Geometric phase and sequential measurements in quantum mechanics

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A rule is proposed to calculate the geometric phase accumulating in the course of a sequential measurement process. This rule is based on the assumptions of a sequential conditioning and a geodesic hypothesis. The dependence of this phase on the type of the applied measurements is explicated.

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I. INTRODUCTION

In their seminal paper [1] Samuel and Bhandari have put forward the idea that the geometric phase arising in a cyclic sequence of measurements is a physically meaningful quantity that can be measured. Hence, one is faced with the problem of a theoretical calculation of this phase.

The first step in solving this problem is the observation that the change of the phase of a state vector in a sequence of measurements can be divided into a sum of two contributions: a phase due to the Hamiltonian evolution of the measured system during the time intervals between the measurements and the phase due *exclusively* to the measurement acts. If we suppose that the dynamical phase can be calculated separately, we can forget it in the sequel and focus our analysis only on the second one.

Samuel and Bhandari have determined the geometric phase generated during the sequence of measurements in the particular case of the von Neumann–Lüders schematization of a measurement process. There are, however, both physical needs [2] and mathematical arguments [2,3] to go beyond that scheme. Therefore, it is the aim of this paper to give a more general rule for the calculation of geometric phase (in the measurement context) which applies to *arbitrary* measurements, and which gives, *as a particular case,* the one of Samuel and Bhandari [1].

One further point has to be clarified: when the problem of the accumulation of a phase in a course of a cyclic sequence of measurements is brought on the basis of the quantum theory of measurement, it becomes clear that this theory does not lead in itself to a determination of a phase. Some further assumptions are necessary. In their paper Samuel and Bhandari have implicitly used a strengthening for vector states of the von Neumann projection postulate [cf. items (1) and (2) of Sec. III], which cannot be used in a more general setting.

The strategy we have used in our paper is the following. First of all, there is the assumption of *sequential conditioning,* that is, the assumption that after each individual measurement a result is registered. This rather common assumption gives us a sequence of states from the initial to the final state of the system. However, such a sequence of states does not yet determine the accumulation of a phase. A rule has to be posed to connect the intermediate states with each other. We apply a *geodesic hypothesis* which claims that the intermediate states are connected to each other along the geodesic lines of the projective space of the pure states of the system. These two rules, the sequential conditioning and the geodesic hypothesis, allow one to attach a geometric phase to any cyclic sequential measurement process.

The theory of sequential measurements is an important part of the quantum theory of measurement, in particular, since it gives measurement dependent predictions. The possibility of performing sequential measurements would therefore open also the possibility for an experimental study of geometric phase. Some experimental results on this topic are already available in the literature [4,5]. However, these investigations deal with photons in a semiclassical state, a fact which may cause some additional interpretational problems. It would therefore be important to have such measurements done also with individual objects, either photons or massive particles.

II. SEQUENTIAL MEASUREMENTS

Let A be a discrete observable, with the eigenvalues a_i and the eigenprojections P_i, and let $(\varphi_{ij})_{i,j\geq 1}$ be an orthonormal basis of eigenvectors of A:

$$A=\sum_i a_i P_i=\sum_i a_i \sum_j P[\varphi_{ij}] \ . \tag{2.1}$$

The probability that a measurement of A leads to a result a_k if the system is prepared in the state $P[\varphi]$ is

$$p_\varphi(a_k):=\langle \varphi|P_k\varphi\rangle=\sum_j |\langle \varphi|\varphi_{kj}\rangle|^2 \ . \tag{2.2}$$

Clearly, $p_\varphi(a_k)$ depends only on the state $P[\varphi]$ and not on its vector representative φ.

In the frame of the quantum theory of measurement a (minimal) measurement of A consists of a (nondegenerate) pointer observable $A_M=\sum \bar{a}_i P[\Phi_i]$ [$(\Phi_i)_{i\geq 1}$ an orthonormal basis of the Hilbert space $\mathcal{H}_M$ of the "pointer states" [6], $\bar{a}_i \leftrightarrow a_i$], an initial state $P[\Phi]$ of the measuring apparatus, and a unitary mapping (measurement coupling),

$$U:\varphi\otimes\Phi\mapsto U(\varphi\otimes\Phi) \ , \tag{2.3}$$

satisfying the probability reproducibility condition

$$p_\varphi(a_k)=\langle U(\varphi\otimes\Phi)|I\otimes P[\Phi_k]U(\varphi\otimes\Phi)\rangle \tag{2.4}$$

for all $k=1,2,\ldots$ and for all initial states $P[\varphi]$ of the measured system [3,7]. It can be shown [7] that any unitary operator U satisfying (2.4) acts on the vector $\varphi_{ij}\otimes\Phi$ as

$$U(\varphi_{ij}\otimes\Phi)=\psi_{ij}\otimes\Phi_i\ , \tag{2.5}$$

where $\psi_{ij}=\sum_{m,n}\langle\varphi_{mn}\otimes\Phi_i|U(\varphi_{ij}\otimes\Phi)\rangle\varphi_{mn}$. Conversely, given any set of unit vectors $(\psi_{ij})_{i,j\geq 1}\subset\mathcal{H}$ which fulfill the orthogonality conditions $\langle\psi_{ij}|\psi_{ij'}\rangle=\delta_{jj'}$ for all $i=1,2,\ldots$ and $\dim(\{\varphi_{ij}\otimes\Phi\}^\perp)=\dim(\{\psi_{ij}\otimes\Phi_i\}^\perp)$, the action (2.5) can be extended to a unitary operator U, on the Hilbert space $\mathcal{H}\otimes\mathcal{H}_M$, satisfying (2.4). In view of these results we say that (2.5) defines the measurement coupling U.

For any unit vector φ we then have

$$\begin{aligned}U(\varphi\otimes\Phi)&=\sum_{ij}\langle\varphi_{ij}|\varphi\rangle U(\varphi_{ij}\otimes\Phi)\\&=\sum_{ij}\langle\varphi_{ij}|\varphi\rangle\psi_{ij}\otimes\Phi_i\\&=\sum_i\sqrt{p_\varphi(a_i)}\gamma_i\otimes\Phi_i\ ,\end{aligned} \tag{2.6}$$

where

$$\gamma_i:=\sum_j\langle\varphi_{ij}|\varphi\rangle\psi_{ij}/\sqrt{p_\varphi(a_i)}$$

[whenever $p_\varphi(a_i)\neq 0$; otherwise we put $\gamma_i=0$].

At this stage an important fact has to be stressed. The defining condition (2.4) of the mapping U shows, in fact, that we have to consider as equivalent different operators U for which the final state of the compound system $P[U(\varphi\otimes\Phi)]$, and thus also the final states of the system and the apparatus, are the same, that is,

$$U\sim U'\Longleftrightarrow U'(\varphi\otimes\Phi)=e^{i\alpha}U(\varphi\otimes\Phi)\quad\forall\varphi\ . \tag{2.7}$$

The physically relevant results must then be invariant with respect to the transformation

$$U\mapsto U'=e^{i\alpha}U\ . \tag{2.8}$$

There are three, increasingly restrictive choices of the measurement coupling U, or, respectively, the U-generating sets of unit vectors $(\psi_{ij})_{i,j\geq 1}$ worth mentioning separately [7,8]:

(a) $(\psi_{ij})_{i,j\geq 1}$ is an orthonormal set. This choice leads exactly to the so-called strong state correlation measurements, i.e., to measurements which provide strong correlations between the final component states $P[\Phi_k]$ and $P[\gamma_k]$ of the measuring apparatus and the measured system.

(b) $(\psi_{ij})_{i,j\geq 1}$ is an orthonormal set of eigenvectors of A. This choice is characteristic for the strong value correlation measurements of A. Such measurements lead to strong correlations between the eigenprojections $P[\Phi_k]$ and P_k (and thus of the eigenvalues $\bar{a}_k$ and a_k) of the pointer observable A_M and the measured observable A.

(c) Finally, the choice $\psi_{ij}=\varphi_{ij}$ corresponds to a von Neumann–Lüders measurement of A.

In general, a measurement of the observable A on the system in the state $P[\varphi]$ induces a change of the state:

$$\begin{aligned}P[\varphi]&\mapsto\sum_i p_\varphi(a_i)P[\gamma_i]\\&\mapsto P[\gamma_k]\ ,\end{aligned} \tag{2.9}$$

where $\sum_i p_\varphi(a_i)P[\gamma_i]$ is the state of the system after the measurement on the condition that the measurement has been performed (with no further assumption), whereas $P[\gamma_k]$ is the final state of the system on the condition that the pointer observable A_M assumes the value $\bar{a}_k$ [9].

Henceforth, we shall restrict ourselves to consider A measurements with definite outcomes so that the induced state changes are of the form

$$P[\varphi]\mapsto P[\gamma_k]\ , \tag{2.10}$$

where, to emphasize, the final state $P[\gamma_k]$ of the system depends on the pointer value $\bar{a}_k$ but also on the type of the involved A measurement, that is, on the applied measurement coupling U.

Consider now a (finite) sequence of measurements of the observables

$$A^{(s)}=\sum a_i^{(s)}P_i^{(s)}\ ,\quad s=1,\ldots,n \tag{2.11}$$

on the system initially in the state $P[\varphi]$. Omitting the temporal description of the system, the probability that a sequential measurement of the observables $A^{(1)}$, $A^{(2)},\ldots,A^{(n)}$ leads to an outcome sequence $(a_{k_1}^{(1)}, a_{k_2}^{(2)},\ldots,a_{k_n}^{(n)})$ is

$$\begin{aligned}\mathrm{Tr}[\mathcal{J}^{A^{(1)}\cdots A^{(n)}}((a_{k_1}^{(1)},\ldots,a_{k_n}^{(n)}))P[\varphi]]&=\mathrm{Tr}[\mathcal{J}^{A^{(n)}}(\{a_{k_n}^{(n)}\})\circ\cdots\circ\mathcal{J}^{A^{(1)}}(\{a_{k_1}^{(1)}\})P[\varphi]]\\&=p_\varphi(a_{k_1}^{(1)})p_{\gamma_{k_1}^{(1)}}(a_{k_2}^{(2)})\cdots p_{\gamma_{k_{n-1}}^{(n-1)}}(a_{k_n}^{(n)})\\&=\langle\varphi|P_{k_1}^{(1)}\varphi\rangle\langle\gamma_{k_1}^{(1)}|P_{k_2}^{(2)}\gamma_{k_1}^{(1)}\rangle\cdots\langle\gamma_{k_{n-1}}^{(n-1)}|P_{k_n}^{(n)}\gamma_{k_{n-1}}^{(n-1)}\rangle\ ,\end{aligned} \tag{2.12}$$

where $\mathcal{J}^{A^{(1)}\cdots A^{(n)}}$ is the instrument defined by the sequential $A^{(1)}\cdots A^{(n)}$ measurement consisting of the fixed $A^{(s)}$ measurements with the associated instruments $\mathcal{J}^{A^{(s)}}$ [10,11]. The sequential measurement result $(a_{k_1}^{(1)},\ldots,a_{k_n}^{(n)})$ is accompanied with the state change

$$P[\varphi]\rightarrow P[\gamma_{k_n}^{(n)}]\ , \tag{2.13}$$

which, due to

$$\mathcal{J}^{A^{(1)}\cdots A^{(n)}}((a_{k_1}^{(1)},\ldots,a_{k_n}^{(n)}))P[\varphi] = \mathcal{J}^{A^{(n)}}(\{a_{k_n}^{(n)}\})\circ\cdots\circ\mathcal{J}^{A^{(1)}}(\{a_{k_1}^{(1)}\})P[\varphi]\ , \quad (2.14)$$

can formally be decomposed as

$$P[\varphi]\to P[\gamma_{k_1}^{(1)}]\to\cdots\to P[\gamma_{k_n}^{(n)}]\ . \quad (2.15)$$

This sequence of state changes is, however, not given by the above sequential $A^{(1)}\cdots A^{(n)}$ measurement with the result $(a_{k_1}^{(1)},\ldots,a_{k_n}^{(n)})$ but it is subject to a further assumption, namely the sequential conditioning. This is the assumption that after each involved $A^{(s)}$ measurement the conditioning with respect to the pointer value $\bar{a}_{k_s}$ is obtained. In other words, the sequential state change (2.15) is obtained in the course of the sequential $A^{(1)}\cdots A^{(n)}$ measurement if after each $A^{(s)}$ measurement the outcome $\bar{a}_{k_s}$ is registered. It is to be emphasized that in both cases (intermediate conditioning or not) the probability to obtain the outcome sequence $(a_{k_1}^{(1)},\ldots,a_{k_n}^{(n)})$ as well as the final (conditional) state $P[\gamma_{k_n}^{(n)}]$ of the system are the same; they do not depend on the type of conditioning, whether obtained stepwisely or only terminal. To obtain also the intermediate states $P[\gamma_{k_s}^{(s)}]$ requires, however, intermediate conditionings.

For the interest of this paper we close the above measurement sequence with a measurement of the simple observable

$$A^{(0)}=a_+^{(0)}P[\varphi]+a_-^{(0)}(I-P[\varphi]) \quad (2.16)$$

choosing a $U^{(0)}$-generating set $(\psi_{ij}^0)_{i,j\geq 1}$ such that $\psi_1^{(0)}=\varphi$. The cyclic state change

$$P[\varphi]\to P[\varphi] \quad (2.17)$$

is then associated with the outcome sequence $(a_{k_1}^{(1)},\ldots,a_{k_n}^{(n)},a_+^{(0)})$, the probability of which being

$$p_\varphi(a_{k_1}^{(1)})p_{\gamma_{k_1}^{(1)}}(a_{k_2}^{(2)})\cdots p_{\gamma_{k_n}^{(n)}}(a_+^{(0)}) = \langle\varphi|P_{k_1}^{(1)}\varphi\rangle\langle\gamma_{k_1}^{(1)}|P_{k_2}^{(2)}\gamma_{k_1}^{(1)}\rangle\cdots\langle\gamma_{k_n}^{(n)}|P[\varphi]\gamma_{k_n}^{(n)}\rangle\ . \quad (2.18)$$

With the assumption of sequential conditioning the state change (2.17) decomposes to a cycle

$$P[\varphi]\to P[\gamma_{k_1}^{(1)}]\to\cdots\to P[\gamma_{k_n}^{(n)}]\to P[\varphi]\ . \quad (2.19)$$

III. GEODESIC HYPOTHESIS

We shall study next the geometric phase arising in a sequential $A^{(1)}\cdots A^{(n)}A^{(0)}$ measurement leading to a state change (2.19).

We proceed in analogy with a Hamiltonian cyclic evolution. In this case, the time evolution gives us a closed path $\mathcal{C}$ in the projective space $\mathcal{P}(\mathcal{H}):=\{P[\varphi]|\varphi\in\mathcal{H},\ \varphi\neq 0\}$ of the pure states of the system. The curve $\mathcal{C}$ is given by the map

$$[t_0,t_1]\ni t\mapsto P[\varphi(t)]\ ,\quad P[\varphi(t_0)]=P[\varphi(t_1)]\ ,$$

and the geometric phase developed during the cyclic evolution from t_0 to t_1 is given by

$$\int_{\mathcal{S}}\omega\ ,$$

where $\mathcal{S}$ is a surface in $\mathcal{P}(\mathcal{H})$ that has the curve $\mathcal{C}$ as its boundary and ω is the "geometric phase 2-form" [12] given by

$$\omega_{P[\varphi]}(X,Y):=i\langle\varphi|[X,Y]\varphi\rangle\ .$$

To explain this formula, we recall a few things. The projective space $\mathcal{P}(\mathcal{H})$ of pure states is a differentiable manifold and, at the same time, a subset of the vector space of bounded operators on $\mathcal{H}$. Hence, the elements of the tangent space $X,Y\in T_{P[\varphi]}(\mathcal{P}(\mathcal{H}))$ are bounded operators, self-adjoint and traceless, and a 2-form on $\mathcal{P}(\mathcal{H})$ is a map $P[\varphi]\mapsto\omega_{P[\varphi]}$, where $\omega_{P[\varphi]}$ is an antisymmetric bilinear function

$$\omega_{P[\varphi]}:T_{P[\varphi]}(\mathcal{P}(\mathcal{H}))\times T_{P[\varphi]}(\mathcal{P}(\mathcal{H}))\to\mathbf{C}\ .$$

In this way the standard theory of integration of p forms on differentiable manifolds can be used to give meaning to the integral $\int_{\mathcal{S}}\omega$ [13].

To use the above analogy, we have a fundamental problem: the sequence (2.19) gives only the $n+1$ points in $\mathcal{P}(\mathcal{H})$ and not any curve connecting these points. However, the projective space of pure states has a natural metric arising from the inner product of $\mathcal{H}$ [1]. This metric allows one to define the notion of a geodesic line in $\mathcal{P}(\mathcal{H})$.

We call *geodesic hypothesis* the assumption that the $n+1$ points in the projective space of the pure states have to be connected by geodesic lines. Explicitly, we denote $\mathcal{G}_i$ the shortest geodesic connecting the states $P[\gamma_{k_i}^i]$ and $P[\gamma_{k_{i+1}}^{i+1}]$, i runs from 0 to n, with the convention that $P[\gamma_{k_0}^0]=P[\varphi]=P[\gamma_{k_{n+1}}^{n+1}]$. [We recall that, given two nonorthogonal elements $P[\varphi]$ and $P[\psi]$ in $\mathcal{P}(\mathcal{H})$, there are two geodesic lines connecting them.] We let

$$\mathcal{G}=\cup_{i=0}^n\mathcal{G}_i\ . \quad (3.1)$$

In this way we have a simple closed curve $\mathcal{G}$, in the space of pure states, that connects the states of the sequence (2.19). In analogy with the Hamiltonian evolution, we propose that the geometric phase acquired during the sequential measurement process (2.19) has to be computed as

$$\Gamma=\int_{\mathcal{S}}\omega\ , \quad (3.2)$$

where $\mathcal{S}$ is a surface that has $\mathcal{G}$ as its boundary.

In order to calculate the phase Γ explicitly we need to express it with a formula that uses only quantities defined in the Hilbert space rather than in the projective space of states.

Indeed, let

$$\varphi:[0,1]\mapsto\mathcal{H}\ ,\quad s\mapsto\varphi(s) \quad (3.3)$$

be a map parametrizing a simple regular closed curve $\bar{\mathcal{G}}$ of unit vectors in $\mathcal{H}$, such that its projection $\pi(\bar{\mathcal{G}})$ on

$\mathcal{P}(\mathcal{H})$ is $\mathcal{G}$. Using the theorem of Stokes and the "pull-back" theorem, one may show [14] that

$$\Gamma=\int_{\mathcal{S}}\omega=i\int_0^1\left\langle\varphi(s)\,\middle|\,\frac{d}{ds}\varphi(s)\right\rangle ds=:i\oint_{\bar{\mathcal{G}}}\langle\varphi|d\varphi\rangle\ . \tag{3.4}$$

This formula can be further explicated using the following property, due to Samuel and Bhandari [1]: for any nonorthogonal unit vectors φ and ψ in $\mathcal{H}$ their phase difference $\arg\langle\varphi|\psi\rangle$ is given by

$$\arg\langle\varphi|\psi\rangle=-i\int_C\langle\varphi|d\varphi\rangle\ , \tag{3.5}$$

where the line integral is calculated along any simple curve C of unit vectors in $\mathcal{H}$ connecting φ and ψ, whose image $\pi(C)$ in $\mathcal{P}(\mathcal{H})$ is the shortest geodesic line connecting the pure states $P[\varphi]$ and $P[\psi]$.

Using this result we have

$$\begin{aligned}\Gamma=i\oint_{\bar{\mathcal{G}}}\langle\varphi|d\varphi\rangle&=i\sum_j\int_{\bar{\mathcal{G}}_j}\langle\varphi|d\varphi\rangle\\&=-\arg\langle\varphi|\gamma_{k_1}^{(1)}\rangle+\cdots+\arg\langle\gamma_{k_n}^{(n)}|\varphi\rangle\\&=-\arg(\langle\varphi|\gamma_{k_1}^{(1)}\rangle\cdots\langle\gamma_{k_n}^{(n)}|\varphi\rangle)\\&=\arg(\langle\varphi|P[\gamma_{k_n}^{(n)}]\cdots P[\gamma_{k_1}^{(1)}]|\varphi\rangle)\ .\end{aligned} \tag{3.6}$$

The last equality makes it clear that the geometric phase depends only on the *states* $P[\gamma_{k_i}^{(i)}]$ and not on the *vectors* $\gamma_{k_i}^{(i)}$.

The geodesic hypothesis deserves some further comments:

(1) In the case of the von Neumann–Lüders measurements, the vectors $\gamma_{k_i}^{(i)}$ are (modulo normalization)

$$\gamma_{k_1}^{(1)}=P_{k_i}^{(1)}\varphi\ ,$$
$$\gamma_{k_i}^{(i)}=P_{k_i}^{(i)}\gamma_{k_{i-1}}^{(i-1)}\ ,$$

where P_{k_i} is the eigenprojection of the observable $A^{(i)}$ belonging to the eigenvalue $a_{k_1}^{(i)}$. Using (3.6) one obtains that the phase associated to this sequence of states is given by

$$\Gamma_{vNL}=\arg(\langle\varphi|P_{k_n}^{(n)}\cdots P_{k_1}^{(1)}\varphi\rangle)\ . \tag{3.7}$$

This result coincides with those of Samuel and Bhandari [1], Anandan and Aharonov [12], and Benedict and Fehér [15]. In their paper Samuel and Bhandari [1] have defined the phase Γ_{vNL} using the following assumption: in a von Neumann–Lüders measurement the vector φ undergoes the change

$$\varphi\mapsto\sum_j\langle\varphi_{kj}|\varphi\rangle\varphi_{kj}=P_k\varphi \tag{3.8}$$

so that in the total cyclic process (2.19) the vector φ changes as

$$\varphi\mapsto\sum_{j_1}\cdots\sum_{j_n}\langle\varphi_{k_1j_1}^{(1)}|\varphi\rangle\cdots\langle\varphi|\varphi_{k_nj_n}^{(n)}\rangle\varphi=\langle\varphi|P_{k_n}^{(n)}\cdots P_{k_1}^{(1)}\varphi\rangle\varphi\ , \tag{3.9}$$

and the phase accumulated is given by (3.7).

(2) We stress that the change of a state vector $\varphi\mapsto P_k\varphi$ is *not* contained in the (non-normalized) state transformation $P[\varphi]\mapsto P[P_k\varphi]$ associated with the von Neumann–Lüders measurement. Formula (3.8) has been extrapolated from the quantum theory of measurement, but it is not a part of that theory.

(3) The definition of the phase of Samuel and Bhandari [1] cannot be used in the general case. Indeed, the natural extension of (3.8) is

$$\varphi\mapsto\sum_j\langle\varphi_{kj}|\varphi\rangle\psi_{kj}\ , \tag{3.10}$$

which in the cyclic process (2.19) would develop a phase

$$\Gamma^*=\arg\left[\sum_{j_1}\cdots\sum_{j_n}\langle\varphi_{k_1j_1}^{(1)}|\varphi\rangle\langle\varphi_{k_2j_2}^{(2)}|\psi_{k_1j_1}^{(1)}\rangle\cdots\langle\varphi|\psi_{k_nj_n}^{(n)}\rangle\right]\ . \tag{3.11}$$

However, Γ^* is not invariant under the phase transformations (2.8), rather it undergoes the change

$$\Gamma^*\mapsto\Gamma^*+\sum_{i=1}^n d(i)\alpha_i\ ,$$

where α_i is the phase factor of (2.8) for the ith step of the measurement and $d(i)$ is the dimension of $P_{k_i}^{(i)}\mathcal{H}$. This fact leads us to consider Γ^* as unphysical. Of course, in the von Neumann–Lüders scheme, the condition $\varphi_{ij}=\psi_{ij}$ forces α_i to zero and so Γ^* turns out to be invariant.

(4) In the case of the von Neumann–Lüders measurements, the geodesic hypothesis and the assumption (3.8) give rise to the same value of the accumulated geometric phase. Nevertheless, they are quite different because the latter allows one to define a phase after each measurement of the sequence, whereas the former considers the geometric phase as a global property of the cyclic sequence of the measurements and not of the single state. In fact, if the sequence is not cyclic one cannot define a phase under the geodesic hypothesis.

(5) The sequential measurement process (2.19) is associated with the measurement result sequence $(a_{k_1}^{(1)}, a_{k_2}^{(2)},\ldots,a_{k_n}^{(n)}, a_+^{(0)})$. This outcome has the sequential probability (2.18). The state change (2.19) may also be associated with a transition probability

$$|\langle\varphi|\gamma_{k_1}^{(1)}\rangle|^2|\langle\gamma_{k_1}^{(1)}|\gamma_{k_2}^{(2)}\rangle|^2\cdots|\langle\gamma_{k_n}^{(n)}|\varphi\rangle|^2\ . \tag{3.12}$$

In general, the two probabilities (2.18) and (3.12) are different. In other words, the probability for the process (2.19) is not, in general, the same as the transition probability for this chain of state changes. In fact, assuming that the two probabilities are always the same is equivalent to the assumption that the involved $A^{(s)}$ measurements are von Neumann–Lüders ones. As already noted above, in this case the geodesic hypothesis is equivalent to the rule (3.9) of Samuel and Bhandari [1].

IV. CONCLUDING REMARKS

In this paper we have investigated the assumptions under which a cyclic sequential measurement process gives rise to a geometric phase. The first assumption is the sequential conditioning: after each participating measurement a result (pointer value) is obtained. The second assumption, the geodesic hypothesis, says that the intermediate states thus obtained are connected with each other along the geodesic lines. The two assumptions allow one to attach a geometric phase to a cyclic sequential measurement process. The resulting phase depends also on the involved measurements, that is, on the applied measurement couplings. An experimental determination of the geometric phase in a cyclic sequential measurement process would thus provide a test of the validity of the geodesic hypothesis in the context of sequential measurements. The possibility of an experimental determination of the geometric phase in sequential measurements is anticipated in Ref. [1].

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