense phase. Hence, if the density is isothermally increased beyond the point of coexistence, a metastable supersaturated vapor will be formed. From an experimental point of view the relevant question to be answered is therefore the following: Can Bose-Einstein condensation also occur in the (meta)stable region of the phase diagram (where $\partial p/\partial n > 0$) of an atomic Bose gas with negative scattering length? The answer to this question can still be found within the framework of the ladder approximation and is the main topic of the present paper.

In order to do so we must consider the degenerate regime where $n\Lambda^3 \simeq 1$ and allow for the possibility of a second-order phase transition due to Bose degeneracy. Consequently, we must first of all identify the appropriate order parameter. This is achieved in Sec. II. Although Sec. II is certainly an important part of the paper, the discussion presented there is of a somewhat technical nature and makes use of functional methods which are very convenient for the derivation of the free-energy density of the order parameters of interest. However, after the correct order parameter is found we return in Sec. III to canonical methods in order to arrive at the final results of the paper in a physically more transparent and perhaps more accessible manner. In particular, we derive in Sec. III the equation of state of the gas and show that it indeed, in the region of validity of the T-matrix approximation, leads to a relationship between the pressure and the inverse density which is qualitatively in agreement with Fig. 1. Most importantly, we are then in a position to answer the question mentioned above. We end in Sec. IV with some conclusions and with a discussion of the experimental relevance of our results.

II. ORDER PARAMETER

We consider a homogeneous system of N bosonic atoms in a volume V. Using a functional approach to the imaginary time formalism, the Euclidean action of this system is

$$
\begin{aligned}
S[\psi^*,\psi] = {} & \int_0^{\hbar\beta} d\tau \int d\vec{x}\, \psi^*(\vec{x},\tau)\left(\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2\nabla^2}{2m} - \mu\right)\psi(\vec{x},\tau) \\
& + \frac{1}{2}\int_0^{\hbar\beta} d\tau \int d\vec{x} \int d\vec{x}'\, \psi^*(\vec{x},\tau)\psi^*(\vec{x}',\tau)V(\vec{x}-\vec{x}')\psi(\vec{x}',\tau)\psi(\vec{x},\tau)\,, \qquad (1)
\end{aligned}
$$

where $\beta = 1/k_BT$, μ is the chemical potential, $V(\vec{x}-\vec{x}')$ is the interatomic interaction potential, and possible three-body forces are neglected since we are interested in the dilute limit. In addition, the grand canonical partition function is then given by the functional integral

$$
Z_G = \int d[\psi^*]d[\psi]\, \exp\left(-\frac{1}{\hbar}S[\psi^*,\psi]\right) \qquad (2)
$$

over the periodic c-number fields $\psi^*(\vec{x},\tau)$ and $\psi(\vec{x},\tau)$. For the ultracold gases considered here the range of the interaction is always much smaller than the thermal de Broglie wavelength ($r_V/\Lambda \ll 1$), which implies that we can replace the potential $V(\vec{x}-\vec{x}')$ by the contact interaction $V_0\delta(\vec{x}-\vec{x}')$ with $V_0 \equiv \int d\vec{x}\, V(\vec{x})$. Together with the thermodynamic potential $\Omega = -(\ln Z_G)/\beta$ this completely determines the thermodynamic properties of the weakly interacting Bose gas.

A. Bose-Einstein condensation

Nevertheless, to extract useful information from Eqs. (1) and (2) we must resort to perturbation theory and thus incorporate the correct symmetry of the homogeneous and at high densities metastable state of the gas. Clearly, both objectives can be achieved if we are able to identify the correct order parameter. An immediate choice is the thermal average $\langle\psi(\vec{x},\tau)\rangle$. However, this choice turns out to be inappropriate for the atomic Bose gas with negative scattering length as can already be seen from the following *ad absurdum* argument: Assuming $\langle\psi(\vec{x},\tau)\rangle$ to be nonzero, the application of the usual Bogoliubov theory shows that the long-wavelength fluctuations in the order parameter are unstable for $a < 0$ [9], which invalidates the initial assumption.

A more rigorous argument is based on the free-energy functional for the time and space-independent value of the above order parameter

$$
F[\langle\psi^*\rangle,\langle\psi\rangle] = V\sum_{n=1}^{\infty}\frac{\Gamma_0^{(2n)}}{(n!)^2}|\langle\psi\rangle|^{2n}\,, \qquad (3)
$$

where $\Gamma_0^{(2n)}$ is the $2n$-point vertex function with all $2n-1$ momentum and frequency arguments equal to zero [14,15]. Using the T-matrix approximation the latter can easily be evaluated in the symmetric (low density or high temperature) phase with the result [15]

$$\Gamma_0^{(2)} = \hbar\Sigma(\vec{0};0) - \mu \equiv -\mu' \,, \tag{4a}$$

$$\Gamma_0^{(4)} = 2T^{\rm MB}(\vec{0},\vec{0},\vec{0};2\mu') \,, \tag{4b}$$

$$\Gamma_0^{(2n)} = 0 \ \text{ for } n \geq 3. \tag{4c}$$

In deriving Eq. (4b) we also made use of the fact that for the momenta $\hbar k \ll \hbar/r_V$ and Matsubara frequencies $\omega_n \ll \hbar/mr_V^2$ of interest the self-energy $\hbar\Sigma(\vec{k};\omega_n)$ is well approximated by the constant $2nT^{\rm 2B}(\vec{0},\vec{0};0) = 8\pi na\hbar^2/m$. The self-energy thus only leads to a shift in the chemical potential and we find that μ' is obtained from the equation of state of the ideal Bose gas

$$n = \frac{1}{V}\sum_{\vec{k}} N(\epsilon_{\vec{k}} - \mu') \,, \tag{5}$$

with $\epsilon_{\vec{k}} = \hbar^2\vec{k}^2/2m$ the kinetic energy of the atoms and $N(\epsilon)$ the Bose distribution $1/(e^{\beta\epsilon} - 1)$. In addition, the many-body T matrix can now be expressed in terms of the two-body T matrix $T^{\rm 2B}(\vec{0},\vec{0};0) = 4\pi a\hbar^2/m$ via [15]

$$\frac{1}{T^{\rm MB}(\vec{0},\vec{0},\vec{0};2\mu')} = \frac{1}{T^{\rm 2B}(\vec{0},\vec{0};0)} + \frac{1}{V}\sum_{\vec{k}} \frac{N(\epsilon_{\vec{k}} - \mu')}{\epsilon_{\vec{k}} - \mu'} \,. \tag{6}$$

Through Eqs. (4)–(6) we observe that for an effectively repulsive interaction ($a > 0$) the free-energy functional has the same form as in the Landau theory of second-order phase transitions [16]. In particular, the free energy is bounded from below and acquires at temperatures below the critical temperature of the ideal Bose gas

$$T_0 = \frac{2\pi\hbar^2}{mk_B}\left(\frac{n}{\zeta(\frac{3}{2})}\right)^{2/3} \tag{7}$$

a minimum for $\langle\psi\rangle \equiv \sqrt{n_0}e^{i\vartheta}$, which determines the condensate density n_0. However, for an effectively attractive interaction ($a < 0$) this is not true, because the coefficient of the quartic term in the free-energy density is negative. We are therefore again led to the conclusion that in the case of an atomic Bose gas with negative scattering length $\langle\psi(\vec{x},\tau)\rangle$ is not the correct order parameter and that Bose-Einstein condensation in this canonical sense does not take place.

B. Evans-Rashid transition

Instead, an analogy with the BCS theory of superconductivity suggests that if the atoms have an effectively attractive interaction, a phase transition with the order parameter $\Delta(\vec{x},\tau) \equiv V_0\langle\psi(\vec{x},\tau)\psi(\vec{x},\tau)\rangle$ occurs. To see if this is indeed the case we must derive and solve the equation for the equilibrium value of $\Delta(\vec{x},\tau)$ in the ladder approximation, which requires the inclusion of fluctuations around the usual BCS (mean-field) theory. As a first step towards this goal we perform a Hubbard-Stratonovich transformation by multiplying the grand canonical partition function Z_G by

$$1 = \mathcal{N}\int d[\Delta^*]d[\Delta]\, \exp\left(\frac{1}{2\hbar V_0}\int_0^{\hbar\beta} d\tau \int d\vec{x}\, |\Delta(\vec{x},\tau) - V_0\psi(\vec{x},\tau)\psi(\vec{x},\tau)|^2\right) \tag{8}$$

and integrating over the fields $\psi^*(\vec{x},\tau)$ and $\psi(\vec{x},\tau)$. The latter is conveniently accomplished in Nambu space and requires the introduction of the two-component field $\phi(\vec{x},\tau) \equiv (\psi(\vec{x},\tau),\psi^*(\vec{x},\tau))$ and the corresponding matrix of exact (normal and anomalous) one-particle Green's functions $G(\vec{x},\tau;\vec{x}',\tau') \equiv -\langle T_\tau[\phi(\vec{x},\tau)\phi^\dagger(\vec{x}',\tau')]\rangle$, obeying the Gorkov equation

$$\begin{pmatrix} -\hbar\frac{\partial}{\partial\tau} + \frac{\hbar^2\nabla^2}{2m} + \mu & -\Delta(\vec{x},\tau) \\ -\Delta^*(\vec{x},\tau) & \hbar\frac{\partial}{\partial\tau} + \frac{\hbar^2\nabla^2}{2m} + \mu \end{pmatrix} G(\vec{x},\tau;\vec{x}',\tau') = \hbar\delta(\vec{x} - \vec{x}')\delta(\tau - \tau') \,. \tag{9}$$

In this manner we arrive at an effective action for the order parameter $\Delta(\vec{x},\tau)$, which is formally given by [17]

$$S[\Delta^*,\Delta] = \frac{\hbar}{2}{\rm Tr}[\ln(-G^{-1})] - \frac{1}{2V_0}\int_0^{\hbar\beta} d\tau \int d\vec{x}\, |\Delta(\vec{x},\tau)|^2 \tag{10}$$

and contains all the desired information on the possibility of a BCS-like phase transition. In particular, it can be expanded in powers of $\Delta(\vec{x},\tau)$ by using the Dyson equation $G^{-1} = G_0^{-1} - \Sigma$, leading to

$${\rm Tr}[\ln(-G^{-1})] = {\rm Tr}[\ln(-G_0^{-1})] - \sum_{n=1}^{\infty} \frac{{\rm Tr}[(G_0\Sigma)^n]}{n} \,, \tag{11}$$

and by taking the self-energy matrix equal to

$$\hbar\Sigma(\vec{x},\tau;\vec{x}',\tau') = \begin{pmatrix} 0 & \Delta(\vec{x},\tau) \\ \Delta^*(\vec{x},\tau) & 0 \end{pmatrix} \delta(\vec{x} - \vec{x}')\delta(\tau - \tau') \,. \tag{12}$$

For our purposes we are especially interested in the quadratic term in this expansion. After performing the trace over both coordinate and Nambu space it is found to be

$$S^{(2)}[\Delta^*,\Delta] = -\int_0^{\hbar\beta} d\tau \int d\vec{x} \int_0^{\hbar\beta} d\tau' \int d\vec{x}'\; \Delta^*(\vec{x},\tau)\hbar G_\Delta^{-1}(\vec{x},\tau;\vec{x}',\tau')\Delta(\vec{x}',\tau') , \tag{13}$$

where the "noninteracting" Green's function of the order parameter obeys

$$G_\Delta^{-1}(\vec{x},\tau;\vec{x}',\tau') = \frac{1}{2\hbar^2}G_{0,11}(\vec{x},\tau;\vec{x}',\tau')G_{0,11}(\vec{x},\tau;\vec{x}',\tau') + \frac{1}{2\hbar V_0}\delta(\vec{x}-\vec{x}')\delta(\tau-\tau') \tag{14}$$

or equivalently

$$\begin{aligned} G_\Delta(\vec{x},\tau;\vec{x}',\tau') &= 2\hbar V_0 \delta(\vec{x}-\vec{x}')\delta(\tau-\tau') \\ &\quad -\frac{V_0}{\hbar}\int_0^{\hbar\beta} d\tau'' \int d\vec{x}''\; G_{0,11}(\vec{x},\tau;\vec{x}'',\tau'')G_{0,11}(\vec{x},\tau;\vec{x}'',\tau'')G_\Delta(\vec{x}'',\tau'';\vec{x}',\tau') . \end{aligned} \tag{15}$$

Transforming to frequency-momentum space then gives

$$G_\Delta(\vec{K};\Omega_n) = 2\hbar V_0 + \frac{1}{V}\sum_{\vec{k}} V_0 \frac{1 + N(\epsilon_{\vec{K}/2+\vec{k}} - \mu) + N(\epsilon_{\vec{K}/2-\vec{k}} - \mu)}{i\hbar\Omega_n + 2\mu - \epsilon_{\vec{K}/2+\vec{k}} - \epsilon_{\vec{K}/2-\vec{k}}} G_\Delta(\vec{K};\Omega_n) , \tag{16}$$

having the solution

$$G_\Delta(\vec{K};\Omega_n) = 2\hbar T^{\mathrm{MB}}(\vec{0},\vec{0},\vec{K};i\hbar\Omega_n + 2\mu) . \tag{17}$$

In mean-field theory we neglect all fluctuations and as a result conclude that the Evans-Rashid transition [17,19] occurs at a temperature determined by

$$\begin{aligned} &\int_0^{\hbar\beta} d\tau \int d\vec{x}\, G_\Delta^{-1}(\vec{x},\tau;\vec{x},\tau) \\ &\qquad = G_\Delta^{-1}(\vec{0};0) = \frac{1}{2\hbar T^{\mathrm{MB}}(\vec{0},\vec{0},\vec{0};2\mu)} = 0 , \end{aligned} \tag{18}$$

which corresponds exactly to the Thouless criterium for the onset of the BCS instability [18]. For a dilute gas, however, mean-field theory is not sufficiently accurate and we must also consider the fluctuations. Fortunately, Eq. (17) shows how we can apply the T-matrix approximation to the Evans-Rashid transition. Indeed, due to this relationship the renormalization of the quadratic term in the action caused by the presence of the $|\Delta|^4$ and the $|\Delta|^6$ "interactions" can be accounted for by replacing in the above discussion the noninteracting Green's function $G_{0,11}(\vec{k};\omega_n) = \hbar/(i\hbar\omega_n - \epsilon_{\vec{k}} + \mu)$ by its renormalized (within the ladder approximation) value $\hbar/[i\hbar\omega_n - \epsilon_{\vec{k}} - \hbar\Sigma(\vec{k};\omega_n) + \mu]$. Since $\hbar\Sigma(\vec{k};\omega_n)$ is well approximated by the constant $2nT^{2\mathrm{B}}(\vec{0},\vec{0};0)$ this implies just a replacement of the chemical potential μ by μ'. The onset of the BCS-like instability is therefore in the dilute limit determined by

$$\begin{aligned} \frac{1}{T^{\mathrm{MB}}(\vec{0},\vec{0},\vec{0};2\mu')} &= \frac{1}{T^{2\mathrm{B}}(\vec{0},\vec{0};0)} + \frac{1}{V}\sum_{\vec{k}} \frac{N(\epsilon_{\vec{k}} - \mu')}{\epsilon_{\vec{k}} - \mu'} \\ &= 0 . \end{aligned} \tag{19}$$

Consequently, the critical temperature of the Evans-Rashid transition is slightly above the critical temperature of the ideal Bose gas, i.e., $T_{\mathrm{ER}} = T_0[1 - O(a/\Lambda_0)]$ [15].

Notice that the above discussion shows that the pair field $\psi(\vec{x},\tau)\psi(\vec{x},\tau)$ can be used even if the interatomic potential has a negative scattering length without having a bound state. This is a truly many-body effect which occurs because bosons prefer to scatter into (momentum) states that are already occupied. As a result the binding between two particles is effectively increased and a resonance, which must always lie just above the continuum threshold for a to be negative, becomes bound at a density-temperature combination determined by Eq. (19).

III. STABILITY OF THE GASEOUS PHASES

In the preceding section we have argued that in the degenerate regime the relevant order parameter for an atomic Bose gas with negative scattering length is $\Delta(\vec{x},\tau) = V_0\langle\psi(\vec{x},\tau)\psi(\vec{x},\tau)\rangle$. In order to proceed and to discuss whether the corresponding phase transition can take place in the (meta)stable region of the phase diagram we must now also consider the gas below the critical temperature T_{ER}. This can of course be achieved by the functional approach used above, but to make the following more transparent we will from now on use canonical methods.

Denoting the equilibrium value of $\Delta(\vec{x},\tau)$ by Δ_0 and applying the BCS approach in combination with the T-matrix approximation, the Hamiltonian of the gas is approximated by

$$\begin{aligned} H = \int d\vec{x}\, \Big\{ &\psi^\dagger(\vec{x}) \left(-\frac{\hbar^2\nabla^2}{2m} - \mu'\right)\psi(\vec{x}) \\ &+ \frac{\Delta_0}{2}\psi^\dagger(\vec{x})\psi^\dagger(\vec{x}) + \frac{\Delta_0^*}{2}\psi(\vec{x})\psi(\vec{x}) \\ &- \frac{|\Delta_0|^2}{2V_0} - n^2 T^{2\mathrm{B}}(\vec{0},\vec{0};0)\Big\} \end{aligned} \tag{20}$$

in the Schrödinger picture. After a diagonalization of this Hamiltonian by means of a Bogoliubov transformation

[20] the density $n = \langle \psi^\dagger(\vec{x})\psi(\vec{x})\rangle$ and the order parameter $\Delta_0 = V_0\langle \psi(\vec{x})\psi(\vec{x})\rangle$ are easily calculated, resulting in the equation of state

$$n = \frac{1}{V}\sum_{\vec{k}}\left\{\frac{\epsilon_{\vec{k}}-\mu'}{\hbar\omega_{\vec{k}}}N(\hbar\omega_{\vec{k}}) + \frac{\epsilon_{\vec{k}}-\mu'-\hbar\omega_{\vec{k}}}{2\hbar\omega_{\vec{k}}}\right\} \tag{21}$$

and the BCS gap equation [17,19]

$$\frac{1}{V_0} + \frac{1}{V}\sum_{\vec{k}}\frac{1+2N(\hbar\omega_{\vec{k}})}{2\hbar\omega_{\vec{k}}} = 0\,,$$

respectively.

As will become clear in a moment it is important to note that in the dispersion $\hbar\omega_{\vec{k}} = \sqrt{(\epsilon_{\vec{k}}-\mu')^2 - |\Delta_0|^2}$ of the Bogoliubov quasiparticles there is a minus sign in front of $|\Delta_0|^2$ instead of the usual plus sign since we are dealing with paired bosons and not with paired fermions. Moreover, the gap equation has an ultraviolet divergence due to the neglect of the momentum dependence of the interaction. However, anticipating the fact that $|\Delta_0|$ is at most of $O(|nT^{2B}(\vec{0},\vec{0};0)|)$ and thus much smaller than $\hbar^2/mr_V^2$, we find from the Lippmann-Schwinger equation [21] for the two-body T matrix that this divergence is canceled by a renormalization of $1/V_0$ to $1/T^{2B}(\vec{0},\vec{0};0)$. Therefore the gap equation becomes

$$\frac{1}{T^{2B}(\vec{0},\vec{0};0)} + \frac{1}{V}\sum_{\vec{k}}\frac{N(\hbar\omega_{\vec{k}})}{\hbar\omega_{\vec{k}}} = 0\,, \tag{22}$$

which is free of divergencies. Together with Eq. (21) it determines both μ' and $|\Delta_0|$ for a fixed density and temperature.

At high temperatures the gap equation has no solution and we must use $|\Delta_0| = 0$. In that case Eq. (21) reduces to the expected equation of state for a dilute Bose gas in the normal phase [cf. Eq. (5)]. Below the critical temperature $T_{\rm ER}$, determined by a linearization of Eq. (22) which correctly leads to Eq. (19), the order parameter $|\Delta_0|$ becomes nonzero and the gas is in a superfluid phase with paired atoms. Lowering the temperature, both $|\Delta_0|$ and μ' increase, but in such a manner that the gap in the dispersion of the Bogoliubov quasiparticles decreases since otherwise the system would not be able to accommodate the same number of particles. Evidently, this behavior is possible due to the above mentioned minus sign in the dispersion relation.

At a second critical temperature $T_{\rm BEC}$ the gap closes and the number of particles with zero momentum diverges, which signals a Bose-Einstein condensation (BEC). Below that temperature we have $|\Delta_0| = -\mu'$ and $\hbar\omega_{\vec{k}}$ equals the famous Bogoliubov dispersion $\sqrt{\epsilon_{\vec{k}}^2 - 2\mu'\epsilon_{\vec{k}}}$ [20]. Furthermore, Eqs. (21) and (22) are replaced by

$$n = n_0 + \frac{1}{V}\sum_{\vec{k}\neq\vec{0}}\left\{\frac{\epsilon_{\vec{k}}-\mu'}{\hbar\omega_{\vec{k}}}N(\hbar\omega_{\vec{k}}) + \frac{\epsilon_{\vec{k}}-\mu'-\hbar\omega_{\vec{k}}}{2\hbar\omega_{\vec{k}}}\right\} \tag{23}$$

and

$$\frac{1}{T^{2B}(\vec{0},\vec{0};0)} - \frac{n_0}{\mu'} + \frac{1}{V}\sum_{\vec{k}\neq\vec{0}}\frac{N(\hbar\omega_{\vec{k}})}{\hbar\omega_{\vec{k}}} = 0\,, \tag{24}$$

determining now μ' and the condensate density n_0 for a fixed density and temperature. Notice that the mechanism for Bose-Einstein condensation is identical to the mechanism causing Bose-Einstein condensation in the ideal Bose gas. In particular, there is no spontaneous breaking of symmetry associated with the second transition and the order parameter is n_0 and not $\langle \psi(\vec{x},\tau)\rangle$, which is zero also below $T_{\rm BEC}$.

To discuss the question of the mechanical stability of the various phases and consequently the experimental observability of the two transitions we must calculate the pressure of the gas. Since the Hamiltonian in Eq. (20) is quadratic this is easily accomplished by evaluating the thermodynamic potential $\Omega = -pV$ and we find

$$\begin{aligned} p = {} & \frac{1}{2V}\sum_{\vec{k}}\left\{\epsilon_{\vec{k}} - \mu' - \frac{|\Delta_0|^2}{2(\epsilon_{\vec{k}}-\mu')} - \hbar\omega_{\vec{k}}\right\} \\ & + \frac{|\Delta_0|^2}{2T^{2B}(\vec{0},\vec{0};0)} + n^2 T^{2B}(\vec{0},\vec{0};0) \\ & - \frac{k_B T}{V}\sum_{\vec{k}}\ln\left(1 - e^{-\beta\hbar\omega_{\vec{k}}}\right)\,, \end{aligned} \tag{25}$$

where we have again canceled an ultraviolet divergence in the expression by renormalizing $1/V_0$ to $1/T^{2B}(\vec{0},\vec{0};0)$. The above expression is valid for all temperatures if we use $|\Delta_0| = 0$ above $T_{\rm ER}$ and $|\Delta_0| = -\mu'$ below $T_{\rm BEC}$.

In Fig. 2 we show the behavior of the pressure as a function of inverse density at a fixed temperature. At high densities the pressure stays negative, which is unphysical and caused by the fact that the T-matrix approximation incorporates the hard core of the interatomic poten-

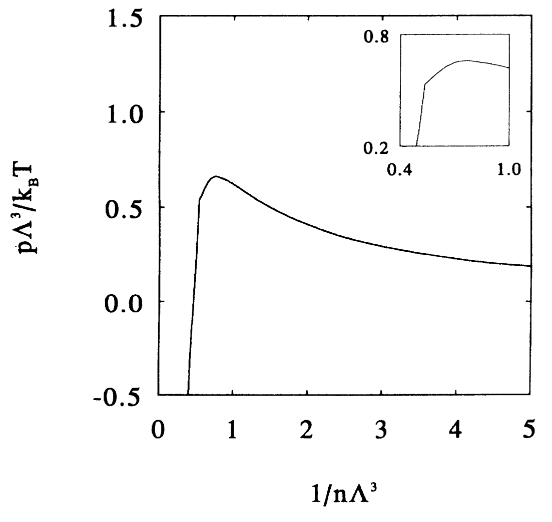

FIG. 2. Pressure as a function of inverse density for a fixed temperature such that $|a/\Lambda| = 10^{-1}$. The inset shows the cusp in the pressure caused by the Evans-Rashid transition.

tial only in an effective way. For a dilute gas $nr_C^3 \ll 1$ and this is indeed justified. At higher densities, however, the hard core becomes of the utmost importance to avoid the complete collapse of the system and must be treated with more care. Nevertheless, we can conclude from Fig. 2 that the gas passes in thermal equilibrium through a first-order transition from a gaseous phase to a phase with high density. The dense phase cannot be described accurately within the framework of the ladder approximation and a more advanced theory, capable of describing a strongly interacting system, is required to discuss whether it is liquid or solid and also its possible superfluid properties.

However, as explained in the Introduction, for the experimental observability of the Evans-Rashid transition we only need to answer the question if it can take place in the (meta)stable region of the phase diagram, where $\partial p/\partial n > 0$. It is not difficult to show [by comparing Eq. (26) with Eq. (19)] that this is never the case. Therefore, we expect the following behavior of the atomic system: By increasing the density or lowering the temperature, the gas will evolve from a stable to a metastable state until, when $\partial p/\partial n = 0$, a point on the spinodal line is reached. Increasing the density further the gas is quenched into the unstable region of the phase diagram and will experience a phase separation by means of a spinodal decomposition.

IV. CONCLUSIONS AND DISCUSSION

In the case of trapped atomic gas samples such a spinodal decomposition should be easily visible experimentally because it causes an increase in the density at the center of the trap and consequently leads to a sudden increase in the number of two-body relaxation and three-body recombination processes that severely limit the lifetime of the system. Therefore, we present in Fig. 3 the spinodal line, following from the condition

$$\frac{1}{T^{2\mathrm{B}}(\vec{0},\vec{0};0)} + \frac{1}{V}\sum_{\vec{k}} \frac{N(\epsilon_{\vec{k}} - \mu')}{\epsilon_{\vec{k}}} = 0\,. \tag{26}$$

We notice that the degeneracy parameter $n\Lambda^3$ is always smaller than $\zeta(\frac{3}{2}) \simeq 2.612$, implying that it is easier to obtain the required condition for phase separation than the condition for Bose-Einstein condensation in the case of an atomic gas with positive scattering length. This is particularly true for atomic cesium in the $|F = 3, M_F = -3\rangle$ state, which can have large negative values of the scattering length by tuning the magnetic field strength. Whether the dynamics of the spinodal decomposition is also observable is unclear at this moment because it requires a detailed study of the growth of the liquid or solid phase taking into account the release of latent heat and the enhanced importance of inelastic collision processes that lead to decay of the sample.

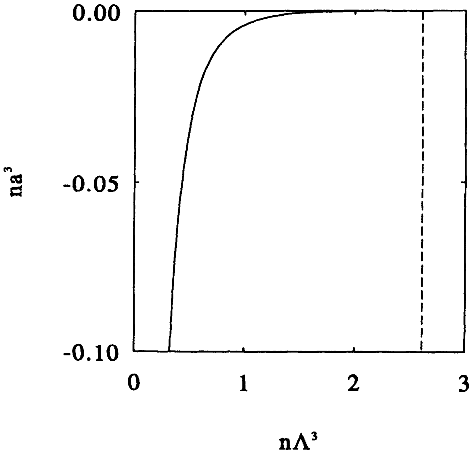

FIG. 3. Spinodal line in the $(n\Lambda^3, na^3)$ plane, separating the (meta)stable and unstable regions of the phase diagram. The dashed line corresponds to the condition for Bose-Einstein condensation in a Bose gas with repulsive interactions.

Summarizing, we have studied the dilute atomic Bose gas with negative scattering length and have shown that although Bose-Einstein condensation can in principle also occur in this case, it does not take place in the gas phase. In contrast, we argue that the gas separates into a normal gas and a (possibly superfluid) liquid or solid. For magnetically trapped atomic lithium or cesium the most likely scenario is phase separation into a gas and a solid since both the thermal energy as well as the energy due to zero-point motion are much smaller than the depth of the potential well. However, a more elaborate discussion is necessary to confirm this conjecture. In any case the separation proceeds by means of a spinodal decomposition and gives a clear experimental signal if one monitors the decay of the atomic density.

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