Collision integral for ionization by electron impact for nonisotropic electron distribution functions

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A collision integral is derived that gives the rate of change of an electron distribution function due to ionizing collisions made by electrons with gas atoms. The treatment is valid for nonisotropic electron distribution functions. The development of the collision integral in this paper uses doubly differential cross sections (DDCS's) for ionization since accurate analytical approximations for different target gases will be more easily available for DDCS's than for the other cross sections that give a more complete description of the ionizing collisions. The treatment takes advantage of the inherent incompleteness in the description provided by the DDCS's and reduces (the incompletely specified) three-body ionization problem into a set of two, fictitious, binary collision processes. These binary collisions are valid in mutually exclusive ranges of parameter space and complement each other in their contributions to the collision integral. The price for this simplification is that these binary collisions conserve neither kinetic energy nor orbital angular momentum. To handle this, the phase space of the gas atoms has been expanded by introducing fictitious scalar and vector parameters, which lend extra internal degrees of freedom to the atoms and destroy the isotropy of space. The overall approach is based on the treatment of S. Chapman and T. G. Cowling [*The Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, London, 1952)].

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I. INTRODUCTION

For a realistic modeling of the numerous plasma physics experiments and of the various plasma based technologies that are coming into increasing use, it is necessary to account for the ionization process with reasonable accuracy and in a self-consistent manner. Ionization by electron impact is one of the most important ionization mechanisms in plasmas and this can be accounted for by introducing an appropriate collision term in the Boltzmann equation for electrons. Such a collision term (also called a collision integral) would give the rate of increase of the electron distribution function due to ionizing collisions, at each point of the electron phase space.

A collision integral for ionization by electron impact was first constructed and used in the electron Boltzmann equation by Holstein [1]. The integral uses the total and differential scattering cross sections for ionization of a gas by electrons. The differential scattering cross sections used in this work are for the processes in which an incoming primary electron of given energy produces a "secondary" (or a "scattered") electron of specified energy. The treatment assumes that electron scattering and production during ionization are isotropic and that the electron distribution function is isotropic in the velocity. This form of the collision integral is frequently used by workers [2] for computing theoretically the ionization and transport coefficients of electron swarms moving through a gas. In such applications it is customary to use the two-term spherical harmonic expansion for the electron distribution function (Lorentz approximation) [1,3], and the collision integral for ionization is used only in the equation for the zeroth-order isotropic part of the distribution function [2].

In most plasma experiments and applications, however, a magnetic field is usually used to improve the confinement of the plasma. For such plasmas the electron distribution functions are highly anisotropic in the velocity and the use of the Lorentz approximation is no longer justified. This in turn implies that, in such cases, the use of collision integrals which assume the electron distribution function to be isotropic in the velocity is also not valid.

The construction of collision integrals for ionization that do not have the above limitation requires not only a careful analysis of the kinematics of the collision process but also more detailed information on the nature of the collisions involved. The latter information is supplied through the collision cross sections. The most detailed information on an ionizing collision is provided by the triply differential cross section (TDCS) [4,5], written formally as $\partial^3\sigma/\partial E_1\partial\Omega_1\partial\Omega_2$. Here dE_1 and $d\Omega_1$ give, respectively, the energy range (E_1, E_1+dE_1) and the solid angle element into which one of the outgoing electrons from the ionization process is scattered or ejected, while $d\Omega_2$ specifies the solid angle element into which the other electron is ejected or scattered. The cross sections are specified in a reference frame in which the gas atoms are initially at rest before the collision. Due to conservation of energy and momentum, this description of the collision (with the energy E_p of the incoming electron as a parameter) suffices to fix the final momenta of all the outgoing particles (including that of the ion). Cross sections are also specified as doubly differential cross sections (DDCS's) [4,5], written as $\partial^2\sigma/\partial E_1\partial\Omega_1$ and obtained formally from the TDCS by an integration over Ω_2, the

direction variables of the second electron. A further integration over Ω_1 yields the singly differential cross section [1,2] ($\equiv \partial\sigma/\partial E_1$), used in the work of Holstein [1].

While it is difficult to obtain complete information on TDCS's for even the simplest target gases such as hydrogen, fairly detailed DDCS data are already available for a large number of target gases [4,5]. Using the available data, therefore, it is possible to construct semiempirical models which give accurate analytical approximations to DDCS's for a large number of gases. Such a semiempirical model was presented recently by Rudd [6] for hydrogen and helium and similar calculations for argon are underway [7]. Thus, even though TDCS's yield more information than DDCS's, it is the latter that are more convenient to use in applications requiring suitable analytical approximations to the cross sections. The collision integral developed in the present work uses DDCS's for modeling ionization by electron impact.

Although the two outgoing electrons are indistinguishable, an artificial distinction is made in the literature to aid in bookkeeping [2,4,6]. In this nomenclature, the outgoing electron with the lower energy is called the secondary electron ($e_{\rm sec}$), while the other with the higher energy is called the scattered electron ($e_{\rm sca}$). It can be seen that a description of the collision in terms of E_1 and Ω_1 (as is done in the case of DDCS's) does not permit the determination of the momenta of the ion and the second electron. Therefore, when using DDCS's, it is customary to regard the atom and ion as infinitely heavy or massive and treat the ionizing collision in the fixed reference frame formed by the immobile atoms and ions. In such a frame, all the available energy is shared between the two outgoing electrons. There is no correlation between the momenta of the scattered and secondary electrons, however, since the fixed atom or ion can absorb or impart arbitrary amounts of momentum in any direction. The energy of the secondary electron ($=E_{\rm sec}$) lies in the range $0 \le E_{\rm sec} \le (E_p - E_i)/2$, while that for the scattered electron ($=E_{\rm sca}$) lies in the range $(E_p - E_i)/2 \le E_{\rm sca} \le (E_p - E_i)$. Here E_p is the energy of the incoming electron and E_i is the ionization energy.

It can be seen that with the above terminology, the DDCS data become partitioned into two parts: one part pertaining only to the secondary electron and the other only to the scattered electron. This partitioning of the DDCS data leads to considerable simplification in describing the ionizing collision. Thus, in considering data for the secondary electron, say, one may replace the actual collision reaction $e + A \rightarrow (I + e_{\rm sca}) + e_{\rm sec}$ by an equivalent binary collision, viz., $e + A \rightarrow S + e_{\rm sec}$. Here A (I) is the atom (ion) and S is a fictitious particle which replaces the composite system ($I + e_{\rm sca}$). Such a replacement can be made only because this partition of the DDCS contains no information regarding the ion or the scattered electron, except for their total energy (deduced from energy conservation). The energy of S is equal to that of the system ($I + e_{\rm sca}$). For a massive atom or ion, however, S will also be massive and hence immobile. In such a case the energy of S ($=E_s$) will correspond to its internal energy, which has to be continuously variable in order that $E_{\rm sec}$ may vary continuously. To allow this we introduce a single, continuously variable, fictitious parameter to characterize both A and S. A change in the internal energy of A (in transforming to S) may then be effected by assuming an interaction law which depends on this fictitious parameter. This would produce a corresponding change in the kinetic energy of the incoming electron, as required by the transition from e to $e_{\rm sec}$ in the binary collision reaction proposed above.

Orbital angular momentum will be conserved in an ionizing collision if the interaction does not involve any quantity that disturbs the isotropy of space [8]. For such cases, the orbital angular momentum of each of the outgoing electrons will, in general, be different from that of the incoming electron. This will hold for massive atoms and ions also. Therefore, in the corresponding binary collision $e + A \rightarrow S + e_{\rm sec}$, the transition from e to $e_{\rm sec}$ (assuming fixed A and S) will not conserve orbital angular momentum. This violation may be permitted, however, by introducing a fictitious vector parameter attached to A and S whose orientation in space is continuously variable. (Its magnitude is of no consequence and may be kept fixed.) To violate conservation of orbital angular momentum, we break the isotropy of space by requiring that the interaction law depend on this vector parameter as well.

The above arguments apply equally well to the part of the DDCS data pertaining to the scattered electron. For this case one may introduce the equivalent binary reaction $e + A \rightarrow S' + e_{\rm sca}$, where $S' \equiv I + e_{\rm sec}$.

The paper is organized as follows. Based on the ideas introduced above, the loss term of the collision integral is formulated in Sec. II. Following that the gain term is set up in Sec. III. The overall approach is that of Chapman and Cowling [9].

II. THE LOSS TERM

Let the electron distribution function be given by $f(\mathbf{r},\mathbf{v},t)$. Here $\mathbf{r}$ represents the spatial coordinates, $\mathbf{v}$ the electron velocity, and t the time. We shall let dt denote the time interval between t and $t+dt$ and $d^3\mathbf{r}d^3\mathbf{v}$ the infinitesimal phase-space volume element located in the neighborhood of the electron phase space point $(\mathbf{r},\mathbf{v})$. Let $(\delta f/\delta t)_i$ be the change in the distribution function per unit time, per unit phase-space volume, due to ionizing collisions. Then the net change in the number of electrons in a time dt, within the volume $d^2\mathbf{r}d^3\mathbf{v}$, is given by $(\delta f/\delta t)_i d^3\mathbf{r}d^3\mathbf{v}dt$. This change is composed of two parts: a loss term, which gives the number of electrons lost from this volume in the time dt, and a gain term, which gives the number of electrons entering this volume within this time. For every electron lost due to ionization at the point $(\mathbf{r},\mathbf{v})$ two electrons will appear simultaneously at two other points $(\mathbf{r},\mathbf{v}_1)$ and $(\mathbf{r},\mathbf{v}_2)$ of phase space. (The electrons appear at the point $\mathbf{r}$, but have velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ which are different from $\mathbf{v}$.) Thus in the loss term, details of the outcome of the ionizing event are not of chief concern, since only changes at $(\mathbf{r},\mathbf{v})$ are being considered. For the gain term, however, it is the outcome of an ionizing collision that is of major interest. Now an electron entering the volume element $d^3\mathbf{r}d^3\mathbf{v}$ at $(\mathbf{r},\mathbf{v})$ in time dt may be either a secondary electron or a

scattered electron. This will depend on the energy range of the incoming electron (which caused the ionization) with respect to the energy of the electron at the phase-space point $(\mathbf{r},\mathbf{v})$ where an electron is added.

Let the electron energy at the phase-space point $(\mathbf{r},\mathbf{v})$ be denoted by E $(=mv^2/2)$. If the electron, enters the phase space at $(\mathbf{r},\mathbf{v})$ as a secondary electron, then $E_{\rm sec}=E$ and as per the discussion of the preceding section, E will lie in the range $0\le E\le(E_p-E_i)/2$, where E_p is the energy of the incoming electron that caused the ionization. This inequality may be inverted to give $E_p\ge 2E+E_i$. It is thus seen that whenever E_p lies in the range $2E+E_i\le E_p<\infty$, the electron produced at $(\mathbf{r},\mathbf{v})$ will be a secondary electron. Likewise, it can be shown that whenever E_p lies in the range $E+E_i\le E_p\le 2E+E_i$, the electron emerging at $(\mathbf{r},\mathbf{v})$ will be a scattered electron. It may be noted that the above two ranges for E_p are nonoverlapping, so that from any ionization event, only one electron (either the scattered one or the secondary, but not both) can contribute to the gain term at $(\mathbf{r},\mathbf{v})$. This means that if the contributing electron is a secondary electron, no information regarding the scattered electron is required and vice versa. It is thus seen that the DDCS's (which are specified with respect to the energy and momentum of any one of the outgoing electrons) would suffice for evaluating the gain term of the collision integral and as discussed in detail in Sec. I, the actual ionizing collision reaction may therefore be replaced by the fictitious but equivalent binary collision reactions $e+A\rightarrow S+e_{\rm sec}$ or $e+A\rightarrow S'+e_{\rm sca}$. For the loss term, to prevent double counting any one of these (but not both) may be used. For the gain term, both reactions have to be used depending on the energy range of the incoming electron.

We now proceed to construct the loss term using the reaction $e+A\rightarrow S+e_{\rm sec}$. We assume initially that A (and hence S) has a finite mass, so that a velocity distribution can be attributed to the gas atoms. (We shall invoke the limit of infinite mass later in the derivation.) As discussed in Sec. I, this reaction does not conserve kinetic energy and orbital angular momentum. We therefore let the distribution function of the gas atoms be denoted by $F(\mathbf{r},\mathbf{V},p,l,t)$. Here $\mathbf{V}$ is the velocity variable of the gas atoms and p is a dummy, continuously variable parameter, describing the inner state of A and S. We assume that it affects the collision interaction and so allows the internal energy of A to change continuously. This permits the total kinetic energy to change during the collision. More specifically, this prescription allows the kinetic energy of $e_{\rm sec}$ to change continuously even when A is treated as massive. The phase space of the gas atoms has therefore been expanded to include p. (In this discussion, p is assumed to be dimensionless, since this does not lead to any loss of generality.) Likewise, l is another dimensionless vector parameter (of fixed magnitude) attached to A and S. l also takes part in the collision interaction and allows the orbital angular momentum to change during the collision, by introducing the required anisotropy into space.

To represent l ($|l|=1$), we assume that a set of Cartesian direction axes has been set up at each point $\mathbf{r}$ of space. The vector l can then be represented by a polar angle θ and an azimuthal angle ϕ at each space point $\mathbf{r}$.

Let the loss term be denoted by $(\delta f/\delta t)_i^{\rm LT}$. The number of electrons lost from the volume $d^2\mathbf{r}d^3\mathbf{v}$ in time dt is $(\delta f/\delta t)_i^{\rm LT}\,d^3\mathbf{r}d^3\mathbf{v}dt$. The number of electrons present in this volume at time t is $f(\mathbf{r},\mathbf{v},t)d^3\mathbf{r}d^3\mathbf{v}$. The number lost from this in time dt (due to ionization) will be equal to the number of ionizing collisions these electrons make with the gas atoms at $\mathbf{r}$, within the time interval dt.

To determine the number of collisions [9] we consider the problem in a reference frame $Q(\mathbf{r},t)$ with its origin attached to the colliding electrons at $(\mathbf{r},t)$ moving with a precollision velocity $\mathbf{v}$. Thus $Q(\mathbf{r},t)$ also moves with a velocity $\mathbf{v}$ in the laboratory frame before collision. (After the collision its velocity becomes $\mathbf{v}'$, the final postcollision velocity of the electrons.) Next consider atoms with velocities in the volume $d^3\mathbf{V}$ (in the neighborhood of $\mathbf{V}$) at time t just before collision occurs. The total spatial density of such atoms at the space-time point $(\mathbf{r},t)$ is given by

$$d^3\mathbf{V}\int F(\mathbf{r},\mathbf{V},p,l,t)dp\,d\Omega_l\ . \tag{1}$$

Here the integration is over all possible values of p and all possible orientations of l. [$d\Omega_l=\sin(\theta)d\theta d\phi$ is the solid angle element at $\mathbf{r}$, within which l lies.] These gas atoms will have a relative velocity $\mathbf{g}=\mathbf{V}-\mathbf{v}$ in the frame $Q(\mathbf{r},t)$. We now choose the polar axis of $Q(\mathbf{r},t)$ to be along the vector $\mathbf{g}$ and construct the azimuthal plane (normal to $\mathbf{g}$) passing through the origin of $Q(\mathbf{r},t)$. Polar angles measured in $Q(\mathbf{r},t)$ will be denoted by χ, while azimuthal angles measured from a suitably chosen axis in the azimuthal plane [of $Q(\mathbf{r},t)$] will be denoted by ϵ. Of the gas atoms having a spatial density given by (1) above at $(\mathbf{r},t)$, consider those which are incident on the azimuthal plane of $Q(\mathbf{r},t)$ (see Fig. 1) with impact parameter and azimuthal angle in the ranges $b,b+db$ and $\epsilon,\epsilon+d\epsilon$. The direction of incidence of these atoms is parallel to the polar axis of $Q(\mathbf{r},t)$. It can be seen, therefore, that the number of such gas atoms which can collide with the electrons at rest at the origin of $Q(\mathbf{r},t)$, within the time interval $t,t+dt$, is given by the number of these atoms contained within the parallelepiped with base area $bdbd\epsilon$ and height gdt ($g=|\mathbf{g}|$), shown in Fig. 1. The volume of such a parallelepiped is $gb\,db\,d\epsilon\,dt$ and multiplying this by the density of atoms given in (1) yields the total number of atoms with velocities in the volume $d^3\mathbf{V}$ (in the neighborhood of $\mathbf{V}$), contained within the parallelepiped at time t. It is given by

$$gb\,db\,d\epsilon\,dt\,d^3\mathbf{V}\int F(\mathbf{r},\mathbf{V},p,l,t)dp\,d\Omega_l\ . \tag{2}$$

To obtain the number of such gas atoms incident over the entire azimuthal plane of $Q(\mathbf{r},t)$, we integrate (2) over b and ϵ as well. Multiplying this by $f(\mathbf{r},\mathbf{v},t)d^3\mathbf{r}d^3\mathbf{v}$ and integrating over all possible gas velocities $\mathbf{V}$ gives the total number of collisions which take place within the time $(t,t+dt)$. Hence the loss term will be given by

$$\begin{aligned}(\delta f/\delta t)_i^{\rm LT}d^3\mathbf{r}d^3\mathbf{v}dt&=f(\mathbf{r},\mathbf{v},t)d^3\mathbf{r}d^3\mathbf{v}dt\\&\times\int F(\mathbf{r},\mathbf{V},p,l,t)d^3\mathbf{V}\,dp\,d\Omega_l gb\,db\,d\epsilon\end{aligned}$$

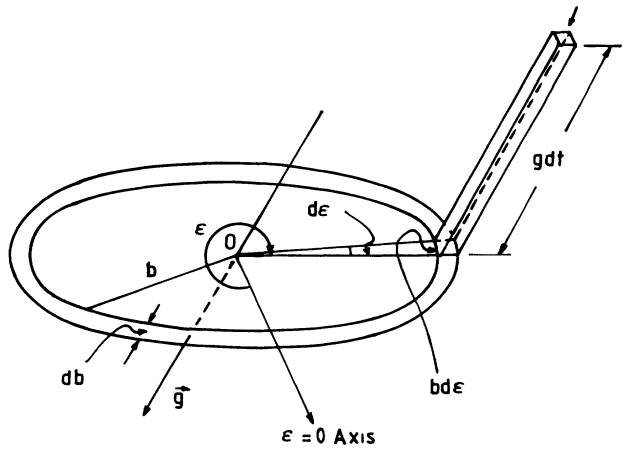

FIG. 1. Atoms are incident along the polar axis (along **g**) in the frame $Q(\mathbf{r},t)$ with impact parameters and azimuthal angles between $(b,b+db)$ and $(\epsilon,\epsilon+d\epsilon)$, respectively. The number of collisions with the electrons located at O, in the time interval $(t,t+dt)$, will be proportional to the number of such atoms contained within the parallelepiped shown in the figure (the volume is $gdtbdbd\epsilon$) at time t. (The figure has been reproduced from Ref. [9], p. 61.)

or

$$(\delta f/\delta t)_i^{\mathrm{LT}} = f(\mathbf{r},\mathbf{v},t)\int F(\mathbf{r},\mathbf{V},p,l,t)d^3\mathbf{V}\,dp\,d\Omega_l gb\,db\,d\epsilon\ , \quad (3)$$

where the integrations on the right-hand side of (3) are over $\mathbf{V}$, p, θ, ϕ, b, and ϵ.

The origin of the system $Q(\mathbf{r},t)$ coincides with the space point $\mathbf{r}$ at time t. Therefore, at time t, l is also located at the origin of $Q(\mathbf{r},t)$ and so it is possible to define polar and azimuthal angles subtended by l in the frame $Q(\mathbf{r},t)$ at this instant of time. It may be noted that the coordinate system in $Q(\mathbf{r},t)$ will in general be rotated with respect to the coordinate system in which θ and ϕ are specified. It can be shown, however, that

$$\sin(\theta)d\theta d\phi = \sin(\chi_l)d\chi_l d\epsilon_l\ , \quad (4)$$

where χ_l and ϵ_l are the polar and azimuthal angles l makes in $Q(\mathbf{r},t)$ at the instant of time t. Thus the variables θ and ϕ specifying l in (3) may be replaced by χ_l and ϵ_l with $d\Omega_l = \sin\chi_l d\chi_l d\epsilon_l$. We now specify the distribution of the gas particles over l. Since we are not considering any specific orientation, we may treat the gas particles to be unpolarized with respect to l. Thus all directions of l are equally likely and we will have

$$F(\mathbf{r},\mathbf{V},p,l,t) = \frac{F(\mathbf{r},\mathbf{V},p,t)}{4\pi}\ .$$

We may further assume that the gas particles are prepared in an initial state such that F is uniformly distributed for p within a range $p_1 \le p \le p_2$ and zero outside. This gives

$$F(\mathbf{r},\mathbf{V},p,l,t) = \begin{cases} \dfrac{F(\mathbf{r},\mathbf{V},t)}{4\pi(p_2-p_1)} = \dfrac{F(\mathbf{r},\mathbf{V},t)}{4\pi} & \text{for } p_1 \le p \le p_2 \\ 0 \quad \text{otherwise}\ , \end{cases} \quad (5)$$

where we have assumed p to be scaled appropriately so that $(p_2-p_1)=1$. Substituting (5) into (3) and integrating over χ_l gives

$$(\delta f/\delta t)_i^{\mathrm{LT}} = f(\mathbf{r},\mathbf{v},t)\int F(\mathbf{r},\mathbf{V},t)d^3\mathbf{V}\,dp\frac{d\epsilon_l}{2\pi}gb\,db\,d\epsilon\ ,$$
$$p_1 \le p \le p_2\ . \quad (6)$$

Further simplification of (6) can be made by introducing the scattering cross section. As shown in Fig. 2, consider a gas atom A impinging with velocity $\mathbf{g}$ on an electron located at the origin of $Q(\mathbf{r},t)$ at time t. The atom has impact parameter and azimuthal angle in the range $(b,b+db)$ and $(\epsilon,\epsilon+d\epsilon)$, respectively. The l vector of the atom lies within a solid angle element $d\Omega_l$ $(=\sin\chi_l d\chi_l d\epsilon_l)$ and its parameter p lies in the range (p_1,p_2). Since scattering takes place by the reaction $e+A \rightarrow S+e_{\mathrm{sec}}$, kinetic energy is not conserved in the reaction. In the frame $Q(\mathbf{r},t)$, the S particle therefore emerges with a velocity $\mathbf{g}'$ such that $g'=|\mathbf{g}'|\neq g$. ($\mathbf{g}'=\mathbf{V}'-\mathbf{v}'$, where $\mathbf{V}'$ and $\mathbf{v}'$ are the final postcollision velocities of the S particle and e_{sec}, respectively, in the laboratory frame.) $\mathbf{g}'$ makes an angle χ_0 with the polar axis $\mathbf{g}$ and since orbital angular momentum is also assumed to change during the collision ϵ_0, the angle subtended by $\mathbf{g}'$ in the azimuthal plane of $Q(\mathbf{r},t)$ is not equal to ϵ (the azimuthal angle at which the atom was originally incident).

To introduce the collision cross section we observe the following. The variables specifying the incoming particle and its trajectory are g, p, l, b, and ϵ. The variables specifying the trajectory of the outgoing particle S are $\mathbf{g}'$, χ_0, and ϵ_0. It may be noted that only the trajectory of S (and not its state) is of interest, since the trajectory of e_{sec} in the laboratory frame will be completely specified by the trajectory of S in $Q(\mathbf{r},t)$, in the limit of infinitely heavy gas particles. Thus the final value of p and the new orientation of l associated with the outgoing S particle are of no consequence.

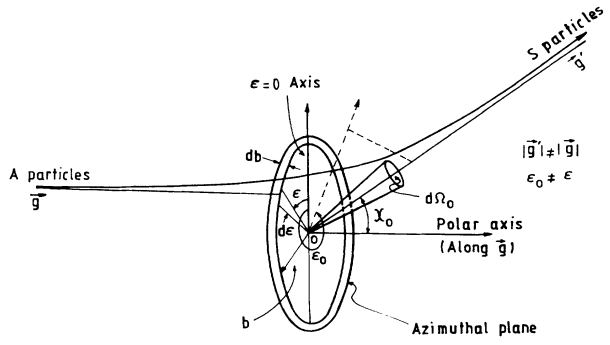

FIG. 2. Gas atoms A impinge on electrons located at O, with velocity $\mathbf{g}$, impact parameters between $(b,b+db)$, and azimuthal angles between $(\epsilon,\epsilon+d\epsilon)$. The atoms are scattered as S particles with velocity $\mathbf{g}'$, into the solid angle element $d\Omega_0$ $(=\sin\chi_0 d\chi_0 d\epsilon_0)$, where χ_0 is the angle $\mathbf{g}'$ makes with the polar axis (along $\mathbf{g}$) and ϵ_0 is the azimuthal angle subtended by $\mathbf{g}'$ in the azimuthal plane with respect to the $\epsilon=0$ axis. Note that $|\mathbf{g}'|\neq|\mathbf{g}|$ and $\epsilon_0\neq\epsilon$, since kinetic energy and orbital angular momentum are not conserved by the reaction $e+A\rightarrow S+e_{\mathrm{sec}}$.

In view of the above, an appropriate set of variables for framing the scattering problem would be the set g, g', χ_0, and ϵ_0. Here the initial speed g of A has been included in the set, since it provides a bound on the total energy available for the collision process. A corresponding initial set of variables may be identified as g, p, b, and ϵ, and the solution to the collision problem will be the transformation laws which connect the initial set of variables $(g;p,b,\epsilon)$ to the new set of variables $(g;g',\chi_0,\epsilon_0)$. It is seen, however, that the transformation laws involve the angles ϵ and ϵ_0, while the $\epsilon=0$ axis in the azimuthal plane of $Q(\mathbf{r},t)$ was chosen completely arbitrarily. To remove this arbitrariness, we make use of $\boldsymbol{l}$ as an internal reference vector and measure all angles in the azimuthal plane with respect to $\boldsymbol{l}_\perp$, the projection of $\boldsymbol{l}$ onto that plane.

Let α and β be the new angles corresponding to ϵ and ϵ_0, respectively. For consistency, we choose the convention that these angles are always measured in the counterclockwise direction from the reference vector $\boldsymbol{l}_\perp$. From Fig. 3 we see therefore that α and β have to be defined as follows:

$$\alpha=\begin{cases}\epsilon+2\pi-\epsilon_l & \text{for } \epsilon<\epsilon_l\\ \epsilon-\epsilon_l & \text{for } \epsilon>\epsilon_l\end{cases},$$
$$\beta=\begin{cases}\epsilon_0+2\pi-\epsilon_l & \text{for } \epsilon_0<\epsilon_l\\ \epsilon_0-\epsilon_l & \text{for } \epsilon_0>\epsilon_l\end{cases}. \tag{7}$$

The integral over ϵ in (6) may now be written as

$$\begin{aligned}\int_0^{2\pi}d\epsilon&=\int_0^{\epsilon_l}d\epsilon+\int_{\epsilon_l}^{2\pi}d\epsilon\\ &=\int_{2\pi-\epsilon_l}^{2\pi}d\alpha+\int_0^{2\pi-\epsilon_l}d\alpha\\ &=\int_0^{2\pi}d\epsilon=\int_0^{2\pi}d\alpha\ ,\end{aligned} \tag{8}$$

so that (6) now reads

$$(\delta f/\delta t)_i^{\mathrm{LT}}=f(\mathbf{r},\mathbf{v},t)\int d^3\mathbf{V}\,F(\mathbf{r},\mathbf{V},t)\frac{d\epsilon_l}{2\pi}gb\,db\,dp\,d\alpha\ ,$$
$$p_1\le p\le p_2\ . \tag{9}$$

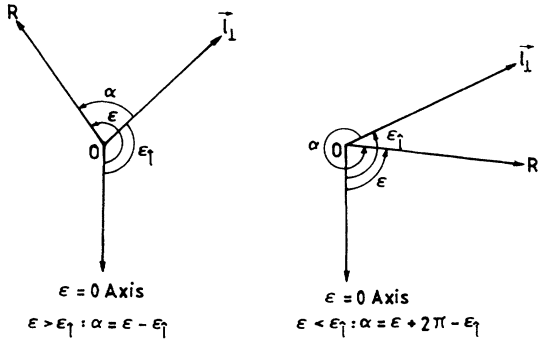

FIG. 3. Determination of α in the azimuthal plane of $Q(\mathbf{r},t)$. $\boldsymbol{l}_\perp$ makes an angle ϵ_l and OR, an angle ϵ with respect to the $\epsilon=0$ axis. α is the angle OR makes with respect to $\boldsymbol{l}_\perp$ (measured counterclockwise). The definitions of α for the cases $\epsilon>\epsilon_l$ and $\epsilon<\epsilon_l$ are given in the figure. Similar considerations hold in defining β from ϵ_0. [Note: The polar axis (along $\mathbf{g}$) points out of the plane of the paper.]

The transformation laws connecting the initial variables to the new ones may be formally written as

$$\begin{aligned}\chi_0&=\chi_0(g;p,b,\alpha)\ ,\\ \beta&=\beta(g;p,b,\alpha)\ ,\\ g'&=g'(g;p,b,\alpha)\ .\end{aligned} \tag{10}$$

These may be inverted to yield

$$\begin{aligned}p&=p(g;g',\chi_0,\beta)\ ,\\ b&=b(g;g',\chi_0,\beta)\ ,\\ \alpha&=\alpha(g;g',\chi_0,\beta)\ .\end{aligned} \tag{11}$$

In (11) we have written $p=p(g;g',\chi_0,\beta)$. Noting, however, that p is an internal variable, giving the internal energy of S, we write

$$p=p(g;g')\ , \tag{12}$$

which yields on inverting

$$g'=g'(g;p)\ . \tag{13}$$

Now p lies in the interval (p_1,p_2). This implies

$$g_1'(g;p_1)\le g'\le g_2'(g;p_2)$$

or

$$g_1'(g;p_2)\le g'\le g_2'(g;p_1)\ . \tag{14}$$

Thus $\mathbf{g}'$ cannot vary arbitrarily, but has to lie within the limits specified by (14).

Using (11) and (12) we may write

$$db\,dp\,d\alpha=|J|dg'd\chi_0d\beta\ , \tag{15a}$$

where $J=\partial(p,b,\alpha)/\partial(g',\chi_0,\beta)$ is the Jacobian of the transformation. Thus

$$b\,db\,dp\,d\alpha=\left[\frac{b(g;g',\chi_0,\beta)}{\sin\chi_0}|J|\right]dg'\sin(\chi_0)d\chi_0d\beta$$

or

$$b\,db\,dp\,d\alpha=\sigma_{g',\Omega}(g;g',\chi_0,\beta)dg'd\Omega\ , \tag{15b}$$

where $\sigma_{g',\Omega}$ is the doubly differential scattering cross section with respect to g' and Ω and is defined as

$$\sigma_{g',\Omega}(g;g',\chi_0,\beta)=\frac{b(g;g',\chi_0,\beta)}{\sin\chi_0}|J|$$

and

$$d\Omega=\sin(\chi_0)d\chi_0d\beta\ . \tag{15c}$$

Substituting (15b) into (9) gives

$$(\delta f/\delta t)_i^{\mathrm{LT}}=f(\mathbf{r},\mathbf{v},t)\int d^3\mathbf{V}\,gF(\mathbf{r},\mathbf{V},t)\times\frac{d\epsilon_l}{2\pi}\sigma_{g',\Omega}(g;g',\chi_0,\beta)dg'd\Omega\ ,$$
$$g_1'\le g'\le g_2'\ . \tag{16}$$

with g_1' and g_2' defined by (13) and (14).

The integral over β in (16) may be converted to one over ϵ_0 as follows. Writing for brevity

$$\sigma(\beta)\equiv\sigma_{g',\Omega}(g;g',\chi_0,\beta) ,$$

we obtain, considering only the integral over β in (16),

$$\int_0^{2\pi}\sigma(\beta)d\beta=\int_0^{2\pi-\epsilon_l}\sigma(\beta)d\beta+\int_{2\pi-\epsilon_l}^{2\pi}\sigma(\beta)d\beta . \tag{17}$$

Noting that $\beta=2\pi-\epsilon_l$ corresponds to the $\epsilon=0$ axis and using (7) gives

$$\beta=\begin{cases}\epsilon_0-\epsilon_l , & 0\le\beta\le 2\pi-\epsilon_l\\ \epsilon_0+2\pi-\epsilon_l , & 2\pi-\epsilon_l\le\beta\le 2\pi .\end{cases} \tag{18}$$

Further, invoking the periodicity of σ in β, i.e., $\sigma(\beta)=\sigma(\beta-2\pi)$, and using this for transforming $\sigma(\beta)$ in the second integral in (17) [with the help of the second of the transformations in (18)], we obtain

$$\sigma(\beta)=\sigma(\beta-2\pi)=\sigma(\epsilon_0+2\pi-\epsilon_l-2\pi)=\sigma(\epsilon_0-\epsilon_l) ,$$
$$2\pi-\epsilon_l\le\beta\le 2\pi .$$

Thus (17) transforms into,

$$\int_0^{2\pi}\sigma(\beta)d\beta=\int_{\epsilon_l}^{2\pi}\sigma(\epsilon_0-\epsilon_l)d\epsilon_0+\int_0^{\epsilon_l}\sigma(\epsilon_0-\epsilon_l)d\epsilon_0 ,$$
$$=\int_0^{2\pi}\sigma(\epsilon_0-\epsilon_l)d\epsilon_0 ,$$

so that (16) reads

$$(\delta f/\delta t)_i^{\mathrm{LT}}=f(\mathbf{r},\mathbf{v},t)\times\int d^3\mathbf{V}\,gF(\mathbf{r},\mathbf{V},t)\times\frac{d\epsilon_l}{2\pi}\sigma_{g',\Omega_0}(g;g',\chi_0,\epsilon_0-\epsilon_l)dg'd\Omega_0$$
$$\text{for } g_1'\le g_2', \quad d\Omega_0=\sin(\chi_0)d\chi_0 d\epsilon_0 . \tag{19}$$

At this stage we invoke the limit of infinitely heavy gas atoms and S particles. In this limit both $|\mathbf{V}|$ and $|\mathbf{V}'|\to 0$, so that

$$g\to v , \quad g'\to v' , \quad \int F(\mathbf{r},\mathbf{V},t)d^3\mathbf{V}\to N(\mathbf{r},t) ,$$

where $N(\mathbf{r},t)$ is the density of gas atoms at the space-time point $(\mathbf{r},t)$. Further, (12) and (13) become in this limit

$$p=p(v;v') , \quad v'=v'(v;p) \tag{20}$$

and (14) reads

$$v_1'(v;p_1)\le v'\le v_2'(v;p_2)$$

or

$$v_1'(v;p_2)\le v'\le v_2'(v;p_1) . \tag{21}$$

Also, $\sigma_{g',\Omega_0}(g;g',\chi_0,\epsilon_0-\epsilon_l)$ transforms to $\sigma_{v',\Omega_0}(v;v',\chi_0,\epsilon_0-\epsilon_l)$. It may be noted that the polar axis $\mathbf{g}\to-\mathbf{v}$ and the vector $\mathbf{g}'\to-\mathbf{v}'$. Thus χ_0 now is the angle between the vectors $-\mathbf{v}'$ and $-\mathbf{v}$ and ϵ_0 is the azimuthal angle subtended by the vector $-\mathbf{v}'$ with respect to the $\epsilon=0$ axis. Equation (19) now reads

$$(\delta f/\delta t)_i^{\mathrm{LT}}=vf(\mathbf{r},\mathbf{v},t)N(\mathbf{r},t)\times\int\sigma_{v',\Omega_0}(v;v',\chi_0,\epsilon_0-\epsilon_l)dv'\frac{d\epsilon_l}{2\pi}d\Omega_0 \tag{22}$$

with $v_1'\le v'\le v_2'$ and $d\Omega_0=\sin(\chi_0)d\chi_0 d\epsilon_0$.

We now evaluate the integral over ϵ_l. Again writing for brevity

$$\sigma(\epsilon_0-\epsilon_l)\equiv\sigma_{v',\Omega_0}(v;v',\chi_0,\epsilon_0-\epsilon_l) ,$$

we obtain for the integral over ϵ_l in (22)

$$\int_0^{2\pi}\sigma(\epsilon_0-\epsilon_l)\frac{d\epsilon_l}{2\pi}=\int_0^{\epsilon_0}\sigma(\epsilon_0-\epsilon_l)\frac{d\epsilon_l}{2\pi}+\int_{\epsilon_0}^{2\pi}\sigma(\epsilon_0-\epsilon_l)\frac{d\epsilon_l}{2\pi} . \tag{23}$$

We use Fig. 4 to define the variable γ. We have

$$\gamma=\begin{cases}\epsilon_0-\epsilon_l , & 0\le\epsilon_l\le\epsilon_0\\ \epsilon_0+2\pi-\epsilon_l , & \epsilon_0\le\epsilon_l\le 2\pi .\end{cases} \tag{24}$$

As before we invoke the periodicity of σ in ϵ_0 or ϵ_l to write $\sigma(\epsilon_0-\epsilon_l)=\sigma(\epsilon_0-\epsilon_l+2\pi)$. Using this for transforming σ in the second integral of (23) gives, with the help of the second of the transformations in (24),

$$\sigma(\epsilon_0-\epsilon_l)=\sigma(\epsilon_0-\epsilon_l+2\pi)=\sigma(\gamma) , \quad \epsilon_0\le\epsilon_l\le 2\pi$$

so that (23) becomes

$$\int_0^{2\pi}\sigma(\epsilon_0-\epsilon_l)\frac{d\epsilon_l}{2\pi}=-\int_{\epsilon_0}^{0}\sigma(\gamma)\frac{d\gamma}{2\pi}-\int_{2\pi}^{\epsilon_0}\sigma(\gamma)\frac{d\gamma}{2\pi}$$
$$=\int_0^{2\pi}\sigma(\gamma)\frac{d\gamma}{2\pi} . \tag{25}$$

The integral in (25) represents an averaging over ϵ_l, the azimuthal angle made by the projection $l_\perp$ of the reference vector l, in the azimuthal plane of $Q(\mathbf{r},t)$. As a result of this averaging, all dependence on the angle ϵ_0 is eliminated. This is to be expected since our original dis-

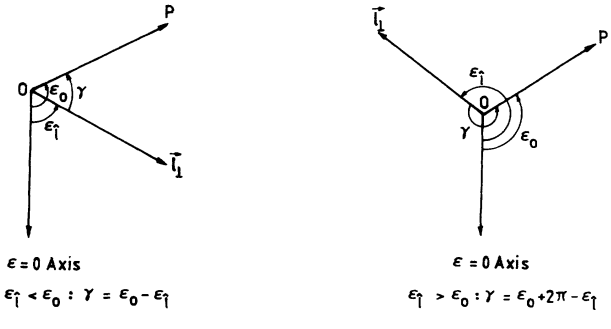

FIG. 4. Determination of γ in the azimuthal plane of $Q(\mathbf{r},t)$. $l_\perp$ and OP make angles ϵ_l and ϵ_0, respectively, with respect to the $\epsilon=0$ axis and γ is the angle OP makes with respect to $l_\perp$ (measured counterclockwise). The definitions of γ in the different ranges $\epsilon_l<\epsilon_0$ and $\epsilon_l>\epsilon_0$ are given in the figure. [Note: The polar axis (along $\mathbf{g}$) points out of the plane of the paper].

tribution of atoms was unpolarized in *I*. Equation (25) may therefore be used to define an averaged σ as follows:

$$\tilde{\sigma}_{v',\Omega_0}(v;v',\chi_0)=\int_0^{2\pi}\sigma_{v',\Omega_0}(v;v',\chi_0,\epsilon_0-\epsilon_l)\frac{d\epsilon_l}{2\pi}$$
$$=\int_0^{2\pi}\sigma_{v',\Omega_0}(v;v',\chi_0,\gamma)\frac{d\gamma}{2\pi}\ . \tag{26}$$

Substituting (26) into (22) gives

$$(\delta f/\delta t)_i^{\mathrm{LT}}=vf(\mathbf{r},\mathbf{v},t)N(\mathbf{r},t)\int dv'\tilde{\sigma}_{v',\Omega_0}(v;v',\chi_0)d\Omega_0 \tag{27}$$

with $v_1'\le v'\le v_2'$ and $d\Omega_0=2\pi\sin(\chi_0)d\chi_0$. $\tilde{\sigma}_{v',\Omega_0}(v;v',\chi_0)$ in (27) is measured in $Q(\mathbf{r},t)$. In a frame in which the atoms are at rest, the electrons approach the atoms with a velocity $+\mathbf{v}$ and are scattered off with speed v' at an angle $(\pi-\chi_0)$ with respect to the original polar axis (the $-\mathbf{v}$ direction). The cross section for the scattering of the electrons in this frame is identical to the cross section $\tilde{\sigma}_{v',\Omega_0}(v;v',\chi_0)$ given above, for the scattering of the S particle in $Q(\mathbf{r},t)$. Under a rotation of the polar axis from the $-\mathbf{v}$ direction to the $+\mathbf{v}$ direction (in the rest frame of the atoms) $\tilde{\sigma}_{v',\Omega_0}(v;v',\chi_0)$ becomes

$$\tilde{\sigma}_{v',\Omega_0}(v;v',\chi_0)\rightarrow\sigma'_{v',\Omega_0}(v;v',\chi_0)=\tilde{\sigma}_{v',\Omega_0}(v;v',\pi-\chi_0)\ , \tag{28}$$

where χ_0 is now the angle made by $+\mathbf{v}'$ with respect to the $+\mathbf{v}$ direction. Under this rotation of the polar axis in the rest frame of the atoms, the element of solid angle $d\Omega_0$ also remains unchanged. It may be noted that this frame, in which the atoms are at rest, is also the laboratory frame since the atoms have been assumed to be infinitely heavy and hence immobile both before and after the collision.

Connection with the ionization problem can be made by setting

$$\sigma'_{v',\Omega_0}(v;v',\chi_0)=\sigma^{\mathrm{sec}}_{v',\Omega_0}(v;v',\chi_0)$$
$$\text{for } 0\le v'\le\left[\frac{v^2}{2}-\frac{E_i}{m}\right]^{1/2}, \tag{29}$$

where $\sigma^{\mathrm{sec}}_{v',\Omega_0}(v;v',\chi_0)$ is the doubly differential cross section for ionization with the primary electron impinging with velocity $\mathbf{v}$ on the massive immobile atoms and the secondary electrons being scattered with velocity $\mathbf{v}'$ into a solid angle element $d\Omega_0=2\pi\sin\chi_0 d\chi_0$ (χ_0 is the angle between the $+\mathbf{v}$ and $+\mathbf{v}'$ directions). In (27) the limits on v' were those for the fictitious reaction $e+A\rightarrow S+e_{\mathrm{sec}}$, derived in (21). In (29) these are replaced by the actual limits which arise in the ionization problem. These can be deduced from the inequality $0\le E_{\mathrm{sec}}\le(E_p-E_i)^{1/2}$ by setting $E_{\mathrm{sec}}=m(v')^2/2$ and $E_p=mv^2/2$.

Likewise, we could also have set

$$\sigma'_{v',\Omega_0}(v;v',\chi_0)=\sigma^{\mathrm{sca}}_{v',\Omega_0}(v;v',\chi_0)$$
$$\text{for } \left[\frac{v^2}{2}-\frac{E_i}{m}\right]^{1/2}\le v'\le\sqrt{2}\left[\frac{v^2}{2}-\frac{E_i}{m}\right]^{1/2} \tag{30}$$

if we had used the reaction $e+A\rightarrow S'+e_{\mathrm{sca}}$ for setting up the loss term. The limits on v' in (30) are deduced from the inequality $(E_p-E_i)/2\le E_{\mathrm{sca}}\le(E_p-E_i)$, by setting $E_{\mathrm{sca}}=m(v')^2/2$ and $E_p=mv^2/2$.

Using (29) in (28) and substituting for $\tilde{\sigma}_{v',\Omega_0}(v;v',\chi_0)$ into (27) yields

$$(\delta f/\delta t)_i^{\mathrm{LT}}=vf(\mathbf{r},\mathbf{v},t)N(\mathbf{r},t)\int dv'\sigma^{\mathrm{sec}}_{v',\Omega_0}(v;v',\chi_0)d\Omega_0 \tag{31}$$

with $0\le v'\le(v^2/2-E_i/m)^{1/2}$. We now define

$$\sigma^{\mathrm{sec}}_{v'}(v;v')=\int\sigma^{\mathrm{sec}}_{v',\Omega_0}(v;v',\chi_0)d\Omega_0$$

and

$$\sigma_T(v)=\int dv'\sigma^{\mathrm{sec}}_{v'}(v;v') \tag{32}$$

with $0\le v'\le(v^2/2-E_i/m)^{1/2}$. Here $\sigma^{\mathrm{sec}}_{v'}(v;v')$ is the singly differential cross section for the secondary electron to be emitted with speed between v' and $v'+dv'$ (used in the work of Holstein). $\sigma_T(v)$ is the total ionization cross section with the incoming electron having a speed v. Analogous to (32) we will have

$$\sigma^{\mathrm{sca}}_{v'}(v;v')=\int\sigma^{\mathrm{sca}}_{v',\Omega_0}(v;v',\chi_0)d\Omega_0$$

and

$$\sigma_T(v)=\int\sigma^{\mathrm{sca}}_{v'}(v;v')dv' \tag{33}$$

with $(v^2/2-E_i/m)^{1/2}\le v'\le\sqrt{2}(v^2/2-E_i/m)^{1/2}$. It may be noted that [2]

$$\sigma^{\mathrm{sca}}_{v_1}(v;v_1)=\sigma^{\mathrm{sec}}_{v'}(v;v') \tag{34}$$

with $v_1=\sqrt{2}[v^2/2-E_i/m-(v')^2/2]^{1/2}$ and $0\le v'\le(v^2/2-E_i/m)^{1/2}$. Using the definition in (32) and carrying out the integration over all solid angles and v' in (31) gives

$$(\delta f/\delta t)_i^{\mathrm{LT}}=vN(\mathbf{r},t)f(\mathbf{r},\mathbf{v},t)\sigma_T(v) \tag{35}$$

for the loss term.

III. THE GAIN TERM

We now try to determine the gain term for the electrons at the phase space point $(\mathbf{r},\mathbf{v})$ at time t, due to ionizing collisions. It was noted in the preceding section that the emerging electron may either be a secondary electron or a scattered electron depending on the range in which the energy of the incoming electron lies. It was seen that for the emerging electron (energy $E=mv^2/2$) to be a secondary electron, the energy of the incoming primary electron must lie in the range

$$2E+E_i\le E_p<\infty\ . \tag{36}$$

Likewise, for the emerging electron to be a scattered electron, one must have

$$E+E_i\le E_p\le 2E+E_i\ . \tag{37}$$

Let $\mathbf{v}_i$ be the velocity of the incoming electron, so that

$E_p = m(v_i)^2/2$. The inequalities (36) and (37) then read, respectively,

$$\sqrt{2}\left[v^2+\frac{E_i}{m}\right]^{1/2} \le v_i < \infty \tag{38}$$

for the electron emerging at $(\mathbf{r},\mathbf{v})$ to be a secondary electron and

$$\sqrt{2}\left[\frac{v^2}{2}+\frac{E_i}{m}\right]^{1/2} \le v_i \le \sqrt{2}\left[v^2+\frac{E_i}{m}\right]^{1/2} \tag{39}$$

for the electron emerging at $(\mathbf{r},\mathbf{v})$ to be a scattered electron. Corresponding to (38) and (39) we shall use the fictitious binary reactions $e+A \rightarrow S+e_{\text{sec}}$ and $e+A \rightarrow S'+e_{\text{sca}}$, respectively, for formulating the gain term.

To construct the gain term, consider electrons in the phase-space volume element $d^3\mathbf{r}d^3\mathbf{v}_i$ at $(\mathbf{r},\mathbf{v}_i)$. The number of such electrons at time t is $f(\mathbf{r},\mathbf{v}_i,t)d^3\mathbf{r}d^3\mathbf{v}_i$. Let these electrons collide with the gas atoms located at $\mathbf{r}$ at time t. Consider gas atoms with velocities in the volume $d^3\mathbf{V}$ at $\mathbf{V}$, parameter p in the range $(p',p'+dp')$ and the l vector lying within the solid angle element $d\Omega_l' = \sin(\theta')d\theta'd\phi'$ (θ' and ϕ' are polar and azimuthal angles subtended by l in a coordinate system in the laboratory frame at $\mathbf{r}$). Here, as before, we have temporarily assumed the gas atoms to have a finite mass. The spatial density of such atoms at the space-time point $(\mathbf{r},t)$ is given by

$$F(\mathbf{r},\mathbf{V},p',l,t)d^3\mathbf{V}dp'd\Omega_l' \ . \tag{40}$$

We simplify F above, using the assumptions made in writing (5) in Sec. II. The spatial density of gas atoms in (40) is then given by

$$\frac{F(\mathbf{r},\mathbf{V},t)}{4\pi}d^3\mathbf{V}dp'd\Omega_l' \ , \quad p_1 \le p' \le p_2 \tag{41}$$

with $(p_2-p_1)=1$ and $d\Omega_l' = \sin(\theta')d\theta'd\varphi'$.

We introduce as before the reference frame $Q'(\mathbf{r},t)$ with its origin attached to the colliding electrons at $(\mathbf{r},t)$, moving with a precollision velocity $\mathbf{v}_i$. In this reference frame, atoms with velocity $\mathbf{V}$ (in the laboratory frame) before collision have a relative velocity $\mathbf{h}=\mathbf{V}-\mathbf{v}_i$. We choose the polar axis of $Q'(\mathbf{r},t)$ along $\mathbf{h}$. We also choose a suitable $\epsilon=0$ axis in the azimuthal plane of $Q'(\mathbf{r},t)$. Let χ_l' and ϵ_l' be, respectively, the polar and azimuthal angles subtended by l in the frame $Q'(\mathbf{r},t)$ at time t (the instant its origin coincides with the space point $\mathbf{r}$). Invoking (4) of Sec. II, we may set $d\Omega_l'$ in (41) equal to $\sin\chi_l' d\chi_l' d\epsilon_l'$ and introduce χ_l' and ϵ_l' as the new variables for describing l.

Of the atoms at $(\mathbf{r},t)$ having a spatial density given by (41) above, consider those that are incident on the azimuthal plane of $Q'(\mathbf{r},t)$ with impact parameters in the range $(b',b'+db')$ and azimuthal angle within $(\epsilon',\epsilon'+d\epsilon')$. These atoms are all incident along the polar axis of $Q'(\mathbf{r},t)$ and collide with the electrons at rest at the origin of $Q'(\mathbf{r},t)$. The number of such atoms that collide within the time interval $(t,t+dt)$ is equal to the number of these atoms that are contained within the parallelepiped of volume $hb'db'd\epsilon'dt$ and is given by

$$\frac{F(\mathbf{r},\mathbf{V},t)}{4\pi}d^3\mathbf{V}dp'd\Omega_l'hb'db'd\epsilon'dt \ , \quad p_1 \le p' \le p_2 \tag{42}$$

with $h=|\mathbf{h}|$. The total number of collisions (labeled dN_c below) is therefore given by

$$dN_c = f(\mathbf{r},\mathbf{v}_i,t)\frac{F(\mathbf{r},\mathbf{V},t)}{4\pi}d^3\mathbf{v}_i d^3\mathbf{r}d^3\mathbf{V}dp'd\Omega_l'hb'db'd\epsilon'dt \ , \quad p_1 \le p' \le p_2 \ . \tag{43}$$

It is assumed that $\mathbf{v}_i,\mathbf{V},p',l,b',\epsilon'$ are such that the velocity of the electron after collision lies in the volume $d^3\mathbf{v}$ about $\mathbf{v}$ while that of the S or S' particle lies within the volume $d^3\mathbf{V}'$ about $\mathbf{V}'$. Thus the postcollision relative velocity of the S (or S') particle in the frame $Q'(\mathbf{r},t)$ (attached to the electrons) is given by $\mathbf{h}'=(\mathbf{V}'-\mathbf{v})$.

The azimuthal angle ϵ' in (43) is measured with respect to the arbitrarily chosen $\epsilon=0$ axis. As before, we may transform it to a new variable α', measured with respect to $l_\perp$ [the projection of l onto the aximuthal plane of $Q'(\mathbf{r},t)$], by following the recipe given in (7). Noting that we will have $d\alpha'=d\epsilon'$, (43) may be written as

$$dN_c = f(\mathbf{r},\mathbf{v}_i,t)\frac{F(\mathbf{r},\mathbf{V},t)}{4\pi}d^3\mathbf{v}_i d^3\mathbf{r}d^3\mathbf{V}dp'd\Omega_l'hb'db'd\alpha'dt \ , \quad p_1 \le p' \le p_2 \ . \tag{44}$$

Following the procedure outlined in Eqs. (10)–(15), we may introduce the collision cross section into (44). This gives

$$dN_c = f(\mathbf{r},\mathbf{v}_i,t)\frac{F(\mathbf{r},\mathbf{V},t)}{4\pi}d^3\mathbf{v}_i d^3\mathbf{r}d^3\mathbf{V}d\Omega_l'hdt \times \sigma_{h',\Omega'}(h;h',\chi_0',\beta')dh'd\Omega' \tag{45}$$

with $h_1 \le h \le h_2$ and $d\Omega' = \sin(\chi_0')d\chi_0'd\beta'$. Here $\sigma_{h',\Omega'}(h;h',\chi_0',\beta')$ is the doubly differential cross section introduced earlier; χ_0' and β' are the polar and azimuthal angles (the latter being measured with respect to $l_\perp$) subtended by $\mathbf{h}'$ in the frame $Q'(\mathbf{r},t)$. The limits (h_1,h_2) of h are deduced from the analog of (12), i.e.,

$$p'=p'(h;h') \ , \tag{46}$$

which may be solved for h to yield the analog of (15):

$$h=h(p',h') \ . \tag{47}$$

Corresponding to the range (p_1,p_2) in which p' can vary, one obtains from (47)

$$h_1(p_1,h') \le h \le h_2(p_2,h')$$

or

$$h_1(p_2,h') \le h \le h_2(p_1,h') \ . \tag{48}$$

Unlike the situation described by (14) where g' could vary for fixed g, we let h vary here for fixed h'. Thus (48) specifies the limits on h in (45).

To transform β' in (45) to angles measured with respect to the $\epsilon=0$ axis, we let the aximuthal angle made by $\mathbf{h}'$ with respect to the $\epsilon=0$ axis be ϵ_0'. β' and ϵ_0' will be relat-

ed by equations identical to that for β and ϵ_0, given in the second set of (7). When using the first of these (i.e., $\beta'=\epsilon_0'+2\pi-\epsilon_l'$ for $\epsilon_0'<\epsilon_l'$), we invoke the periodicity of $\sigma(\beta')[\equiv\sigma_{h',\Omega'}(h;h',\chi_0',\beta')]$ in β' and write

$$\sigma(\beta')=\sigma(\beta'-2\pi)=\sigma(\epsilon_0'+2\pi-\epsilon_l'-2\pi)=\sigma(\epsilon_0'-\epsilon_l') , \quad \epsilon_0'<\epsilon_l' .$$

As a result, β' in (45) may be replaced by $\epsilon_0'-\epsilon_l'$ and using $d\beta'=d\epsilon_0'$, (45) becomes

$$dN_c=f(\mathbf{r},\mathbf{v}_i,t)\frac{F(\mathbf{r},\mathbf{V},t)}{4\pi}d^3\mathbf{v}_i d^3\mathbf{r}d^3\mathbf{V}d\Omega_l' h dt \times\sigma_{h',\Omega_0'}(h;h',\chi_0',\epsilon_0'-\epsilon_l')dh'd\Omega_0' \tag{49}$$

with $h_1\le h\le h_2$ and $d\Omega_0'=\sin(\chi_0')d\chi_0'd\epsilon_0'$. We now invoke the limit of infinite mass for the gas atoms A and the S/S' particles. In this limit both $|\mathbf{V}|$ and $|\mathbf{V}'|\to 0$ so that $h\to v_i$, $h'\to v$, and $F(\mathbf{r},\mathbf{V},t)\to N(\mathbf{r},t)\delta^3(\mathbf{V})$, where $N(\mathbf{r},\mathbf{t})$ is the density of gas atoms at $(\mathbf{r},t)$ and $\delta^3(\mathbf{V})$ is the three-dimensional Dirac δ function in the velocity $\mathbf{V}$. Further (46) and (47) become in this limit

$$p'=p'(v_i;v) , \quad v_i=v_i(p',v) \tag{50}$$

so that (48) reads

$$v_{i1}(p_1,v)\le v_i\le v_{i2}(p_2,v)$$

or (51)

$$v_{i1}(p_2,v)\le v_i\le v_{i2}(p_1,v)$$

and $\sigma_{h',\Omega_0'}(h;h',\chi_0',\epsilon_0'-\epsilon_l')$ transforms to $\sigma_{v,\Omega_0'}(v_i;v,\chi_0',\epsilon_0'-\epsilon_l')$. Again, as before, the polar axis $\mathbf{h}\to-\mathbf{v}_i$ and the vector $\mathbf{h}'\to-\mathbf{v}$. Thus χ_0' is now the angle between the vectors $-\mathbf{v}$ and $-\mathbf{v}_i$ and ϵ_0' is the azimuthal angle subtended by $-\mathbf{v}$ (with respect to the $\epsilon=0$ axis). Equation (49) now reads

$$dN_c=v_i f(\mathbf{r},\mathbf{v}_i,t)N(\mathbf{r},t)\delta^3(\mathbf{V})d^3\mathbf{v}_i d^3\mathbf{r}d^3\mathbf{V}\frac{d\Omega_l'}{4\pi}dt \times\sigma_{v,\Omega_0'}(v_i;v,\chi_0',\epsilon_0'-\epsilon_l')dvd\Omega_0' \tag{52}$$

with $v_{i1}\le v_i\le v_{i2}$, $d\Omega_0'=\sin(\chi_0')d\chi_0'd\epsilon_0'$, and the limits v_{i1} and v_{i2} being given by (51).

Equation (52) gives the contribution to the gain term for fixed $\mathbf{v}_i$, $\mathbf{V}$, and l at the phase-space point $(\mathbf{r},\mathbf{v})$ at time t. To obtain the contribution for all $\mathbf{V}$ and l we integrate (52) over $\mathbf{V}$ and all possible orientations of l (i.e., χ_l' and ϵ_l'). The integral over $\mathbf{v}$ yields unity, while the integral over χ_l' is also straightforward. Finally, the integral over ϵ_l' may be handled as in Sec. II (see Fig. 4) and Eqs. (23)–(25). Analogously to (26) we define an averaged σ as follows:

$$\tilde{\sigma}_{v,\Omega_0'}(v_i;v,\chi_0')=\int_0^{2\pi}\sigma_{v,\Omega_0'}(v_i;v,\chi_0',\epsilon_0'-\epsilon_l')\frac{d\epsilon_l'}{2\pi} . \tag{53}$$

Integrating (52) over $\mathbf{V}$ and all possible orientations of l yields the gain term at $(\mathbf{r},\mathbf{v})$ at time t for a fixed value of $\mathbf{v}_i$. Denoting this part gain term by $(\delta f/\delta t)_{i,\mathbf{v}_i}^{GT}d^3\mathbf{r}\,d^3\mathbf{v}\,dt$ (the subscript $\mathbf{v}_i$ indicates a fixed value of $\mathbf{v}_i$) we have [on integrating (52) and using (53)]

$$(\delta f/\delta t)_{i,\mathbf{v}_i}^{\mathrm{GT}}d^3\mathbf{r}d^3\mathbf{v}dt = v_i f(\mathbf{r},\mathbf{v}_i,t)N(\mathbf{r},t)d^3\mathbf{v}_i d^3\mathbf{r}dt\tilde{\sigma}_{v,\Omega_0'}(v_i;v,\chi_0')dvd\Omega_0' \tag{54}$$

with $v_{i1}\le v_i\le v_{i2}$ and $d\Omega_0'=2\pi\sin(\chi_0')d\chi_0'$. $\tilde{\sigma}_{v,\Omega_0'}(v_i;v,\chi_0')$ in (54) is measured in $Q'(\mathbf{r},t)$. In a frame of reference at rest with the atoms the electrons approach the atoms with a velocity $+\mathbf{v}_i$ and are scattered off with a velocity $+\mathbf{v}$, making an angle $(\pi-\chi_0')$ with respect to the $\mathbf{v}_i$ direction (the polar axis). The cross section for the scattering of electrons in this frame is identical to the cross section $\tilde{\sigma}_{v,\Omega_0'}(v_i;v,\chi_0')$ for the scattering of the S/S' particles in $Q'(\mathbf{r},t)$. Under a rotation of the polar axis from the $-\mathbf{v}_i$ direction to the $+\mathbf{v}_i$ direction (in the rest frame of the atoms) $\tilde{\sigma}_{v,\Omega_0'}(v_i;v,\chi_0')$ becomes

$$\tilde{\sigma}_{v,\Omega_0'}(v_i;v,\chi_0')\to\sigma'_{v,\Omega_0'}(v_i;v,\chi_0')=\tilde{\sigma}_{v,\Omega_0'}(v_i;v,\pi-\chi_0') , \tag{55}$$

where χ_0' is the angle made by $+\mathbf{v}$ with respect to the $+\mathbf{v}_i$ direction. Under the rotation of the polar axis in the rest frame of the atoms, the element of the solid angle $d\Omega_0'$ also remains unchanged. Noting that the rest frame of the atoms is also the laboratory frame and that

$$d^3\mathbf{v}=v^2dvd\Omega_0' , \quad d\Omega_0'=2\pi\sin(\chi_0')d\chi_0' , \tag{56}$$

we have on substituting for $\tilde{\sigma}_{v,\Omega_0'}(v_i;v,\chi_0')$ from (55) into (54) and using (56)

$$(\delta f/\delta t)_{i,\mathbf{v}_i}^{\mathrm{GT}}=\frac{N(\mathbf{r},t)}{v^2}v_i f(\mathbf{r},\mathbf{v}_i,t)\sigma'_{v,\Omega_0'}(v_i;v,\chi_0')d^3\mathbf{v}_i . \tag{57}$$

To obtain the total gain term we sum over all $\mathbf{v}_i$. This gives

$$(\delta f/\delta t)_i^{\mathrm{GT}}=\sum_{\mathbf{v}_i}(\delta f/\delta t)_{i,\mathbf{v}_i}^{\mathrm{GT}} =\frac{N(\mathbf{r},t)}{v^2}\int d^3\mathbf{v}_i f(\mathbf{r},\mathbf{v}_i,t)v_i\sigma'_{v,\Omega_0'}(v_i;v,\chi_0') , \quad v_{i1}\le v_i\le v_{i2} \tag{58}$$

where χ_0' is the angle between $\mathbf{v}_i$ and the fixed vector $\mathbf{v}$. Correspondence with the ionization problem is made by setting

$$\sigma'_{v,\Omega_0'}(v_i;v,\chi_0')=0 \quad \text{for } 0\le v_i\le\left[v^2+\frac{2E_i}{m}\right]^{1/2} , \tag{59}$$

$$\sigma'_{v,\Omega_0'}(v_i;v,\chi_0')=\sigma^{\mathrm{sca}}_{v,\Omega_0'}(v_i;v,\chi_0') \quad \text{for } \left[v^2+\frac{2E_i}{m}\right]^{1/2}\le v_i\le\sqrt{2}\left[v^2+\frac{E_i}{m}\right]^{1/2} , \tag{60}$$

$$\sigma'_{v,\Omega_0'}(v_i;v,\chi_0')=\sigma^{\mathrm{sec}}_{v,\Omega_0'}(v_i;v,\chi_0')$$

$$\text{for } \sqrt{2}\left[v^2+\frac{E_i}{m}\right]^{1/2} \le v_i \le \infty \ . \quad (61)$$

The limits on v_i in (60) and (61) are obtained from the inequalities (39) and (38) discussed earlier. The total collision term is obtained by combining the gain term in (58) with the loss term given by (35). Thus

$$(\delta f/\delta t)_i(\mathbf{r},\mathbf{v},t) = \frac{N(\mathbf{r},t)}{v^2}\int d^3\mathbf{v}_i f(\mathbf{r},\mathbf{v}_i,t) v_i \sigma'_{v,\Omega_0'}(v_i;v,\chi_0') - vN(\mathbf{r},t) f(\mathbf{r},\mathbf{v},t)\sigma_T(v) \ , \quad (62)$$

with χ_0' being the angle between $\mathbf{v}_i$ and $\mathbf{v}$ (a fixed vector), $\sigma'_{v,\Omega_0'}(v_i;v,\chi_0')$ given by Eqs. (59) to (61), and $\sigma_T(v)$ given by (32) or (33).

It is easy to show that for an electron distribution function that is isotropic in the velocity, the form of the collision integral derived by Holstein [1,2] is recovered.

IV. CONCLUSION

In conclusion, we have in this paper, derived a collision integral corresponding to ionizing collisions made by electrons. The form derived is more general than previous ones [1] in that it is not restricted to isotropic electron distribution functions and is based on doubly differential cross sections rather than singly differential cross sections for ionization by electron impact. This is the most detailed description feasible today because there is already sufficiently complete and detailed DDCS data, both analytical and experimental, available for several target gases. On the other hand, the data on triply differential cross sections, which would provide an even more detailed description of electron-impact ionization, are incomplete even for the simplest target gases, so that the construction of a collision integral based on TDCS would not be of much practical significance today.

This work represents a step towards a more realistic modeling of partially ionized gases in which electron-impact ionization plays a dominant role and the electron distribution functions are distinctly nonisotropic. In particular, this is the case for rf or microwave discharge plasmas confined in magnetic mirror fields, to which we intend to apply the present result.

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